Geometric Properties and Exact Travelling Wave Solutions for the Generalized Burger-Fisher Equation and the Sharma-Tasso-Olver Equation*

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Abstract In this paper, we study the dynamical behavior and exact parametric representations of the traveling wave solutions for the generalized Burger-Fisher equation and the Sharma-Tasso-Olver equation under different parametric conditions, the exact monotonic and non-monotonic kink wave solutions, two-peak solitary wave solutions, periodic wave solutions, as well as unbounded traveling wave solutions are obtained.

Keywords Monotonic and non-monotonic kink wave, periodic wave, exact solution, integrable system, Burger-Fisher equation, Sharma-Tasso-Olver equation.

MSC(2010) 34A, 35D, 45G.

1. Introduction

It is well known that finding exact solutions of nonlinear wave equations is of great significance because a lot of mathematical models of describing physical phenomena are arising in physics, mechanics, biology, chemistry and engineering. Various powerful methods for obtaining explicit exact traveling wave solutions to nonlinear equations have been developed such as the inverse scattering method, Darboux transformation method, Hirota bilinear method, homogeneous balance method, tanh-function method and so on. For examples, the generalized Burgers-Fisher equation

$$u_t + \alpha u^m u_x + \beta u_{xx} + \gamma u(1 - u^m) = 0 \tag{1.1}$$

has a wide range of applications in plasma physics, fluid mechanics, capillary-gravity waves, nonlinear optics and chemical physics, where α, β and $\gamma \in \mathbf{R}, m$ is positive constant. By using different method, [1,7,8,11,12,14,16] have obtained some exact explicit traveling wave solutions of equation (1.1).

For the following double nonlinear dispersive equation (it was called the Sharma-Tasso-Olver equation):

$$u_t + \alpha(u^3)_x + \frac{3}{2}\alpha(u^2)_{xx} + \alpha u_{xxx} = 0, \tag{1.2}$$

where α is a real parameter, it was first derived as an example of odd members of the Burgers hierarchy by extending the "linearization" achieved through the Cole-Hopf ansatz to equations containing as highest derivatives odd space derivatives. To find the exact traveling wave solutions of equation (1.2), many physicists and mathematicians have paid much attentions to this equation in recent years due to its importance in mathematical physics (see [**?**, 3, 4, 6, 9–11, 15], et al.).

Letting $u(x,t) = \phi(x - vt) = \phi(\xi)$, where $\xi = x - vt$ and v stand for the velocity of wave, substituting it into equation (1.1), we have

$$\phi'' + \frac{1}{\beta}(\alpha\phi^m - v)\phi' + \frac{\gamma}{\beta}\phi(1 - \phi^m) = 0, \tag{1.3}$$

where "'" stand for the derivative with respect to ξ. Without loss of generality, by a parameter transformation, we can take $\beta = 1$.

Equation (1.3) is equivalent to the planar cubic system:

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = -(\alpha\phi^m - v)y - \gamma\phi(1 - \phi^m). \tag{1.4}$$

Letting $u(x,t) = \phi(x - vt) = \phi(\xi)$, substituting it into equation (1.2) and integrating the obtained equation once, taking the integral constant as 0, we obtain

$$\phi'' + 3\phi\phi' - \frac{v}{\alpha}\phi + \phi^3 = 0. \tag{1.5}$$

Equation (1.5) is equivalent to the planar cubic system:

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = -3\phi y + \frac{v}{\alpha}\phi - \phi^3. \tag{1.6}$$

To the best our knowledge, we notice that the dynamical behavior of travelling wave solutions of systems (1.4) and (1.6) have not be studied before. From the view point of the theory of dynamical systems, we hope to know which orbit corresponds to a known exact solution. In other words, we need to understand the geometric properties of all known exact solutions. In this paper, under some integrable parameter conditions, we consider two traveling wave systems of equation (1.1) and equation (1.2) and answer the above problem. The dynamics and exact parametric representations for the traveling wave solutions of equations (1.1) and (1.2) can be given.

