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Periodic Solutions of the Duffing Differential Equation Revisited via the Averaging Theory*

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Abstract We use three different results of the averaging theory of first order for studying the existence of new periodic solutions in the two Duffing differential equations $\ddot{y} + a\sin y = b\sin t$ and $\ddot{y} + ay - cy^3 = b\sin t$, where a, b and c are real parameters.

Keywords Periodic solution, averaging method, Duffing differential equation, bifurcation, stability.

MSC(2010) 34C15, 34C25.

1. Introduction and statement of the main results

Hamel [6] in 1922 gaves the first general results for the existence of periodic solutions of the periodically forced pendulum equation

$$\ddot{y} + a\sin y = b\sin t, \tag{1.1}$$

where the dot denotes derivative with respect to the independent variable t , also called the time, and $y \in \mathbb{S}^1$ is an angle. Four years earlier this equation was the main subject of a monograph published by Duffing [4], who restricted his study to the periodic solutions of the following approximate equation

$$\ddot{y} + ay - cy^3 = b\sin t. \tag{1.2}$$

This equation is now known as the *Duffing differential equation*. The differential equation (1.2) describes the motion of a damped oscillator with a more complicated potential than in the harmonic motion (i.e. when $c = 0$). As usual the parameter a controls the size of stiffness, b controls the amplitude of the periodic driving force, and c controls the amount of nonlinearity in the restoring force. In particular, equation (1.2) models a spring pendulum such that its spring's stiffness only obey approximately the Hooke's law.

Many other different classes of Duffing differential equations have been investigated by several authors. They are mainly interested in the existence of periodic solutions, in their multiplicity, stability, bifurcation, ... See for instance the good survey of Mawhin [10] and for example the articles [2,3,5,7,8,12,14,17,18].

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In this work we shall study the periodic solutions of the Duffing differential equations (1.1) and (1.2), where a, b and c are real parameters, via the averaging theory.

Our main results on the periodic solutions of the Duffing differential equation (1.1) are the following.

Theorem 1.1. *Let ε be a small parameter. The Duffing differential equation* (1.1) *has*

(*a*) *four periodic solutions $y_1(t) = -b\sin t + O(\varepsilon)$, $y_2(t) = \pi - b\sin t + O(\varepsilon)$, $y_3(t) = O(\varepsilon)$, $y_4(t) = \pi + O(\varepsilon)$ if $ab \neq 0$, $a = O(\varepsilon^2)$ and $b = O(\varepsilon)$;*

(*b*) *two periodic solutions $y_i(t)$ for $i = 3, 4$ if $b = 0$ and $a \neq 0$;*

(*c*) *infinitely many periodic solutions $y(t) = k - b\sin t$ with $k \in \mathbb{R}$ if $a = 0$ and $b \neq 0$;*

(*d*) *no periodic solutions if $a = b = 0$.*

Theorem 1.2. *The generalized eigenfunction systems $v_0^0(x), v_0^1(x)$, $\{u_m^0(x)\}_{m=1}^{+\infty}$ and $\{u_m^1(x)\}_{m=1}^{+\infty}$ of the operator F are complete in the sense of Cauchy principal value in Z .*

Theorem 1.1 will be proved in section 2 using the averaging theorems given in the Appendix.

We note that Theorem 1.1 provides new results with respect to Theorem 2.1 of Tarantello [16] where the author provides conditions for having zero, one or two periodic solutions, while in Theorem 1.1 we provide conditions for having zero, two, four or infinitely many periodic solutions.

Our main results on the periodic solutions of the differential system (1.2) are the following.

Theorem 1.3. *The Duffing differential equation* (1.2) *has*

(*a*) *one periodic solution $y(t) = -\sqrt[3]{4b/(3c)}\sin t + O(\varepsilon)$, if $bc \neq 0$, $a = 1$ and $b = O(\varepsilon)$ and $c = O(\varepsilon)$;*

(*b*) *one periodic solution $y(t) = O(\varepsilon)$, if $b \neq 0$, $a = O(\varepsilon)$, $b = O(\varepsilon)$ and $c = O(\varepsilon^2)$;*

(*c*) *two periodic solutions $y_\pm(t) = \pm\sqrt{a/c}$, if $ac > 0$, $b \neq 0$, $a = O(\varepsilon^2)$, $b = O(\varepsilon)$ and $c = O(\varepsilon^2)$;*

(*d*) *three periodic solutions $y(t) = y_0$, where $y_0 \in \{0, \pm\sqrt{a/c}\}$ if $ac > 0$, $b \neq 0$, $a = O(\varepsilon^2)$, $b = O(\varepsilon^2)$ and $c = O(\varepsilon^2)$.*

Theorem 1.3 will be proved in section 3 using three different averaging theorems.

