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The Blow-up Dynamics for the L^2-Critical Hartree Equation with Harmonic Potential*

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Abstract In this paper, we study the L^2-critical Hartree equation with harmonic potential which arises in quantum theory of large system of nonrelativistic bosonic atoms and molecules. Firstly, by using the variational characteristic of the nonlinear elliptic equation and the Hamilton conservations, we get the sharp threshold for global existence and blow-up of the Cauchy problem. Then, in terms of a change of variables, we first find the relation between the Hartree equation with and without harmonic potential. Furthermore, we prove the upper bound of blow-up rate in $\mathbb{R}^3$ as well as the mass concentration of blow-up solution for the Hartree equation with harmonic potential in $\mathbb{R}^N$.

Keywords Hartree equation, harmonic potential, blow-up rate, upper bound, mass concentration

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1. Introduction

In a non-relativistic setting where the number of boson is very large, the Hartree equation with harmonic potential is a model describing a quantum mechanical boson system, and arises from many-body quantum mechanics in a mean-field limit [7] and in Bose-Einstein condensate (BEC) with long range interaction [8,16]. A Bose condensate can be represented by a wave function that obeys the following Hartree equation with harmonic potential

$$iu_t + \Delta u - \omega^2 |x|^2 u + \left(\frac{1}{|x|^{N-\gamma}} * |u|^p \right) |u|^{p-2} u = 0, \tag{1.1}$$

where $u = u(t,x) : [0,T) \times \mathbb{R}^N \to \mathbb{C}$ is a complex-valued wave function, $2 \le p < 2^*$ ($2^* = 1 + \frac{\gamma+2}{N-2}$ if $N \ge 3$ or $2^* = \infty$ if $N = 1,2$), N is the space dimension, $0 < \gamma < N$, $0 < T \le \infty$, $i = \sqrt{-1}$, Δ is the Laplace operator, $\omega > 0$ and $*$ denotes the convolution operator in $\mathbb{R}^N$.

When $\omega = 0$, Eq.(1.1) reduces to the focusing Schrödinger-Hartree equation

$$iu_t + \Delta u + \left(\frac{1}{|x|^{N-\gamma}} * |u|^p \right) |u|^{p-2} u = 0. \tag{1.2}$$

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For $p = 2$, the above equation (1.2) becomes the well-known standard Hartree equation, which can be considered as a model describing a quantum mechanical boson system in non-relativistic setting where the number of boson is very large. And it arises in the study of long range interaction between the molecules, which goes back to the work of [3, 6, 10, 11]. The equation (1.2) can be written as the Schrödinger-possion system of the form

$$\begin{cases} iv_t + \Delta v + W\,|v|^{p-2}\,v = 0, \\ -\Delta W = (N-2)|S^{N-1}|\,|v|^p\,, \end{cases}$$

where $|S^{N-1}|$ denotes the surface of the unit sphere in $\mathbb{R}^N$. This can be viewed as an electrostatic version of the Maxwell-Schrödinger system, it describing the interaction between the electromagnetic field and the wave function related to a quantum non-relativistic charged particle (see [15, 17]).

In [5], Cazenave established the local well-posedness for (1.2) in $H^1(\mathbb{R}^N)$ (also see Ginibre and Velo [10] for $p = 2$). In [1], Arora and Roudenko studied the global and finite time blow-up solution under the mass-energy assumption, and obtained the sharp threshold for global existence and finite time blow-up in mass super-critical and energy sub-critical regime. In [9], Genev and Venkov proved the concentration properties and the existence of standing wave solutions, and got conditions for formation of singularities in dependence of the values of $p \geq 2$ and $\gamma = 2$. In [13], Krieger, Lenzmann and Raphaël showed the existence of critical mass finite time blowup solution $u(t, x)$ that demonstrates the pseudo-conformal blowup rate $||\nabla v(t)||_{L^2} \sim \frac{1}{|t|}$ as $t \to 0$. In [24], Yang, Roudenko and Zhao obtained that a generic blow-up has a self-similar structure and exhibited log-log blow-up rate for $\gamma = 2$ and $p = 1 + \frac{4}{N}$ in $N = 3 \sim 7$ by numerical simulations.In [2], Arora, Roudenko and Yang showed the spectral property for $\gamma = 2$ and $p = 1 + \frac{4}{N}$ in dimension $N = 3$. By using the spectral results above, Arora and Roudenko showed the upper bounds on the blow-up rate for the blow-up solutions: there is a constant $C > 0$ such that

$$||\nabla v(t,x)||_{L^2} \leq C \left(\frac{|\ln(T-t)|}{T-t} \right)^{\frac{1}{2}}, \quad as \quad t \to T.$$

Let $u(t, x) = e^{it}\varphi(x)$ be a standing wave solution of (1.2). Then $\varphi(x)$ satisfies the following nonlinear elliptic equation

$$\Delta\varphi + \left(\frac{1}{|x|^{N-\gamma}} * |\varphi|^p \right) |\varphi|^{p-2}\,\varphi = \varphi. \tag{1.3}$$

The equation (1.3) is also called the nonlinear Choquard or Choquard-pekar equation. In [14], Lieb first proved the existence and uniqueness of the minimizing solution to (1.3) for $p = 2$ and $\gamma = 2$ in $\mathbb{R}^3$. In [19], Moroz and Van Schaftingen proved the general existence of positive solutions along with regularity and radial symmetry of solutions to (1.3) (also see [20]). In [13] , Krieger, Lenzmann and Raphaël proved the uniqueness of the ground state solution in dimension $N = 4$ for $p = 2$ and $\gamma = 2$. In [1], Arora and Roudenko proved the uniqueness of the ground state solution in dimension $2 < N < 6$ for $p = 2$ and $\gamma = 2$. In [23], Xiang proved the uniqueness of ground state solution for $p = 2 + \varepsilon$ and $\gamma = 2$. In the general case, the uniqueness is still an open problem.