This paper is organized as follows. In section 2, we consider the phase portraits of system (1.4) in the integrable cases and give exact kink wave solutions for the generalized Burgers-Fisher equation (1.1). In section 3, we discuss the phase portraits and the graphs of level curves (i.e. the orbits of (1.4) for any fixed h) defined by $H(\phi, y) = h$ of system (1.6) for $v\alpha > 0, v = 0$ and $v\alpha < 0$, respectively. In section 4, we discuss the dynamical behavior of solutions of system (1.4) and figure out exact explicit parametric representations of the traveling wave solutions of the Sharma-Tasso-Olver equation (1.2).

2. The phase portraits of system (1.4) in the integrable cases and the exact kink wave solutions of equation (1.1)

In this section, we first consider the dynamics of system (1.4). Clearly, If m is a non-even positive real number, then system (1.4) has two equilibrium points $E_0(0,0)$ and $E_+(1,0)$. If m is an even integer, then system (1.4) has three equilibrium points $E_0(0,0), E_-(-1,0)$ and $E_+(1,0)$.

Let $M(\phi_i, 0)$ be the coefficient matrix of the linearized system of (1.4) at equilibrium point $E_i(\phi_i, 0)$. Then, we obtain $\det M(\phi_i, 0) = \gamma(1 - (m+1)\phi_i^m)$. Thus, we have

$$J(0,0) = \det M(0,0) = \gamma, \quad J(\pm 1, 0) = \det M(\pm 1, 0) = -m\gamma.$$

$$\Omega = (\text{trace}M(0,0))^2 - 4J(0,0) = v^2 - 4\gamma.$$

By the theory of planar dynamical systems, for an equilibrium point of a planar integrable system, if $J < 0$, then the equilibrium point is a saddle point; If $J > 0$ and $(\text{trace}M)^2 - 4J < 0(> 0)$, then it is a center point (a node point); if $J = 0$ and the Poincaré index of the equilibrium point is 0, then this equilibrium point is a cusp.

By using the results in [2], we know the following two conclusions.

Condition 1. $v > \frac{2\alpha}{m+1}, \gamma = \frac{\alpha((m+1)v-\alpha)}{(m+1)^2}$. Under this parameter condition, system (1.4) has two first integral as follows:

$$I_1(\phi, y) = \left(y + \frac{1}{2}(-v \mp \omega)\phi + \frac{\alpha}{m+1}\phi^{m+1}\right) e^{\frac{1}{2}(-v\pm\omega)t} = h, \qquad (2.1)$$

where $\omega = \sqrt{\Omega} = \sqrt{v^2 - 4\gamma}$.

Condition 2. $\alpha = \frac{(m+1)v}{m+2}, \gamma = \frac{(m+1)v^2}{(m+2)^2}$. Under this parameter condition, system (1.4) has a first integral as follows:

$$I_2(\phi, y) = \left(y - \frac{v}{m+2}\phi + \frac{\alpha}{m+1}\phi^{m+1}\right) e^{-\left(\frac{m+1}{m+2}\right)vt} = h. \qquad (2.2)$$

Under the above two parameter conditions, we know that $\Omega > 0$, equilibrium point $E_0(0,0)$ is a node. Equilibrium points $E_\mp(\mp 1, 0)$ are saddle points. For $m = 1, 2, 3, 4$, respectively, we have the phase portraits of system (1.4) shown in Fig.1 (a)-(d).

We see from (2.1) and (2.2) that when $h = 0$, system (1.4) has three algebraic curve solutions of degree $m + 1$:
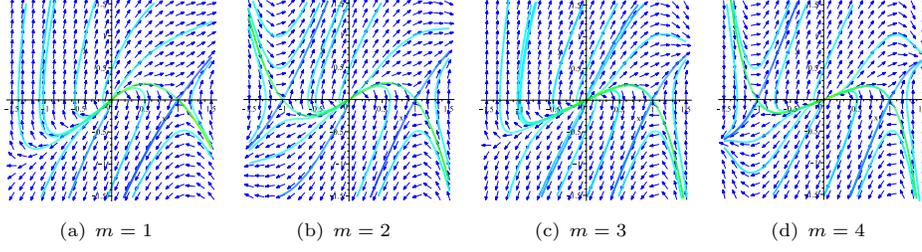
$$y = -\frac{1}{2}(-v \mp \omega)\phi - \frac{\alpha}{m+1}\phi^{m+1}, \quad y = \frac{v}{m+2}\phi - \frac{\alpha}{m+1}\phi^{m+1}. \qquad (2.3)$$