2. Proof of Theorem 1.1

Instead of working with the Duffing differential equation (1.1) we shall work with the equivalent differential systems

$$\begin{aligned} \dot{x} &= -a\sin y + b\sin t, \\ \dot{y} &= x. \end{aligned} \tag{2.1}$$

In order to apply the theorems of the averaging theory of first order, given in the Appendix, for studying the periodic solutions of the differential system (2.1) we scale the variables and the parameters of this differential system.

We start doing a scaling of the variables (x, y) and of the parameter a and b as follows

$$x = \varepsilon^{m_1} X, \quad y = \varepsilon^{m_2} Y, \quad a = \varepsilon^{n_1} A, \quad b = \varepsilon^{n_2} B, \tag{2.2}$$

where m_1, m_2, n_1 and n_2 are integers such that the differential equation (2.1) becomes

$$\begin{aligned} \dot{X} &= -\varepsilon^{n_1 - m_1} A \sin(\varepsilon^{m_2} Y) + \varepsilon^{n_2 - m_1} B \sin t, \\ \dot{Y} &= \varepsilon^{m_1 - m_2} X, \end{aligned} \tag{2.3}$$

where $m_1 - m_2$, $n_1 + m_2 - m_1$ (because $\sin(\varepsilon^{m_2} Y) = \mathcal{O}(\varepsilon^{m_2})$) and $n_2 - m_1$ must be non–negative integers such that

$$\{m_1 - m_2, n_1 + m_2 - m_1, n_2 - m_1\} \cap \{1\} \neq \emptyset,$$

because we want that the differential system (2.3) has some term of order one in ε in order to apply the averaging theory with respect to the small parameter ε of order one. Also we do not consider the case $n_2 - m_1 > 1$, otherwise in the averaging theory the term $b \sin t$ of system (2.1) would not contribute to the existence of periodic solutions, and we want to take it into account. Therefore, we distinguish the following seven cases

Case I: $m_1 - m_2 = 0$, $n_1 + m_2 - m_1 = 0$ and $n_2 - m_1 = 1$;

Case II: $m_1 - m_2 = 0$ and $n_1 + m_2 - m_1 = 1$;

Case III: $m_1 - m_2 = 1$ and $n_1 + m_2 - m_1 = 0$;

Case IV: $m_1 - m_2 = 1$ and $n_1 + m_2 - m_1 = 1$;

Case V: $m_1 - m_2 > 1$ and $n_1 + m_2 - m_1 = 1$;

Case VI: $m_1 - m_2 = 1$ and $n_1 + m_2 - m_1 > 1$;

Case VII: $m_1 - m_2 > 1$, $n_1 + m_2 - m_1 > 1$ and $n_2 - m_1 = 1$;

and every case $\alpha \in \{II, III, \ldots, VI\}$ is separated into the following two subcases:

$$\begin{aligned} &(\alpha.1) \quad n_2 - m_1 = 0, \\ &(\alpha.2) \quad n_2 - m_1 = 1. \end{aligned}$$

We have applied the three theorems of averaging of the Appendix for studying the existence of periodic solutions in the 12 previous subcases of differential systems (2.3). As we shall see the proof of Theorem 1.1 when $ab \neq 0$ will come from the subcase (IV.1), and when $a \neq 0$ and $b = 0$ from the subcase (IV.2). All the other subcases, either do not satisfy the hypotheses of one of the three theorems of the averaging, or provide partial results of the ones stated in Theorem 1.1. Consequently, in what follows we only provide the details of the more positive results, i.e. we shall give the proofs of statements (a) and (b) of Theorem 1.1 only considering the subcases (IV.1) and (IV.2). The proofs of statements (c) and (d) are done without using the averaging theory.

