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The Benjamin-Ono-Burgers Equation: New Ideas and New Results

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Abstract Consider the Cauchy problem for the Benjamin-Ono-Burgers equation. There exists a unique global weak solution under appropriate conditions on the initial function and the external force. Here are many very important and interesting questions.

- Can we accomplish the exact limits for all order derivatives of the global smooth solution of the Benjamin-Ono-Burgers equation, in terms of some given information, representing certain physical mechanisms?
- What are the influences of various physical mechanisms (represented by the initial function, the external force, the order of the derivatives and the diffusion coefficient) on the exact limits?
- Can we establish improved decay estimates with sharp rates for all order derivatives of the solution, so that the most important constants $\mathcal{A}$ and $\mathcal{C}$ are independent of any norm of any order derivatives of the initial function, the external force and the solution, for all sufficiently large $t > 0$? Other positive constants $\mathcal{B}$ and $\mathcal{D}$ in the estimates are much less important because $\mathcal{B}t^{-1}$ and $\mathcal{D}t^{-1}$ becomes arbitrarily small as $t \to \infty$. This kind of decay estimates may play a substantial role in long time, accurate numerical simulations.
- Can we use the solution of the corresponding linear equation to approximate the solution of the Benjamin-Ono-Burgers equation?
- Can we couple together classical ideas (such as the Fourier transformation, the Parseval's identity, Lebesgue's dominated convergence theorem, squeeze theorem, etc) in an appropriate way to establish important and interesting results for the Benjamin-Ono-Burgers equation?
- For very similar dissipative dispersive wave equations, such as the Korteweg-de Vries-Burgers equation and the Benjamin-Bona-Mahony-Burgers equation, can we apply the same ideas developed in this paper to establish the same or very similar results?

We will couple together a few novel ideas, several existing ideas and existing results and use rigorous mathematical analysis to provide positive solutions to these important and interesting questions.

Keywords Benjamin-Ono-Burgers equation, global smooth solution, all order derivatives, exact limits, improved decay estimates with sharp rates

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1. Introduction

1.1. The Mathematical model equations and known related results

Consider the Cauchy problem for the Benjamin-Ono-Burgers equation

$$\frac{\partial}{\partial t}u+\mathcal{H}\frac{\partial^2}{\partial x^2}u-\alpha\frac{\partial^2}{\partial x^2}u+\frac{\partial}{\partial x}(u^2)=f(x,t), \tag{1.1}$$

$$u(x,0)=u_0(x). \tag{1.2}$$

Also consider the Cauchy problem for the corresponding linear equation

$$\frac{\partial}{\partial t}v+\mathcal{H}\frac{\partial^2}{\partial x^2}v-\alpha\frac{\partial^2}{\partial x^2}v=f(x,t), \tag{1.3}$$

$$v(x,0)=u_0(x). \tag{1.4}$$

In these equations, the positive constant $\alpha>0$ represents the diffusion coefficient, the Hilbert operator $\mathcal{H}$ is defined by the principal value of the following singular integral

$$[\mathcal{H}\phi](x)=\frac{1}{\pi}\text{ P. V. }\int_{\mathbb{R}}\frac{\phi(y)}{x-y}\mathrm{d}y, \tag{1.5}$$

for all functions $\phi\in L^2(\mathbb{R})$. The Fourier transformation of the Hilbert operator is given by

$$\widehat{\mathcal{H}\phi}(\xi)=\mathrm{i}\mathcal{S}(\xi)\widehat{\phi}(\xi),\qquad \xi\in\mathbb{R}, \tag{1.6}$$

where $\mathcal{S}=\mathcal{S}(\xi)$ represents the standard sign function

$$\mathcal{S}(\xi)=-1\text{ for }\xi<0,\qquad \mathcal{S}(0)=0,\qquad \mathcal{S}(\xi)=+1\text{ for }\xi>0.$$

It is well known that there exists a global smooth solution

$$u\in C^\infty(\mathbb{R}\times\mathbb{R}^+), \tag{1.7}$$

$$u\in L^\infty(\mathbb{R}^+,H^{2m}(\mathbb{R})),\qquad \frac{\partial}{\partial x}u\in L^2(\mathbb{R}^+,H^{2m}(\mathbb{R})),\forall m>0, \tag{1.8}$$

to the Cauchy problem for the Benjamin-Ono-Burgers equation, if the initial function and the external force satisfy the following assumptions

$$u_0\in H^{2m}(\mathbb{R}), \tag{1.9}$$

$$f\in C^\infty(\mathbb{R}\times\mathbb{R}^+)\cap L^1(\mathbb{R}^+,L^2(\mathbb{R}))\cap L^2(\mathbb{R}^+,H^{2m}(\mathbb{R})), \tag{1.10}$$

for all positive constants $m>0$.

It is also well known that there holds the following elementary decay estimate with a sharp rate

$$\sup_{t>0}\left\{t^{1/2}\int_{\mathbb{R}}|u(x,t)|^2\mathrm{d}x\right\}<\infty, \tag{1.11}$$

if the initial function and the external force satisfy the additional conditions

$$u_0\in L^1(\mathbb{R})\cap H^{2m}(\mathbb{R}), \tag{1.12}$$