Write that

$$y = \phi(A_j - B\phi^m), \quad j = 1, 2, 3, \qquad (2.4)$$

where

$$A_1 = \frac{1}{2}(v + \omega), \ A_2 = \frac{1}{2}(v - \omega), \ A_3 = \frac{v}{m+2}, \ B = \frac{\alpha}{m+1}.$$

(a) $m = 1$ (b) $m = 2$ (c) $m = 3$ (d) $m = 4$

Figure 1. The phase portraits and invariant curves of system (1.4)

By using the first equation of (1.4), we get $\xi = \int_{\phi_0}^{\phi} \frac{d\phi}{\phi(A_j - B\phi^m)}$, $0 < \phi_0 < 1$. It implies that

$$\phi(\xi) = \left(\frac{A_j \Phi_0 e^{mA_j\xi}}{1 + \Phi_0 B e^{mA_j\xi}} \right)^{\frac{1}{m}}, \quad j = 1, 2, 3, \tag{2.5}$$

where $\Phi_0 = \frac{\phi_0^m}{A_j - B\phi_0^m}$.

Hence, we obtain the following conclusion.

Theorem 2.1. *Under the parameter condition 1 or 2, the generalized Burger-Fisher equation* (1.1) *has kink wave solutions given by* (2.5)*, which correspond to the invariant curve solutions of system* (1.4) *defined by* (2.3)*.*

3. The phase portraits of system (1.6) and the graphs of the level curves defined by $H(\phi, y) = h$

We first consider the Liénard equation

$$\phi'' + f(\phi)\phi' + g(\phi) = 0 \tag{3.1}$$

and corresponding planar dynamical system

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = -f(\phi)\phi' - g(\phi). \tag{3.2}$$

If functions $f(\phi)$ and $g(\phi)$ of equation (3.1) satisfy

$$\frac{d}{d\phi} \left(\frac{g(\phi)}{f(\phi)} \right) = kf(\phi), \quad k = \text{constant} \neq 0, \tag{$\mathcal{L}$}$$

we say that the Chiellini's integrability condition holds.

In [5], we stated the following conclusion.

Theorem A(The integrability of Liénard equation via Chiellini's condition). *Suppose that the functions $f(\phi)$ and $g(\phi)$ of equation* (3.1) *satisfy the Chiellini's integrability condition* $(\mathcal{L})$.

(i) If $f(\phi)$ is known, then

$$g(\phi) = f(\phi) \left[C_1 + k \int f(\phi)d\phi \right]; \tag{3.3}$$

if $g(\phi)$ is known, then

$$f(\phi) = \pm \frac{g(\phi)}{\sqrt{C_2 + 2k \int g(\phi)d\phi}}, \tag{3.4}$$

where C_1 and C_2 are two arbitrary constants.

(ii) For a fixed k given by condition $(\mathcal{L})$, system (3.2) has the following first integrals:

$$\left(ky^2 + y\frac{g(\phi)}{f(\phi)} + \frac{g^2(\phi)}{f^2(\phi)}\right) \exp\left(\frac{2}{\sqrt{4k-1}} \arctan\left(\frac{y+2\frac{g(\phi)}{f(\phi)}}{y\sqrt{4k-1}}\right)\right) = h, \quad for \ \ k > \tfrac{1}{4},$$

$$\frac{y}{y+2\frac{g(\phi)}{f(\phi)}} - 2arctanh\left(1 + \frac{4g(\phi)}{yf(\phi)}\right) - \ln\left(\frac{g(\phi)}{f(\phi)}\right) = h, \quad for \ \ k = \tfrac{1}{4}, \tag{3.5}$$

$$\left(y - \kappa_1\frac{g(\phi)}{f(\phi)}\right)^{\sqrt{1-4k}-1}\left(y - \kappa_2\frac{g(\phi)}{f(\phi)}\right)^{\sqrt{1-4k}+1} = h, \quad for \ \ k < \tfrac{1}{4},$$

where $\kappa_1 = \frac{-1+\sqrt{1-4k}}{2k}, \kappa_2 = -\frac{1+\sqrt{1-4k}}{2k}, h$ is an integral constant.