Proof. [Proof of statement (a) of Theorem 1.1.] For the case (IV.1), i.e. for

$$m_1 - m_2 = 1, \quad n_1 + m_2 - m_1 = 1, \quad n_2 - m_1 = 0,$$

we take

$$m_1 = 1, \quad m_2 = 0, \quad n_1 = 2, \quad n_2 = 1. \tag{2.4}$$

Then system (2.3) becomes

$$\begin{aligned} \dot{X} &= -\varepsilon A \sin Y + B \sin t, \\ \dot{Y} &= \varepsilon X. \end{aligned} \tag{2.5}$$

Now we shall apply the averaging Theorem 4.1 to system (2.5). In what follows we use the notation of Theorem 4.1. Thus $\mathbf{x} = (X, Y)^T$ and

$$F_0(t, \mathbf{x}) = \begin{pmatrix} B \sin t \\ 0 \end{pmatrix}, \quad F_1(t, \mathbf{x}) = \begin{pmatrix} -A \sin Y \\ X \end{pmatrix}, \quad F_2(t, \mathbf{x}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

In order to apply some of the three averaging theorems of the appendix for studying the periodic solutions of the differential system (2.5) we must consider that the functions F_i for $i = 0, 1, 2$ are defined in $\mathbb{R} \times \Omega$, where Ω is a bounded open subset of $\mathbb{R}^2$, here we take Ω equal to the disc of center $(0, 0)$ and radius $k+1$, being k the positive integer of the statement of Theorem 1.1.

The unperturbed differential system (4.2) (i.e. in our case system (2.5) with $\varepsilon = 0$) has the solution

$$\mathbf{x}(t, \mathbf{z}, 0) = (X(t), Y(t))^T = (B + X_0 - B \cos t, Y_0)^T. \tag{2.6}$$

such that $\mathbf{x}(0, \mathbf{z}, 0) = \mathbf{z} = \alpha = (X_0, Y_0)^T$. All these solutions are $2\pi-$periodic if and only if $B \neq 0$, here we use the assumption that $b \neq 0$. Then, since $m = n = 2$ using the notation of Theorem 4.1, we have $\xi =$ identity and the conditions (a) and (b) of Theorem 4.1 are satisfied trivially. So system (2.5) satisfies all the assumptions of Theorem 4.1, consequently in what follows we apply this theorem for studying the periodic solutions of system (2.5).

We compute for our system (2.5) the fundamental matrix $M_{\mathbf{z}}(t)$ associated to the first variational system (4.3) such that $M_{\mathbf{z}}(0) = \text{Id}$ of $\mathbb{R}^2$, and we obtain

$$M_{\mathbf{z}}(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Now we must compute the function $\mathcal{F}(\alpha) = \mathcal{F}(X_0, Y_0)$ given in (4.4), i.e.

$$\begin{aligned} \mathcal{F}(X_0, Y_0) = \begin{pmatrix} F_1(X_0, Y_0) \\ F_2(X_0, Y_0) \end{pmatrix} &= \int_0^{2\pi} M_{\mathbf{z}}^{-1}(t) F_1(t, \mathbf{x}(t, \mathbf{z}, 0)) dt, \\ &= \begin{pmatrix} -\displaystyle\int_0^{2\pi} A \sin Y_0 ds \\ \displaystyle\int_0^{2\pi} (X_0 + B - B \cos s)\, ds \end{pmatrix} = 2\pi \begin{pmatrix} -A \sin Y_0 \\ B + X_0 \end{pmatrix}. \end{aligned}$$

Therefore, solving the system $\mathcal{F}(X_0, Y_0) = (0,0)$, we obtain in Ω the solutions

$$\alpha_j = (X_{0,j}, Y_{0,j}) = (-B, j\pi) \quad \text{for } j = -k, \ldots, -1, 0, 1, \ldots, k.$$

Moreover we have that the Jacobian

$$\det\left(\frac{\partial \mathcal{F}}{\partial \alpha}(\alpha_j)\right) = (-1)^k A \neq 0,$$

because by assumption $a \neq 0$. Hence Theorem 4.1 says that if $AB \neq 0$ then for every solution $(X_{0,j}, Y_{0,j}) = (-B, j\pi)$ of the system $\mathcal{F}(X_0, Y_0) = 0$, the differential system (2.5) with $\varepsilon = \varepsilon(k) \neq 0$ sufficiently small has a $2\pi-$ periodic solution $(X(t,\varepsilon), Y(t,\varepsilon))$ such that $(X(0,\varepsilon), Y(0,\varepsilon) \to (-B, j\pi)$ when $\varepsilon \to 0$. So, from (2.5) and (2.6) the periodic solution $(X(t,\varepsilon), Y(t,\varepsilon))$ tends to the solution

$$(X(t), Y(t)) \approx \big(-B\cos t, j\pi\big) \tag{2.7}$$

for ε sufficiently small, i.e.

$$(X(t,\varepsilon), Y(t,\varepsilon)) = \big(-B\cos t + O(\varepsilon), j\pi - \varepsilon B \sin t + O(\varepsilon^2)\big).$$

After the change of variables (2.2) satisfying (2.4), i.e.

$$x = \varepsilon X, \quad y = Y, \quad b = \varepsilon B, \quad a = \varepsilon^2 A,$$

we obtain that the 2π–periodic solution (2.7) of system (2.5) becomes the 2π–periodic solution

$$(x(t), y(t)) \approx \big(-b\cos t, j\pi - b\sin t\big)$$

of system (2.1). Now taking into account that y is an angle, doing modulo 2π these 2π–periodic solutions of system (2.1) provide the following two 2π–periodic solutions

(i) $y(t) \approx -b\sin t$ if j is even,

(ii) $y(t) \approx \pi - b\sin t$ if j is odd,

of the differential equation (1.1). This ends the proof of statement (a) of Theorem 1.1.

