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Limit Cycles for a Class of Continuous-Discontinuous Piecewise Differential Systems

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Abstract During this century, an increasing interest appeared in studying the planar piecewise differential systems. This is due to their numerous applications for modelling many natural phenomena. For understanding the dynamics of the planar differential systems we must control the existence or non-existence of periodic orbits and limit cycles. So many papers have been published studying the existence or non-existence of periodic orbits and limit cycles for continuous or discontinuous piecewise differential systems. But until now very few papers have studied the periodic orbits and limit cycles of piecewise differential systems where two differential systems of the piecewise differential system are continuous and discontinuous respectively. We study the periodic orbits and limit cycles of the planar continuous–discontinuous piecewise differential systems separated by two parallel straight lines, such that either in one of these straight lines the piecewise differential system is continuous and in the other one discontinuous. In two pieces of these piecewise differential systems there are arbitrary Hamiltonian systems of degree two and in the third piece there is an arbitrary Hamiltonian system of degree one forming the continuous-discontinuous piecewise differential systems. We determine the limit cycles of these piecewise differential systems by considering two cases. In the first the Hamiltonian system of degree one can be in the middle of the three zones, and in the second it is on one side of the three zones.

Keywords Limit cycles, Hamiltonian system, continuous-discontinuous piecewise linear differential systems, first integrals

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1. Introduction

Poincaré's works started the qualitative study of the differential systems instead of finding exact or approximative solutions of themselves. With him also appeared the notion of the limit cycles which became one of the most important objects for understanding the dynamics of the differential systems in the plane, see [27].

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The concept of a limit cycle is a concept whose importance is not hidden from any researcher in the area of the differential systems in dimension two and related fields. But in general, to determine the existence or absence of a limit cycle is not an easy task. See for instance the Hilbert's 16th problem [10,12,14].

At the beginning of the 1930s the limit cycles started to be studied in the continuous and discontinuous piecewise differential systems due to their importance in many mechanical and electrical applications. For more information on their past and present applications, see the books [1,6,28] and the survey [24]. The continuous piecewise differential systems have been studied for several authors, see for instance [4,9,17,22,23], and for the discontinuous ones see without being exhaustive [2,3,5, 9,11,13,15,16,18–21,25,26].

In this paper we consider continuous-discontinuous piecewise differential systems separated by two parallel straight lines. These systems have quadratic Hamiltonian systems in two different regions and a linear Hamiltonian system in the third region, and we want to study the existence and non–existence of limit cycles, and in the case of the existence of limit cycles, we also want to find their maximum number of limit cycles that these continuous-discontinuous piecewise differential systems can exhibit. See a previous paper on continuous-discontinuous piecewise differential systems separated by two parallel straight lines in [19]. Now we need to take into account two cases: the Hamiltonian system of degree one may be located either in the center of the three zones, or in a lateral zone.

Here we shall study the periodic orbits and the limit cycles which intersect exactly at two points, both parallel straight lines of the continuous-discontinuous piecewise differential systems.

Doing a rescaling of the plane variables and a rotation of themselves, if necessary, we can assume without loss of generality that the two parallel straight lines are $x = -1$ and $x = 1$. Thus we shall study the continuous-discontinuous piecewise differential systems of the form

$$(\dot{x}, \dot{y}) = \begin{cases} \left(-\dfrac{\partial H_i}{\partial y}, \dfrac{\partial H_i}{\partial x}\right) \text{ if } x \geq 1, \\ \left(-\dfrac{\partial H_j}{\partial y}, \dfrac{\partial H_j}{\partial x}\right) \text{ if } -1 \leq x \leq 1, \\ \left(-\dfrac{\partial H_2}{\partial y}, \dfrac{\partial H_2}{\partial x}\right) \text{ if } x \leq -1, \end{cases} \tag{1.1}$$

where $H_2 = H_2(x, y)$ is an arbitrary polynomial of degree 3, $H_i = H_i(x, y)$, $H_j = H_j(x, y)$ are arbitrary polynomials of degree 3 and 2 for $i = 3$, $j = 1$, or $H_i = H_i(x, y)$, $H_j = H_j(x, y)$ are arbitrary polynomials of degree 2 or 3 for $i = 1$, $j = 3$. In the straight line $x = -1$ the piecewise differential system is continuous and in $x = 1$ discontinuous.

This paper studies the existence or non-existence of periodic orbits and limit cycles that such kinds of continuous-discontinuous piecewise differential systems can exhibit. And in the case of the existence of limit cycles we determine their maximum numbers.

In what follows we give explicitly the Hamiltonian systems which form the