Clearly, equation (1.5) satisfies Chiellini's integrability condition with $k = \frac{2}{9}$. In fact, $\frac{d}{d\phi}\left(\frac{g}{f}\right) = \frac{d}{d\phi}\left(\frac{1}{3}\phi^2 - \frac{v}{\alpha}\right) = \frac{2}{9}(3\phi) = \frac{2}{9}f(\phi)$. Therefore, we know from Theorem A that system (1.6) is an integrable system. It has the first integral as follows (also see [2]):

$$H(\phi, y) = \left(y + \phi^2 - \frac{v}{\alpha}\right)^{-2}\left(y + \frac{1}{2}\left(\phi^2 - \frac{v}{\alpha}\right)\right) = h. \tag{3.6}$$

For planar dynamical system (1.6), when $v\alpha > 0$, it has three equilibrium points $E_1\left(-\sqrt{\frac{v}{\alpha}}, 0\right), E_2(0,0)$ and $E_3\left(\sqrt{\frac{v}{\alpha}}, 0\right)$. When $v\alpha \le 0$, system (1.6) has only one equilibrium point $E(0,0)$.

Let $M(\phi_i, 0)$ be the coefficient matrix of the linearized system of (1.6) at equilibrium point $(\phi_i, 0)$. Then, we have $\det M(\phi, 0) = 3\phi_i - \frac{v}{\alpha}$. Thus, we have

$$J(0,0) = \det M(0,0) = -\frac{v}{\alpha}, \quad J\left(\pm\sqrt{\frac{v}{\alpha}}, 0\right) = \det M\left(\sqrt{\frac{v}{\alpha}}, 0\right) = \frac{2v}{\alpha},$$

$$\left(\text{trace}M\left(\pm\sqrt{\frac{v}{\alpha}}, 0\right)\right)^2 - 4J\left(\pm\sqrt{\frac{v}{\alpha}}, 0\right) = \frac{v}{\alpha}.$$

We see from the above discussion that for $v\alpha > 0, E_1$ and E_3 are node points, E_2 is a saddle point. For $v\alpha < 0$, $E(0,0)$ is a center point. For $v = 0$, it is a three-multiple singular point.

Let $h_2 = H(0,0) = -\frac{\alpha}{2v}$ and $h_{1,3} = H\left(\pm\sqrt{\frac{v}{\alpha}}, 0\right) = \infty$. By qualitative analysis, corresponding to $v\alpha > 0, v = 0$ and $v\alpha < 0$, respectively, we have the following three phase portraits shown in Fig.1. We also give the graphs of some level curves defined by $H(\phi, y) = h$ when h varies, which are shown in Fig.2 and Fig.3.

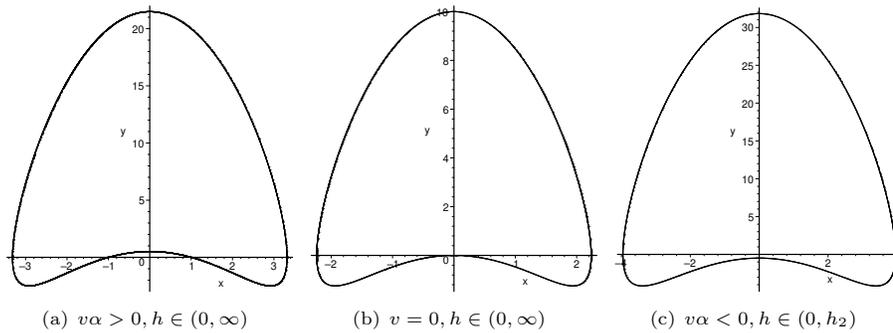# 4. Exact traveling wave solutions of equation (1.2) and their dynamics

In this section, we consider the exact traveling wave solutions of equation (1.2) and their dynamics.

