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Generalized Ito Formula and Some Stochastic Inclusions*

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Abstract This work is on a general integration by parts formula generalizing the Ito formula. An application is given to stochastic inclusions which have a different form than usual.

Keywords Ito formula, stochastic evolution inclusions

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1. Introduction

The Ito formula [3, 4, 7, 13] is really about integration by parts in the setting where there is a stochastic integral. It is of fundamental importance in SPDE and related fields [3, 4, 7, 9, 17]. We establish an implicit Ito formula in this work and apply it to study a specific stochastic evolution inclusion.

1.1. The situation

Let $V \subseteq W, W' \subseteq V'$ be separable Banach spaces, such that V is dense in W and $B \in \mathcal{L}(W, W')$ satisfies

$$\langle Bw, w\rangle \geq 0, \ \langle Bu, v\rangle = \langle Bv, u\rangle . \tag{1.1}$$

Note that B does not need to be one to one. Also allowed is the case where B is the Riesz map. It could also happen that $V = W$. Let X have values in V and satisfy the following

$$BX(t) = BX_0 + \int_0^t Y(s)\, ds + B\int_0^t Z(s)\, dW(s), \tag{1.2}$$

$X_0 \in L^2(\Omega; W)$ and is $\mathcal{F}_0$ measurable, where Z is $\mathcal{L}_2\left(Q^{1/2}U, W\right)$ progressively measurable and

$$\|Z\|_{L^2\left([0,T]\times\Omega, \mathcal{L}_2\left(Q^{1/2}U, W\right)\right)} < \infty.$$

Here Q is a nonnegative self adjoint operator defined on U. See [21] for stochastic integrals.

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Assume that X, Y satisfy

$$BX, Y \in K' \triangleq L^{p'}([0,T] \times \Omega; V'),$$

the σ algebra of measurable sets defining K' will be the progressively measurable sets. Here $1/p' + 1/p = 1,\ p > 1$. In the following sense (1.2) holds: For a.e. ω, the equation holds in V' for all $t \in [0,T]$. Thus we are considering a particular representative X of K for which this happens. Also it is only assumed that $BX(t) = B(X(t))$ for a.e. t. Thus BX is the name of a function having values in V' for which $BX(t) = B(X(t))$ for a.e. t. Assume that X is progressively measurable and

$$X \in L^p([0,T] \times \Omega, V) \triangleq K.$$

Also $W(t)$ is a JJ^* Wiener process on U_1 in the above diagram. U_1 can be assumed to be U.

The **goal** is to prove the following Ito formula valid for a.e. t for each ω off a set of measure zero.

$$\langle BX(t), X(t)\rangle = \langle BX_0, X_0\rangle + \int_0^t \left(2\langle Y(s), X(s)\rangle + \langle BZ, Z\rangle_{\mathcal{L}_2}\right) ds + \int_0^t \left(Z \circ J^{-1}\right)^* BX \circ J dW. \tag{1.3}$$

The most significant feature of the last term is that it is a local martingale.

To understand the **goal**, the following fundamental deterministic result will be very helpful. It says essentially that if $(Bu)' \in L^{p'}(0,T;V')$ and $u \in L^p(0,T;V)$ then the map $u \to Bu(t)$ is continuous as a map from

$$X_1 \triangleq \left\{u \in L^p([0,T];V) : (Bu)' \in L^{p'}([0,T];V')\right\}$$

with

$$\|u\|_{X_1} \triangleq \|u\|_{L^p(0,T,V)} + \left\|(Bu)'\right\|_{L^{p'}(0,T;V')}$$

to W'.

Proposition 1.1. *Let $Y \in L^{p'}(0,T;V')$ and*

$$Bu(t) = Bu_0 + \int_0^t Y(s)\, ds \text{ in } V',\ u_0 \in W, Bu(t) = B(u(t)) \text{ for a.e. } t. \tag{1.4}$$

Thus $Y = (Bu)'$ as a weak derivative in the sense of V' valued distributions. It is known that $u \in L^p(0,T,V)$ for $p > 1$. Then $t \to Bu(t)$ is continuous into W' for t off a set of measure zero N and there exists a continuous function $t \to \langle Bu, u\rangle(t)$ such that for all $t \notin N$, $\langle Bu, u\rangle(t) = \langle B(u(t)), u(t)\rangle, Bu(t) = B(u(t))$, and for all t,

$$\frac{1}{2}\langle Bu, u\rangle(t) = \frac{1}{2}\langle Bu_0, u_0\rangle + \int_0^t \langle Y(s), u(s)\rangle\, ds.$$

Note that the formula (1.4) shows that $Bu_0 = Bu(0)$. Also it shows that $t \to \langle Bu, u\rangle(t)$ is continuous. To emphasize this a little more, Bu is the name of a function. $Bu(t) = B(u(t))$ for a.e. t and $t \to Bu(t)$ is continuous into V' on $[0,T]$ because of the integral equation.

The Ito formula to be developed in this paper is a probabilistic version of the above. Specifically, our main results about the Ito formula is as follows.