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An Example of Piecewise Linear Systems with Infinitely Many Limit Cycles Separated by a Piecewise Linear Curve and Its Perturbations*

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Abstract In this paper, we present an example of piecewise linear systems with infinitely many crossing limit cycles defined in two zones separated by a piecewise linear curve with countable corners. Then we prove that under piecewise linear perturbations, the perturbed system can have infinitely many limit cycles, or exactly ℓ limit cycles for any given nonnegative integer ℓ.

Keywords Piecewise linear system, crossing limit cycle, piecewise linear perturbation, piecewise smooth switching curve, stability

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1. Introduction

A classical problem in the theory of dynamical systems is to determine the number and relative configuration of limit cycles. For example, the second part of the Hilbert's 16th problem is to find a uniform upper bound $\mathcal{H}(n)$ for the maximum number of limit cycles of planar polynomial systems of degree n such that $\mathcal{H}(n)$ depends only on n. This problem is extremely difficult and is still open. In fact, it remains unsolved whether $\mathcal{H}(n)$ is finite even for $n = 2$.

In the past decades, stimulated by real world applications from applied science such as mechanics, electronic engineering and control theory, it is natural to consider the same problem for planar piecewise smooth (PWS) systems. In particular, there is considerable interest in finding a uniform upper bound $\mathcal{H}_p^c(n)$ depending only on n of the maximum number of crossing limit cycles in the planar piecewise polynomial systems of degree n defined in two zones Σ_L^+ and Σ_L^- separated by exactly one switching line Σ_L given by the following form:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} (P_n^+(x,y),\, Q_n^+(x,y))^T, \text{ if } (x,y) \in \Sigma_L^+, \\ (P_n^-(x,y),\, Q_n^-(x,y))^T, \text{ if } (x,y) \in \Sigma_L^-, \end{cases} \tag{1.1}$$

where $P_n^{\pm}(x,y)$ and $Q_n^{\pm}(x,y)$ are real polynomials of degree n, Σ_L^+ and Σ_L^- are disjoint open sets of $\mathbb{R}^2$, $\Sigma_L \subset \mathbb{R}^2$ is a straight line, $\Sigma_L^+ \cup \Sigma_L \cup \Sigma_L^- = \mathbb{R}^2$. Here the Fillipov's convention is assumed for the solutions of system (1.1) on Σ_L. A

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crossing limit cycle of system (1.1) refers to an isolated periodic orbit of (1.1) which intersects Σ_L transversally. Thus no sliding or grazing occurs. All of the limit cycles mentioned in the sequel are crossing limit cycles.

To date, most attentions have been paid to the cases when $n = 1$ and $n = 2$. The simplest case of (1.1) is piecewise linear (PWL) systems, i.e. when $n = 1$. In 1991, Lum and Chua conjectured that system (1.1) has at most one limit cycle under the continuity hypothesis when $n = 1$ [28]. This conjecture was proved in 1998 by Freire et al. in [13]. In [19], Han and Zhang constructed examples of (1.1) without the continuity hypothesis that have two limit cycles when $n = 1$ and conjectured that the upper bound for the maximum number of limit cycles of such a kind of systems is two. This conjecture was disproved by Huan and Yang in [21] by presenting concrete examples of such a kind of systems with three limit cycles. Since then, it had remained an open question whether there exists a unform upper bound for the maximum number of limit cycles of PWL systems defined in two zones separated by a straight line. Progress has been made recently by Carmona et al. in [3]. They provided a positive answer to this question by obtaining a unform upper bound $L^* \leq 8$ for the maximum number of limit cycles such a type of PWL systems has. Moreover, it was proved in [33] that two limit cycles can be bifurcated from the perturbations of PWL Hamiltonian systems with two saddles.

Like for smooth systems, the case of system (1.1) is intractable when $n = 2$ for the present time. Thus many researchers paid their attention to the center-focus and cyclicity problems under the assumption that $(0, 0)$ is a nondegenerate *pseudo-focus* of (1.1). That is to determine whether $(0, 0)$ is a center or a focus of (1.1) and to find the maximum number of small amplitude limit cycles bifurcating from $(0, 0)$. Those problems were first studied by Coll et al. in [7]. They pointed out that a pseudo-focus of system (1.1) can be classified into four types. They are *focus-focus* (FF), *focus-parabolic* (FP), *parabolic-focus* (PF) and *parabolic-parabolic* (PP). In [7, 16], planar PWS quadratic systems having four and five limit cycles were presented respectively. The center-focus and cyclicity problems for switching Bautin systems were further investigated in [4, 32] and it was proved in [32] that at least ten small amplitude limit cycles can bifurcate from a center in planar PWS quadratic systems of FF type. In [25], Llibre and Mereu proved that five limit cycles can bifurcate from the isochronous centers of planar PWS quadratic systems by using the generalized averaging theory of first order. Gouveia and Torregrosa proved in [17] that at least thirteen small amplitude limit cycles can bifurcate from an equilibrium of planar PWS quadratic systems with one switching line. In [8] da Cruz et al. proved that sixteen limit cycles can bifurcate from the period annulus of some isochronous quadratic centers in a class of planar PWS quadratic systems with one switching line. Thus $\mathcal{H}_p^c(2) \geq 16$. As far as we know, this is the best lower bound of the cyclicity of system (1.1) for $n = 2$.

In recent years, the attention has been paid to planar PWS quadratic systems with FP (or PF) and PP type critical points. Concrete examples of planar PWS quadratic systems with a FP type critical point that have at least four limit cycles and with a PP type critical point that have at least one limit cycle were constructed in [7]. It was proved in [31] that at least six limit cycles can bifurcate from a FP type critical point in a planar PWS quadratic system. Small amplitude limit cycles in planar PWS Hamiltonian systems with invisible fold-fold (i.e. PP type) critical points were discussed in [10]. In the work of Novaes and Silva [29], an example of planar PWS quadratic systems with a PP type critical point $(0, 0)$ that has five