□

Proof. [Proof of statement (b) of Theorem 1.1.] For the case (IV.2), i.e. for

$$m_1 - m_2 = 1, \quad n_1 + m_2 - m_1 = 1, \quad n_2 - m_1 = 1;$$

we take

$$m_1 = 1, \quad m_2 = 0, \quad n_1 = 2, \quad n_2 = 2. \tag{2.8}$$

Then system (2.3) becomes

$$\begin{aligned} \dot{X} &= -\varepsilon A \sin Y + \varepsilon B \sin t, \\ \dot{Y} &= \varepsilon X. \end{aligned} \tag{2.9}$$

We shall apply the averaging Theorem 4.2 to system (2.9). In what follows we shall use the notation of system (4.5) and of Theorem 4.2. Thus $\mathbf{x} = (X, Y)^T$ and

$$F_1(t, \mathbf{x}) = \begin{pmatrix} -A\sin Y + B\sin t \\ X \end{pmatrix}, \qquad F_2(t, \mathbf{x}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We must compute the function $g(\mathbf{y})$ given in (4.7), i.e.

$$g(\mathbf{y}) = \frac{1}{2\pi}\int_0^{2\pi}\begin{pmatrix} -A\sin Y_0 + B\sin s \\ X_0 \end{pmatrix} ds = \begin{pmatrix} -A\sin Y_0 \\ X_0 \end{pmatrix}.$$

Then, due to the fact that Y is an angle we have two solutions for (X_0, Y_0), namely $(X_1, Y_1) = (0, 0)$ and $(X_2, Y_2) = (0, \pi)$. Since the Jacobian (4.8) in these two solutions is A and $-A$ respectively, and by assumptions $a \neq 0$, by Theorem 4.2 the system (2.9) has two periodic solutions $(X_i(t, \varepsilon), Y_i(t, \varepsilon))$ such that $(X_i(0, \varepsilon), Y_i(0, \varepsilon))$ tends to (X_i, Y_i) when $\varepsilon \to 0$. So by statement (a) of Theorem 4.2 we have

$$(X_i(t, \varepsilon), Y_i(t, \varepsilon)) \approx \big(O(\varepsilon), Y_i + O(\varepsilon)\big) \tag{2.10}$$

for ε sufficiently small.

After the change of variables (2.2) satisfying (2.8), i.e.

$$x = \varepsilon X, \quad y = Y, \quad b = \varepsilon^2 B, \quad a = \varepsilon^2 A,$$

we obtain that the two periodic solutions (2.10) of system (2.9) becomes the two periodic solutions

$$(x_1(t), y_1(t)) \approx \big(O(\varepsilon), O(\varepsilon)\big), \quad (x_2(t), y_2(t)) \approx \big(O(\varepsilon), \pi + O(\varepsilon)\big),$$

of system (2.1). This ends the proof of statement (b) of Theorem 1.1. □

Proof. [Proof of statement (c) of Theorem 1.1.] Now the differential system (2.1) becomes

$$\dot{x} = b\sin t, \qquad \dot{y} = x.$$

Its general solution is $x(t) = k_1 - b\cos t$ and $y(t) = k_1 t - b\sin t + k_2$, where k_1 and k_2 are arbitrary constants. So, clearly the unique periodic solutions of the differential equation (1.1) are $y(t) = -b\sin t + k_2$. This proves the statement (c) Theorem 1.1. □

Proof. [Proof of statement (c) of Theorem 1.1.] Now the differential system (2.1) becomes

$$\dot{x} = b\sin t, \qquad \dot{y} = x.$$

Its general solution is $x(t) = k_1 - b\cos t$ and $y(t) = k_1 t - b\sin t + k_2$, where k_1 and k_2 are arbitrary constants. So, clearly the unique periodic solutions of the differential equation (1.1) are $y(t) = -b\sin t + k_2$. This proves the statement (c) Theorem 1.1. □

