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New Fixed Point Results over Orthogonal $\mathcal{F}$-Metric Spaces and Application in Second-Order Differential Equations

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Abstract In this article, we introduce the notion of cyclic α-admissible mapping with respect to θ with its special cases, which are cyclic α-admissible mapping with respect to θ^* and cyclic α^*-admissible mapping with respect to θ. We present the notion of orthogonal $(\alpha\theta - \beta F)$-rational contraction and establish new fixed point results over orthogonal $\mathcal{F}$-metric space. The study includes illustrative examples to support our results. We apply our results to prove the existence and uniqueness of solutions for second-order differential equations.

Keywords Fixed point, orthogonal $(\alpha\theta - \beta F)$-rational contraction, cyclic α-admissible mapping with respect to θ, orthogonal $\mathcal{F}$-metric space, second-order differential equation

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1. Introduction

In 1922, [9] the Polish mathematician Banach presented the most important of the fixed point theorems known as the Banach contraction principle, which proves the existence and uniqueness of the fixed point for any contraction mapping $\mathcal{H} : \Xi \to \Xi$, and (Ξ, D) should be a complete metric space, where $D : \Xi \times \Xi \to [0, \infty)$. In 2012, Wardowski [7] presented the concept of F-contraction, a generalization of the Banach contraction principle. This means that Banach contractions can be seen as a particular case of F-contractions. Therefore, many researchers used this concept to study the existence and uniqueness of the fixed point in a different way instead of using Banach's theorem. [10–16] is recommended for readers interested in fixed point findings obtained using the notion of F-contraction. Some authors modified or reformulated some of the conditions of this concept and studied some fixed point theorems (see [17–20]). Kanokwan et al. [27] introduced the notion of orthogonal F-contraction and established some fixed point results over orthogonal

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metric space. Jleli and Samet [1] introduced the concept of $\mathcal{F}$-metric space in 2018. Researchers have since paid great attention to this space and have used it to study several fixed point theorems (see, e.g., [21–24]). Some even used F-contraction to study some fixed point theorems over $\mathcal{F}$-metric space (see [25, 26]). In 2020, T. Kanwal et al. [3] presented the notion of orthogonal $\mathcal{F}$-metric spaces and proved some fixed point theorems and [8,29,30] presented some new fixed point results over orthogonal $\mathcal{F}$-metric space. M. Taleb et al. studied some fixed point results and applied their results to study the existence and uniqueness of solutions of nonlinear neutral differential equations (see [23]) and in [28] also studied the existence and uniqueness of solutions to first-order differential equations.

In this paper, we present a modification of the cyclic (α, θ)-admissible mapping, which is done by introducing the concept of cyclic α-admissible mapping with respect to θ, presenting the notion of orthogonal $(\alpha\theta - \beta F)$-rational contraction. Additionally, we establish new fixed point results over orthogonal $\mathcal{F}$-metric space. The paper is organized as follows. In Sect.(3), the concept of cyclic α-admissible mapping with respect to θ is introduced (see Definition (3.1)) which is a modification to what was stated in definition (2.11) and we provide an illustrative example to support this result (see Example (3.1)). We deduce two special concepts from definition (3.1), namely cyclic α-admissible mapping with respect to θ^* and cyclic α^*-admissible mapping with respect to θ (see Remark (3.1)). We also introduce a concept of orthogonal $(\alpha\theta - \beta F)$-rational contraction (see Definition (3.2)) and we establish new fixed point results over orthogonal $\mathcal{F}$-metric space (see Theorem (3.1)) and (Corollaries (3.1), (3.2) and (3.3)). These results are supported by an example (3.2). In Sect. (4), we apply our results to show the existence and uniqueness of solutions for second-order differential equations (see Theorem (4.1)). Our results generalize and advance existing literature, such as [5], [23] and also present a novel approach to establish the existence and uniqueness of solutions for second-order differential equations.

2. Preliminaries

In 2012, Wardowski [7] introduced the notion of F- contraction as follows.

Definition 2.1 ([7]). Let $\Xi \neq \emptyset$, and (Ξ, D) be a metric space. A mapping $\mathcal{H} : \Xi \to \Xi$ is called F- contraction if $\exists\ \tau > 0,\ \forall\ \omega, \nu \in \Xi,\ D(\mathcal{H}\omega, \mathcal{H}\nu) > 0$, and we have

$$\tau + F\left(D(\mathcal{H}\omega, \mathcal{H}\nu)\right) \leq F\left(D(\omega, \nu)\right),$$

where $F : (0, \infty) \to \mathbb{R}$ satisfies the following conditions:

$(\mathcal{F}_1)$ $0 < \mathfrak{s} < \iota \quad \Rightarrow \quad F(s) \leq F(\iota)$.

$(\mathcal{F}_2)$ $\forall\ \{\iota_\mathfrak{n}\} \subset (0, +\infty)$, we have

$$\lim_{\mathfrak{n}\to+\infty} F(\iota_\mathfrak{n}) = -\infty \quad \Leftrightarrow \quad \lim_{\mathfrak{n}\to+\infty} \iota_\mathfrak{n} = 0.$$

$(\mathcal{F}_3)$ For some $r \in (0, 1)$, $\lim_{\iota\to 0^+} \iota^r F(\iota) = 0$.

The class of functions F is denoted by Φ.

Definition 2.2 ([1]). Let $D_\mathcal{F} : \Xi \times \Xi \to [0, \infty)$ be a given mapping. If $\exists\ a \in [0, \infty)$, such that