(a) $v\alpha > 0$ (b) $v = 0$ (c) $v\alpha < 0$
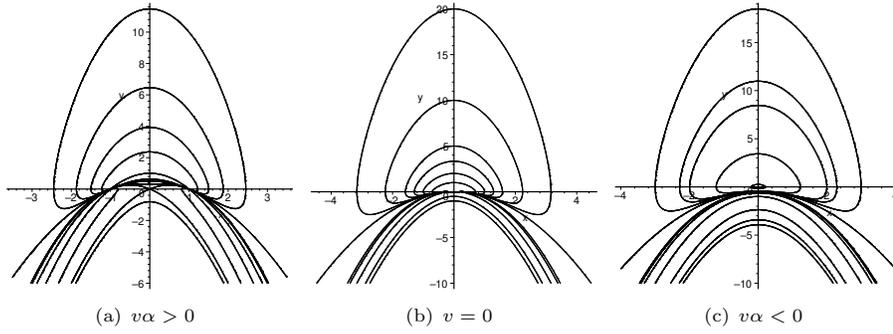
Figure 2. The phase portraits of system (1.6)

(a) $h = h_2$ (b) $h \in (h_2, 0)$ (c) $h = 0$
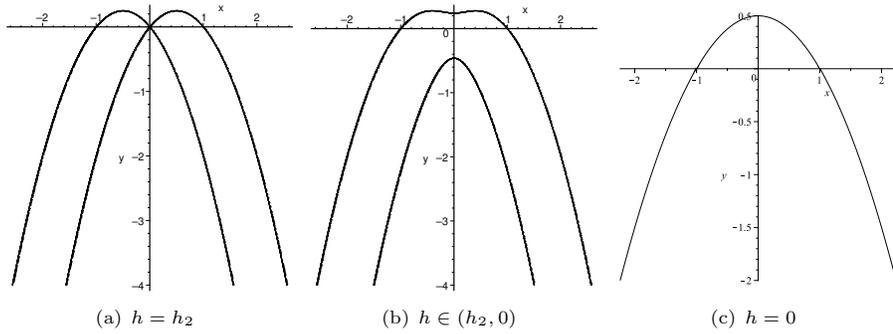
Figure 3. The level curves defined by $H(\phi, y) = h$ of system (1.6) when $v\alpha > 0$

(a) $v\alpha > 0, h \in (0, \infty)$ (b) $v = 0, h \in (0, \infty)$ (c) $v\alpha < 0, h \in (0, h_2)$

Figure 4. The level curves defined by $H(\phi, y) = h$ of system (1.6) in three cases

1. The case $v\alpha > 0$ (see Fig.2 (a)).

(i) Corresponding to the level curves defined by $H(\phi, y) = h_2$ in Fig.3 (a), we have six orbits of system (1.6) to the three equilibrium points, for which four orbits give rise to unbounded solutions of equation (1.2), two heteroclinic orbits connecting the equilibrium points E_1, E_2 and E_3 give rise to two kink wave solutions of equation (1.2). In fact, $H(\phi, y) = h_2$ can be written as

$$\left(y - \sqrt{\frac{v}{\alpha}}\phi + \phi^2\right)\left(y + \sqrt{\frac{v}{\alpha}}\phi + \phi^2\right) = 0.$$

Thus, by taking $y = \sqrt{\frac{v}{\alpha}}\phi - \phi^2$ and $y = -\sqrt{\frac{v}{\alpha}}\phi - \phi^2$, respectively, using the first equation of system (1.6) to integrate, we obtain the parametric representations of two monotonic kink wave solutions as follows:

$$\phi(\xi) = \phi_+(\xi) \equiv \frac{\sqrt{\frac{v}{\alpha}}}{1 + a_0 e^{-\sqrt{\frac{v}{\alpha}}\xi}}, \quad \phi(\xi) = \phi_-(\xi) \equiv -\frac{\sqrt{\frac{v}{\alpha}}}{1 + a_0 e^{\sqrt{\frac{v}{\alpha}}\xi}}, \tag{4.1}$$

where $a_0 = \frac{\sqrt{\frac{v}{\alpha}} - \phi_0}{\phi_0}$, $0 < \phi_0 < \sqrt{\frac{v}{\alpha}}$.

