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Discontinuous Fractional Sturm-Liouville Problems with Hilfer Derivatives*

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Abstract In this paper, we study discontinuous Sturm-Liouville problem with fractional Hilfer derivatives. By defining an operator A in the Hilbert space $L_2[-1,1]$, this research shows that the eigenvalues and corresponding eigenfunctions of the main problem coincide with the eigenvalues and corresponding eigenfunctions of the constructed operator. Moreover, the characteristic function is also constructed such that the eigenvalues of the problem are coincide with the zeros of this function.

Keywords Sturm-Liouville problem, Hilfer derivatives, eigenparameter, fractional boundary conditions, fractional transmission conditions

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1. Introduction

Named after mathematicians Jacques Charles Francois Sturm and Joseph Liouville, Sturm-Liouville(S-L) problem is a mathematical concept that deals with the eigenvalue problem of a differential equation. In related fields such as quantum mechanics, heat transfer, and vibration analysis, it has attracted much attention and plays an important role in mathematical physics. Although first proposed more than 170 years ago, Sturm-Liouville theory has produced numerous research papers and monographs, it remains one of the most thriving areas of research [1–3], and many references with respect to physics and mechanics problems are contained in [4–6].

In general, as a class of boundary value problems, the Sturm-Liouville problem involves finding the eigenfunctions and eigenvalues of a second-order linear differential equation of the form:

$$-\frac{\mathrm{d}}{\mathrm{d}x}[p(x)\frac{\mathrm{d}y}{\mathrm{d}x}]+q(x)y=\lambda w(x)y,$$

where $p(x)$, $q(x)$, and $w(x)$ are specified functions defined on an interval $[a,b]$. λ is the spectral parameter with certain boundary value conditions, while y is

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the eigenfunction. The differential equation mentioned above, combined with the boundary conditions in forms

$$c_1 y(a) + c_2 y'(a) = 0, (c_1^2 + c_2^2 > 0),$$
$$d_1 y(b) + d_2 y'(b) = 0, (d_1^2 + d_2^2 > 0),$$

is referred to as regular Sturm-Liouville problems (SLPs) if $p(x)$,$w(x) > 0$ and $p'(x)$, $q(x)$ and $w(x)$ are continuous functions over the finite interval $[a, b]$. To solve this problem, it is necessary to apply several techniques such as separation of variables, Fourier series, and Green's functions in order to obtain the eigenvalues and eigenfunctions that meet the boundary conditions.

While a significant amount of research has been conducted on the general theory and methodologies for boundary value problems with continuous coefficients, little is currently known about similar problems that involve discontinuities. Several studies, such as those detailed in [7–10], have explored the discontinuous boundary value problems with transmission conditions and its potential application to boundary value problems in parabolic equations. Additionally, some research has also examined transmission condition problems in mechanics. A particular focus has been placed on studying the discontinuous Sturm-Liouville problem, especially when the eigenparameter is presented in both the differential equation and the boundary and transmission conditions, as discussed in [11–14]. One common method for solving discontinuous Sturm-Liouville problems is to divide the domain into multiple sub-domains, each with its own set of continuous coefficients. The solutions for each sub-domain can then be matched at the points of discontinuity using boundary conditions. Another approach is to use a weighted inner product to define the space of functions over which the problem is defined. This allows for a wider range of functions to be used in the solution, including those that are not continuous. This problem can provide valuable insights into a wide range of physical phenomena.

Fractional Sturm-Liouville problem is a type of differential equation problem that involves fractional derivatives in the Sturm-Liouville operator. The history of fractional calculus can be found in [15,16]. One important application of fractional calculus is in modeling problems involving anomalous diffusion, viscoelasticity, and other phenomena that exhibit fractal behavior. Researches [17–21] have demonstrated that fractional derivative models typically provide more accurate solutions for real processes of anomalous systems compared to models based on integer-order derivatives. The traditional methods used to solve ordinary Sturm-Liouville problems may not be applicable to fractional Sturm-Liouville problems. Specialized techniques, such as the fractional calculus and spectral methods, are often employed to solve these problems.

There exist various definitions for fractional derivatives and integrals in fractional calculus, such as Riemann-Liouville, Caputo, Grunwald-Letnikov, and others. Each of these has its own advantages and disadvantages. The selection of the appropriate definition to use relies on the specific problem being investigated.The Riemann-Liouville and Caputo derivative definitions are among the most commonly utilized tools in fractional calculus, particularly for the purpose of modeling physical systems. Additionally, a new generalized definition of fractional derivatives has been proposed by Hilfer, which has garnered significant attention in recent years. The Hilfer derivative is defined by two parameters α and β, with Riemann-Liouville and Caputo fractional derivatives being specific cases of $\beta = 0$ and $\beta = 1$ ([22–24]), respectively. Although the Hilfer derivative has only recently been introduced, it has