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On Nodal Solutions of the Schrödinger-Poisson System with a Cubic Term*

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Abstract In this paper, we consider the following Schrödinger-Poisson system with a cubic term

$$\begin{cases} -\Delta u + V(|x|)u + \lambda\phi u = |u|^2 u \quad \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2 \quad \text{in } \mathbb{R}^3, \end{cases} \tag{0.1}$$

where $\lambda > 0$ and the radial function $V(x)$ is an external potential. By taking advantage of the Gersgorin disc theorem and Miranda theorem, via the variational method and blow up analysis, we prove that for each positive integer k, problem (0.1) admits a radial nodal solution $U_{k,4}^{\lambda}$ that changes sign exactly k times. Furthermore, the energy of $U_{k,4}^{\lambda}$ is strictly increasing in k and the asymptotic behavior of $U_{k,4}^{\lambda}$ as $\lambda \to 0_+$ is established. These results extend the existing ones from the super-cubic case in [17] to the cubic case.

Keywords Schrödinger-Poisson system, nodal solutions, Gersgorin disc theorem, Miranda theorem, blow-up analysis

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1. Introduction

In the last decades, the following Schrödinger-Poisson system

$$\begin{cases} -\Delta u + V(x)u + \lambda\phi u = |u|^{p-1} u \quad \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2 \quad \text{in } \mathbb{R}^3 \end{cases} \tag{1.1}$$

has attracted much research attention due to its deep physical backgrounds and mathematical challenges. Here $\lambda > 0, 1 < p < 5$ and V represents external potential function. From a physical point of view, system (1.1) comes from semiconductor theory and is used to simulate the evolution of electronic ensemble in semiconductor

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crystals, see [4,20] for instance. In mathematical contents, the appearance of the nonlocal term $\lambda\phi u$ causes some mathematical difficulties and makes the study of (1.1) interesting. As we know, there are many existence results in the literature on the solutions of (1.1), such as ground state solutions [3,15], bound state solutions [1,15,22], positive solutions [5,21], non-radial solutions [9], and semiclassical state solutions [14]. For more related problems, one can refer to [6,27] and references therein.

Recently, some researchers have shown interest in the existence and properties of nodal solutions (or sign-changing solutions) to (1.1). When the nonlinearity $|u|^{p-2}u$ satisfies the super-cubic growth condition that $p \in (3,5)$, via the Nehari manifold method, Wang-Zhou [23] studied the existence of a least energy nodal solution of (1.1) which changes sign only once. Later, the existence of infinitely many radial nodal solutions of (1.1) with any prescribed number of nodal domains was proved by Kim-Seok [17] via the variational method and gluing method for $p \in (3,5)$, see also [13] for a dynamical method. For the more general nonlinearity $f(u)$ satisfying super-cubic condition, one can see [2,7,8,10,16] for instance. For the cubic case $p=3$, Zhong-Tang [28] investigated the existence and asymptotical behaviors of a least energy nodal solution with exactly two nodal domains to (1.1) by the Nehari manifold method. Later, Sun-Wu [22] extended this result to the sub-cubic case $p \in (1,3)$. Furthermore, Liu-Wang-Zhang [18] obtained infinitely many sign-changing solutions for $p \in (2,3]$ by using the perturbation method and the invariant subsets of descending flow. In [14], Ianni-Vaira obtained infinitely many nonradial sign-changing solutions in the semiclassical limit for $p \in (1,3]$ by using the Lyapunov-Schmit reduction method. For more related results and details, one can refer to [11,25,26]. From the above discussions, we see that $p=3$ is a critical value. So a natural question arises that whether equation (1.1) with $p=3$ admits radial nodal solutions with a prescribed number of nodal domains. In this paper, we shall give a confirmative answer to the following cubic case $p=3$ of (1.1), that is,

$$\begin{cases} -\Delta u + V(|x|)u + \lambda\phi u = |u|^2 u \quad \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2 \quad \text{in } \mathbb{R}^3, \end{cases} \tag{1.2}$$

where $\lambda > 0$ and V satisfies

(V) $V(|x|) \in \mathbb{C}([0,+\infty),\mathbb{R})$ is bounded from below by a positive constant V_0.

As is well known, equation (1.2) is equivalent to

$$-\Delta u + V(|x|)u + \lambda\phi_u u = |u|^2 u \quad \text{in } \mathbb{R}^3 \tag{1.3}$$

with $\phi_u(x) = \int_{\mathbb{R}^3} \frac{u^2(y)}{4\pi|x-y|}dy$, which has a variational structure. Let

$$H_V = \{u \in H^1(\mathbb{R}^3) : u(x) = u(|x|),\ \int_{\mathbb{R}^3} V(|x|)u^2 < +\infty\}$$

be endowed with the norm $\|u\|_{H_V} = \left(\int_{\mathbb{R}^3}(|\nabla u|^2 + V(|x|)|u|^2)dx\right)^{\frac{1}{2}}$. Then its energy functional $I_{\lambda,4} : H_V \to \mathbb{R}$ is

$$I_{\lambda,4}(u) := \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + V(|x|)u^2)dx + \frac{\lambda}{4}\int_{\mathbb{R}^3}\phi_u u^2 dx - \frac{1}{4}\int_{\mathbb{R}^3}|u|^4.$$