(ii) Corresponding to the level curves defined by $H(\phi, y) = h, h \in (h_2, 0)$ in Fig.3 (b), for every h, the level curves consist of three unbounded orbits and a heteroclinic orbit of system (1.6) connecting the equilibrium points E_1 and E_3. The heteroclinic orbit gives a monotonic kink wave solution of equation (1.2). Now, $H(\phi, y) = h$ can be solved to get

$$y = \frac{1}{2|h|}\left(1 + \frac{2|h|v}{\alpha} - 2|h|\phi^2 + \sqrt{1 + \frac{2|h|v}{\alpha} - 2|h|\phi^2}\right)$$

and

$$y = \frac{1}{2|h|}\left(1 + \frac{2|h|v}{\alpha} - 2|h|\phi^2 - \sqrt{1 + \frac{2|h|v}{\alpha} - 2|h|\phi^2}\right).$$

Hence, using the first equation of system (1.6), we obtain the parametric representations of the family of heteroclinic orbits of system (1.6) connecting the equilibrium points E_1 and E_3 as follows:

$$\phi(\xi) = \mp\frac{1}{\sqrt{2}}\left[\frac{2v}{\alpha} - \frac{1}{|h|} + \left(\frac{1}{\sqrt{|h|}} - \frac{\frac{2v}{\alpha}}{\sqrt{A}\cosh(\omega\xi) + \frac{1}{\sqrt{|h|}}}\right)^2\right]^{\frac{1}{2}}, \tag{4.2}$$

for $\xi \in (-\infty, 0]$ and $\xi \in [0, \infty)$, respectively,

where $A = \frac{1}{|h|} - \frac{2v}{\alpha} > 0, \omega = \frac{1}{2}\sqrt{2A}$. Therefore, for all $h \in (h_2, 0)$, there exists a family of monotonic kink wave solutions of equation (1.2) given by equation (4.2).

(iii) Corresponding to the level curves defined by $H(\phi, y) = 0$ in Fig.3 (c), we have three orbits of system (1.6), for which two orbits are unbounded, one orbit is the heteroclinic orbits connecting the equilibrium points E_1 and E_3. In this case, $H(\phi, y) = 0$ can be written as $y = -\frac{1}{2}\left(\phi^2 - \frac{v}{\alpha}\right)$. Hence, by using the first equation of system (1.6) to integrate, we have the parametric representation of the heteroclinic orbits as follows:

$$\phi(\xi) = \phi_l(\xi) \equiv \sqrt{\frac{v}{\alpha}}\tanh\left(\frac{1}{2}\sqrt{\frac{v}{\alpha}}\xi\right). \tag{4.3}$$

This parametric representation gives rise to a monotonic kink wave solution(i.e., wavefront solution) of equation (1.2).

The two unbounded orbits has the parametric representations:

$$\phi(\xi) = \pm\sqrt{\frac{v}{\alpha}}\operatorname{ctnh}\left(\frac{1}{2}\sqrt{\frac{v}{\alpha}}\xi\right). \tag{4.4}$$

(iv) Corresponding to the level curves defined by $H(\phi, y) = h, h \in (0, \infty)$ in Fig.4 (a), for every h we have two families of heteroclinic orbits of system (1.6) connecting the equilibrium points E_1 and E_3, for which one orbit family gives a family of monotonic kink wave solutions of equation (1.2), another orbit family give a family of non-monotonic kink wave solutions of equation (1.2).

