Journal of Nonlinear Modeling and Analysis
Volume 6, Number 3, September 2024, 732–745
http://jnma-online.com
DOI:10.12150/jnma.2024.732

Limit Cycles in Piecewise Smooth Van der Pol Equations*

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Abstract This paper focuses on analyzing the properties of equilibrium points and limit cycles in two types of planar piecewise smooth slow-fast systems. The right-half system is a classical van der Pol equation, while the left-half system is either linear or quadratic. Additionally, we provide a detailed description of the characteristics of limit cycles in these systems.

Keywords Limit cycles, slow-fast systems, van der Pol equations

MSC(2010) 34C07, 34C25, 34D15, 34E17.

1. Introduction

Van der Pol equations are a type of nonlinear differential equations that describe the behavior of oscillatory systems. The general form of van der Pol equation is modeled by a second-order differential equation

$$\frac{d^2x}{dt^2} - \mu(1-x^2)\frac{dx}{dt} + x = 0,$$

where x is the position of the oscillator, t is time, and μ is a parameter that controls the strength of the damping and the nonlinearity of the system [1]. It has been used to model a wide range of physical phenomena, including electrical circuits, chemical reactions, and biological systems.

One of the most interesting features of the van der Pol equation is its ability to exhibit limit cycle behavior. There are many results on the existence, stability and uniqueness of limit cycles for smooth Liénard systems [2]. In recent years, stimulated by nonsmooth phenomena in the real world, the relevant research has been extended to nonsmooth case, see for instance [3–6].

Recently, there has been a growing interest in the study of canard phenomena in nonsmooth systems. Recall that a canard is a trajectory of singularly perturbed differential equation that follows both the attracting and repelling slow manifolds for $O(1)$ time. Canard is associated with a bifurcation phenomena, named canard explosion, i.e. a transition from small Hopf cycles to relaxation oscillations through a sequence of canard cycles. It was first discovered in the context of the van der Pol oscillator with constant forcing and has important implications in applications,

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*The authors were supported by the National Natural Science Foundation of China (12271088) and the Natural Science Foundation of Shanghai (21ZR1401000).

we direct the reader to the paper by Benoit [7]. During the past few years, the focus of research has been on piecewise linear (PWL) systems. Desroches et al. [8] summarized the similarities and differences between PWL systems with three zones and smooth van der Pol system. For more studies in PWL canard phenomena, see for instance [9–11]. Andrew and Glendining turned their attention to piecewise smooth (PWS) Liénard systems, and they investigated the canard-like phenomena in PWS van der Pol systems [12] and extended the results in more general cases [13]. We remark that the above research mainly focuses on the existence of canard cycles. However, the asymptotic expansion of the parameter for which canard exists has not been provided.

In this paper we focus our attention on the existence of limit cycles and canard explosion. We consider two types of piecewise smooth van der Pol systems

$$\begin{aligned} \dot{x} &= -y + F(x), \\ \varepsilon\dot{y} &= (x - \lambda), \end{aligned} \qquad F(x) = \begin{cases} (-1)^m k x^m, \, x < 0, \\ x^2 - \frac{1}{3}x^3, \quad x \geqslant 0, \end{cases} \tag{1.1}$$

where $k > 0$, $0 < \varepsilon \ll 1$, $m = 1$(or 2), and the dot denotes the derivative of x and y with respect to the time t. The set $S = \{(x, y) : y = F(x)\}$, called the critical manifold, plays a central role in the analysis of slow-fast system (1.1).

The structure of this paper is as follows: Section 2 examines the stability of equilibrium points and limit cycles for two types of van der Pol equations. Section 3 explores the impact of parameters on the limit cycle. The paper concludes with a discussion in Section 4.

2. Piecewise smooth van der Pol equations

2.1. Van der Pol equations with linear left branch

Considering the first type of piecewise smooth systems

$$\begin{aligned} x' &= -y + F_1(x), \\ y' &= \varepsilon(x - \lambda), \end{aligned} \qquad F_1(x) = \begin{cases} -kx, \qquad x < 0, \\ x^2 - \frac{1}{3}x^3, \, x \geqslant 0, \end{cases} \tag{2.1}$$

where $0 < \varepsilon \ll 1$, $k^2 - 4\varepsilon > 0$. It is clear that system (2.1) has a unique equilibrium point $E(\lambda, k\lambda)$ when $\lambda < 0$, or $E(\lambda, \lambda^2 - \frac{1}{3}\lambda^3)$ when $\lambda \geqslant 0$. For the convenience of discussion, we divide the plane into three regions as follows:

$$L = \{(x, y)|x < 0\}, \quad M = \{(x, y)|0 \leqslant x \leqslant 2\}, \quad R = \{(x, y)|x > 2\}.$$

2.1.1. The stability of the equilibrium point of system (2.1)

Theorem 2.1. *Let $\Delta = (2\lambda - \lambda^2)^2 - 4\varepsilon$. The stability of the equilibrium point E of system (2.1) is as follows.*

(1) *If $\lambda < 0$, then E is a stable node.*

(2) *If $\lambda = 0$, then E is a stable node in the left half-plane and a stable first-order focus in the right half-plane.*

(3) *If $0 < \lambda < 2, \Delta < 0$, then E is an unstable focus.*