Journal of Nonlinear Modeling and Analysis
Volume 6, Number 3, September 2024, 683–692
http://jnma-online.com
DOI:10.12150/jnma.2024.683

Limit Cycle Bifurcations of a Cubic Polynomial System via Melnikov Analysis*

Peixing Yang[1], Jiang Yu [1,†]

Abstract In this paper, a linear perturbation up to any order in ϵ for a cubic center with a multiple line of critical points is considered. By the algorithm of any order Melnikov function, the sharp upper bound of the number of limit cycles is 2.

Keywords Melnikov functions, bifurcations, limit cycles

MSC(2010) 34C05, 34C07, 37G15.

1. Introduction

One of the important problems in the qualitative theory of differential systems is the Hilbert's 16th problem, which aims to find the distributions and numbers of limit cycles of planar polynomials differential systems. The perturbation of the integrable non-Hamiltonian system as follows:

$$\begin{cases} \dot{x} = yh(x,\, y) - \epsilon p(x,\, y,\, \epsilon), \\ \dot{y} = -xh(x,\, y) + \epsilon q(x,\, y,\, \epsilon), \end{cases} \tag{1.1}$$

is closely related to the weak Hilbert's 16th problem of determining an upper bound of the number of zeros for the following integral

$$I(h) = \oint_{\frac{1}{2}(x^2+y^2)=h} \frac{p(x,\, y,\, 0)dy + q(x,\, y,\, 0)dx}{h(x,\, y)},$$

where $p(x,\, y,\, \epsilon)$ and $q(x,\, y,\, \epsilon)$ are polynomials in x, y depending analytically on ϵ, and here $h(x,\, y)$ is a polynomial in x, y with $h(0,0) \neq 0$.

Many researchers focus on system (1.1) with different $h(x,\, y)$ and the difficulty reflects on how to deal with the Abel integral with a denominator of $h(x,\, y)$, as discussed in [1, 3, 5, 8–12] and references therein. For $h(x,\, y) = ax^2 + bx + 1$, the authors in [10–12] studied system (1.1) with different ranges of a and b by the first

†the corresponding author.
Email address:peixing0806@sjtu.edu.cn (P. Yang),
jiangyu@sjtu.edu.cn (J. Yu)
[1]School of Mathematical Sciences, CMA-Shanghai, Shanghai Jiao Tong University, Shanghai 200240, P. R. China.
*The authors are partially supported by the National Natural Science Foundations of China (12271354). The first author is supported by the China Postdoctoral Science Foundation (2021M702144). The second author is supported by the National Natural Science Foundations of China (11931016 and 12161131001), Innovation Program of Shanghai Municipal Education Commission (2021-01-07-00-02-E00087).

order Melnikov function, respectively. Buică et al. [2] considered system (1.1) with $h(x, y) = 1 + x$ by the first three order Melnikov functions. The authors in [13] considered system (1.1) with $h(x) = (1+x)^2$ under the perturbation up to the first order in ϵ by any order Melnikov functions.

Inspired by the works mentioned above, we would like to see the influence of the linear perturbation up to any order in ϵ on the number of limit cycles by higher order Melnikov functions.

More precisely, in this paper, we consider the following system:

$$\begin{cases} \dot{x} = y(1+x)^2 - \sum_{i=0}^{N} \epsilon^{i+1} P_i(x, y), \\ \dot{y} = -x(1+x)^2 + \sum_{i=0}^{N} \epsilon^{i+1} Q_i(x, y), \end{cases} \tag{1.2}$$

where $P_i(x, y) = a_{i0} + a_{i1}x + a_{i2}y$ and $Q_i(x, y) = b_{i0} + b_{i1}x + b_{i2}y$ for $0 \le i \le N$. Here $N \ge 1$ is an integer. System (1.2) can be rewritten as follows,

$$dH = \epsilon(\omega_0 + \epsilon\omega_1 + \cdots + \epsilon^N \omega_N),$$

where

$$H(x, y) = \frac{1}{2}(x^2 + y^2), \quad \omega_i = \frac{P_i(x, y)dy + Q_i(x, y)dx}{(1+x)^2}.$$

When $\epsilon = 0$, there exist a family of periodic orbits $\Gamma_h : \{\frac{1}{2}(x^2+y^2) = h,\ h \in (0, \frac{1}{2})\}$. And we denote $M_k(h)$ as the $k-$th order Melnikov function of system (1.2) by the displacement function

$$d(h, \epsilon) = \epsilon M_1(h) + \epsilon^2 M_2(h) + \cdots + \epsilon^k M_k(h) + \cdots, h \in (0, \frac{1}{2}).$$

Then we give our main result in the following theorem.

Theorem 1.1. *For system (1.2), the following statements hold.*

(i) If the first order Melnikov function $M_1(h)$ is not zero identically, then $M_1(h)$ has at most one isolated zero, multiplicity taken into account.

(ii) If $M_j(h) \equiv 0$ for $1 \le j \le k-1$ and $M_k(h) \not\equiv 0$ with $k \le N+1$, then the $k-$th order Melnikov function $M_k(h)$ has at most two isolated zeros, multiplicity taken into account.

(iii) If $M_j(h) \equiv 0$ for $1 \le j \le N+1$ and $M_{N+2}(h) \not\equiv 0$, then the $N+2-$th order Melnikov function $M_{N+2}(h)$ has no isolated zero.

(iv) If $M_j(h) \equiv 0$ for $1 \le j \le N+2$, system (1.2) is integrable.

In short, the maximum number of limit cycles bifurcated from the cubic center is 2 *by any order Melnikov function, taking into account their multiplicities. All the upper bounds mentioned above can be reached with proper parameters.*

2. The calculation of $M_k(h)$

In this section we shall give the calculation of any order Melnikov function according to the algorithm of higher order Melnikov functions proposed in [4, 7, 13].