Considering the lower arc of level curve defined by $y = \frac{1}{2}\left(\left(\frac{2v}{\alpha} + \frac{1}{h} - 2\phi^2\right) - \sqrt{\frac{1}{h}\left(\frac{2v}{\alpha} + \frac{1}{h} - 2\phi^2\right)}\right)$ from (3.6) and using the first equation of system (1.6), we obtain the parametric representation of monotonic kink wave solutions as follows:

$$\phi(\xi) = \mp\frac{1}{\sqrt{2}}\left[\frac{2v}{\alpha} + \frac{1}{h} - \left(\frac{\frac{2v}{\alpha}}{\sqrt{A}\cosh(\omega\xi) + \frac{1}{\sqrt{h}}} + \frac{1}{\sqrt{h}}\right)^2\right]^{\frac{1}{2}}, \tag{4.5}$$

for $\xi \in (-\infty, 0]$ and $\xi \in [0, \infty)$, respectively,

where $A = \frac{1}{h} + \frac{2v}{\alpha} > 0, \omega = \frac{1}{2}\sqrt{2A}$.

For all $h \in (0, \infty)$, equation (4.5) gives rise to uncountably infinite many monotonic kink wave solutions of equation (1.2) shown in Fig.5 (a).

Considering the upper arc of level curve defined by $y = \frac{1}{2}\left(\left(\frac{2v}{\alpha} + \frac{1}{h} - 2\phi^2\right) + \sqrt{\frac{1}{h}\left(\frac{2v}{\alpha} + \frac{1}{h} - 2\phi^2\right)}\right)$ from (3.6), $h \in (0, \infty)$, using the first equation of system (1.6), we obtain the parametric representation of non-monotonic kink wave solutions as follows:

$$\phi(\xi) = \mp\frac{1}{\sqrt{2}}\left[\frac{2v}{\alpha} + \frac{1}{h} - \left(\frac{\frac{2v}{\alpha}}{\sqrt{A}\cosh(\omega\xi) - \frac{1}{\sqrt{h}}} - \frac{1}{\sqrt{h}}\right)^2\right]^{\frac{1}{2}}, \tag{4.6}$$

for $\xi \in (-\infty, 0]$ and $\xi \in [0, \infty)$, respectively.

For all $h \in (0, \infty)$, equation (4.6) gives rise to uncountably infinite many non-monotonic kink wave solutions of equation (1.2) shown in Fig.5 (b).

2. The case of $v = 0$ (see Fig.2 (b)).

Corresponding to the curves defined by $H(\phi, y) = h, h \in (0, \infty)$ in (3.6) (see Fig.4 (b)), system (1.6) has a family of homoclinic orbits connecting to the origin $E(0, 0)$. We see from (3.6) that in this case, $H(\phi, y) = h$ follows that $y = \frac{1}{2}\left(\frac{1}{h} - 2\phi^2 \pm \sqrt{\frac{\frac{1}{h} - 2\phi^2}{h}}\right)$.

Thus, we have the parametric representations of the family of homoclinic orbits as follows:

$$\phi(\xi) = \phi_u(\xi) = \frac{2\xi}{2h + \xi^2}. \tag{4.7}$$

By using (4.7), for a fixed $h \in (0, \infty)$ and $c = 0$, we obtain the wave profiles of two peak solitary wave solution $\phi(\xi)$ of equation (1.2) shown in Fig.5 (c).

3. The case of $v\alpha < 0$ (see Fig.2(c)).

(i) Corresponding to the level curves defined by $H(\phi, y) = h$, $h \in (-\infty, 0]$ in (3.6) (see Fig.2 (c)), for every fixed h, system (1.6) has three unbounded orbits. They give three unbounded solutions of equation (1.2). Specially, when $h = 0$, $H(\phi, y) = 0$ can be written as $\frac{1}{2}(y + \phi^2 + |\frac{v}{\alpha}|)(2y + \phi^2 + |\frac{v}{\alpha}|) = 0$. Thus, we have the parametric representations of the exact solutions of equation (1.2) as follows:

$$\phi(\xi) = -\sqrt{|\frac{v}{\alpha}|} \tan\left(\sqrt{|\frac{v}{\alpha}|}\xi\right), \quad \phi(\xi) = -\sqrt{|\frac{v}{\alpha}|} \tan\left(\frac{1}{2}\sqrt{|\frac{v}{\alpha}|}\xi\right). \tag{4.8}$$

