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On the Uniqueness of Limit Cycles in Codimension Two Bifurcations

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Abstract In this paper we present a survey on the uniqueness of limit cycles bifurcating from a center, homoclinic loop or a heterclinic loop in planar systems, introducing five bifurcation theorems, and then apply the theorems to the study of codimension two bifurcations, obtaining a complete analysis on the uniqueness of limit cycles.

Keywords Uniqueness, limit cycle, bifurcation, codimension two

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1. Introduction

Consider a 2-dimensional system of the form

$$\begin{aligned} \dot{x} &= f(x, y, \varepsilon), \\ \dot{y} &= g(x, y, \varepsilon), \end{aligned} \tag{1.1}$$

where $(x, y) \in \mathbb{R}^2,\ \varepsilon \in \mathbb{R}^2,\ f, g \in C^\infty$. Suppose

$$\frac{\partial(f, g)}{\partial(x, y)}\Big|_{x=y=\varepsilon=0} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{1.2}$$

Then from [5] we know that under certain conditions (1.1) is equivalent to

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= \mu_1 + \mu_2 y + x^2 + xyP(x, \mu) + y^2 Q(x, y, \mu) \end{aligned} \tag{1.3}$$

for $|x| + |y| + |\varepsilon|$ small, where $\mu = (\mu_1, \mu_2)$, $P, Q \in C^\infty$ with $P(0, 0) = 1$.

If (1.1) is centrally symmetric with respect to the origin, then under (1.2) and certain conditions (1.1) is equivalent to

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= \mu_1 x + \mu_2 y \pm x^3 + x^2 yP(x, \mu) + y^2 Q(x, y, \mu) \end{aligned} \tag{1.4}$$

for $|x| + |y| + |\varepsilon|$ small, where $\mu = (\mu_1, \mu_2)$, $P, Q \in C^\infty$ with

$$P(-x, \mu) = P(x, \mu) = -1 + O(|\mu|),\ Q(-x, -y, \mu) = -Q(x, y, \mu) = O(|\mu|).$$

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For a C^∞ 3-dimensional system of the form

$$\dot{x} = f(x, \varepsilon),\ x \in \mathbb{R}^3, \tag{1.5}$$

where $\varepsilon \in \mathbb{R}^2$, and

$$\frac{\partial f}{\partial x}(0,0) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

the truncated normal form system for (1.5) leads to a 2-dimensional system of the form

$$\begin{aligned} \dot{x} &= \varepsilon_1 x + axy + d_1 x^3 + d_2 x y^2, \\ \dot{y} &= \varepsilon_2 + bx^2 + cy^2 + d_3 x^2 y + d_4 y^3, \end{aligned} \tag{1.6}$$

where $(x, y) \in \mathbb{R}^2$ with $x \geq 0$ and $x + |y|$ small. See [5, 23].

Similarly, under some nonresonant conditions one can get a 2-dimensional system of the form

$$\begin{aligned} \dot{x} &= x(\varepsilon_1 + p_1 x^2 + p_2 y^2 + q_1 x^4 + q_2 x^2 y^2 + q_3 y^4), \\ \dot{y} &= y(\varepsilon_2 + p_3 x^2 + p_4 y^2 + q_4 x^4 + q_5 x^2 y^2 + q_6 y^4) \end{aligned} \tag{1.7}$$

with $x \geq 0, y \geq 0$ small from a C^∞ 4-dimensional system

$$\dot{x} = f(x, \varepsilon_1, \varepsilon_2),\ x \in \mathbb{R}^4,$$

where

$$\frac{\partial f}{\partial x}(0,0,0) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega \\ 0 & 0 & -\omega & 0 \end{pmatrix},\ \omega \neq 0.$$

System (1.3) was studied by Bogdanov [1, 2] and Takens [30]. System (1.4) was studied by Khorozov [25] and Carr [3]. Systems (1.6) and (1.7) were studied by Zoladek [34,35] and others. One can also find detailed studies and related references for systems (1.3), (1.4), (1.6) and (1.7) in the books [5, 13, 28].

In this paper we present a survey on the uniqueness of limit cycles bifurcating from a center, homoclinic loop or a heterclinic loop in planar systems by giving five bifurcation theorems, and then apply them to the study of systems (1.3), (1.4), (1.6) and (1.7) which are absent in many references.

In section 2 we list some general results on the number of limit cycles in Hopf, homoclinic and hetercinic bifurcations. In section 3 we provide an introduction to the study of limit cycles of (1.3), (1.4), (1.6) and (1.7) and an application of the main results in section 2 to the systems.

2. Preliminary theorems

In this section we list five theorems obtained by the author of this paper and his coauthors in different papers. However, all of these theorems have been rewritten