(ii) Corresponding to the curves defined by $H(\phi, y) = h, h \in (0, |\frac{2\alpha}{v}|)$ in (3.6) (see Fig.4 (c)), system (1.6) has a family of periodic orbits enclosing the origin $E(0,0)$. These orbits give rise to a family of periodic solutions of equation (1.2). By using (3.6), we see from $y = \frac{1}{2}\left[\frac{1}{h} - |\frac{2v}{\alpha}| - 2\phi^2 + \sqrt{\frac{1}{h}(\frac{1}{h} - |\frac{2v}{\alpha}| - 2\phi^2)}\right]$ and the first equation of (1.6) that the parametric representation of the exact periodic solutions of equation (1.2) is as follows:

$$\phi(\xi) = \mp\frac{1}{\sqrt{2}}\left[\frac{1}{h} - |\frac{2v}{\alpha}| - \left(\frac{|\frac{2v}{\alpha}|}{\frac{1}{\sqrt{h}} + \sqrt{A}\sin(\sqrt{|\frac{v}{\alpha}|}\xi - \xi_0)} - \frac{1}{\sqrt{h}}\right)^2\right]^{\frac{1}{2}}, \tag{4.9}$$

for $\xi \in (-\infty, 0]$ and $\xi \in [0, \infty)$, respectively,

where $\xi_0 = \arcsin\left(\frac{(\sqrt{A} + \frac{1}{\sqrt{h}}) - |\frac{4v}{\alpha}|\sqrt{h}}{(\sqrt{A}h + 1)\sqrt{A}}\right)$.

By using (4.9), for a fixed $h \in (0, \infty)$ and $v\alpha < 0$, we obtain the wave profiles of periodic wave solution of equation (1.2) shown in Fig.5 (d).
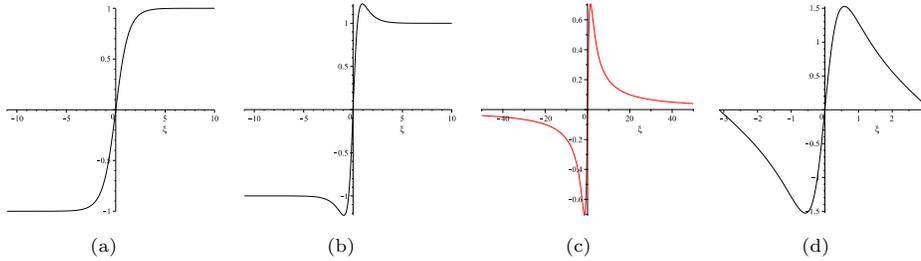
Figure 5. The Profiles of waves of equation (1.2) defined by $H(\phi, y) = h$

To sum up, from the above discussions, we have the following result.

Theorem 4.1. *The traveling wave system* (1.6) *of the Sharma-Tasso-Olver equation* (1.2) *is an integrable system with the first integral* (3.6).

(i) For $v\alpha > 0$, depending on the changes of the level curves defined by $H(\phi, y) = h, h \in (h_2, 0)$ and $h \in (0, \infty)$, equation (1.2) has two families of uncountably infinite many monotonic kink wave solutions given by (4.2), (4.5) and a family nonmonotonic kink wave solutions given by (4.6).

(ii) For $v = 0$, depending on the changes of the level curves defined by $H(\phi, y) = h, h \in (0, \infty)$, equation (1.2) has a family of uncountably infinite many two-peak solitary wave solutions.

(iii) For $v\alpha < 0$, depending on the changes of the level curves defined by $H(\phi, y) = h, h \in (0, h_2)$, equation (1.2) has a family of uncountably infinite many periodic wave solutions given by (4.9).

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