Proof. [Proof of statement (d) of Theorem 1.1.] Under the assumptions of statement (d) the differential system (2.1) becomes

$$\dot{x} = 0, \qquad \dot{y} = x.$$

Its general solution is $x(t) = k_1$ and $y(t) = k_1 t + k_2$, where k_1 and k_2 are arbitrary constants. So the system has no periodic solutions. □

3. Proof of Theorem 1.3

Instead of working with the Duffing differential equation (1.2) we shall work with the equivalent differential system

$$\begin{aligned} \dot{x} &= -ay + cy^3 + b\sin t, \\ \dot{y} &= x. \end{aligned} \tag{3.1}$$

Again in order to apply the three theorems of the averaging theory of first order for studying the periodic solutions of the differential system (3.1) we scale the variables and the parameters of this differential system.

We start doing a rescaling of the variables (x, y) and of the parameter a, b and c as follows

$$x = \varepsilon^{m_1} X, \quad y = \varepsilon^{m_2} Y, \quad a = \varepsilon^{n_1} A, \quad b = \varepsilon^{n_2} B, \quad c = \varepsilon^{n_3} C, \tag{3.2}$$

where m_1, m_2, n_1, n_2 and n_3 are integers such that the differential equation (3.1) becomes

$$\begin{aligned} \dot{X} &= -\varepsilon^{n_1+m_2-m_1} AY + \varepsilon^{n_3+3m_2-m_1} CY^3 + \varepsilon^{n_2-m_1} B\sin t, \\ \dot{Y} &= \varepsilon^{m_1-m_2} X, \end{aligned} \tag{3.3}$$

where $m_1 - m_2$, $n_1 + m_2 - m_1$, $n_3 + 3m_2 - m_1$ and $n_2 - m_1$ must be non–negative integers such that

$$\{m_1 - m_2, n_1 + m_2 - m_1, n_3 + 3m_2 - m_1, n_2 - m_1\} \cap \{1\} \neq \emptyset.$$

We essentially distinguish the same seven cases of section 2, i.e. Therefore, we distinguish the following seven cases

Case I: $m_1 - m_2 = 0$ and $n_1 + m_2 - m_1 = 0$;

Case II: $m_1 - m_2 = 0$ and $n_1 + m_2 - m_1 = 1$;

Case III: $m_1 - m_2 = 1$ and $n_1 + m_2 - m_1 = 0$;

Case IV: $m_1 - m_2 = 1$ and $n_1 + m_2 - m_1 = 1$;

Case V: $m_1 - m_2 > 1$ and $n_1 + m_2 - m_1 = 1$;

Case VI: $m_1 - m_2 = 1$ and $n_1 + m_2 - m_1 > 1$;

Case VII: $m_1 - m_2 > 1$ and $n_1 + m_2 - m_1 > 1$.

Every case $\alpha \in \{II, III, \ldots, VI\}$ is divided into the following four subcases:

$$\begin{aligned} &(\alpha.1) \quad n_3 + 3m_2 - m_1 = 0 \quad \text{and} \quad n_2 - m_1 = 0, \\ &(\alpha.2) \quad n_3 + 3m_2 - m_1 = 0 \quad \text{and} \quad n_2 - m_1 = 1, \\ &(\alpha.3) \quad n_3 + 3m_2 - m_1 = 1 \quad \text{and} \quad n_2 - m_1 = 0, \\ &(\alpha.4) \quad n_3 + 3m_2 - m_1 = 1 \quad \text{and} \quad n_2 - m_1 = 1; \end{aligned}$$

and the cases I and VII are divided only into the following three subcases

$$
\begin{aligned}
&(\alpha.2) \quad n_3 + 3m_2 - m_1 = 0 \quad \text{and} \quad n_2 - m_1 = 1,\\
&(\alpha.3) \quad n_3 + 3m_2 - m_1 = 1 \quad \text{and} \quad n_2 - m_1 = 0,\\
&(\alpha.4) \quad n_3 + 3m_2 - m_1 = 1 \quad \text{and} \quad n_2 - m_1 = 1.
\end{aligned}
$$

We have applied the three theorems of averaging (see the Appendix) for studying the existence of periodic solutions of the 26 previous subcases of differential systems (3.3). As we shall see in the proof of Theorem 1.3 statement (a) will come from the subcase (I.4), statement (b) from the subcase (III.3), statement (c) from the subcase (IV.3), statement (d) from the subcase (IV.4). All the other subcases, either do not satisfy the hypotheses of one of the three theorems of the averaging, or provide partial results of the ones stated in Theorem 1.3. As in the proof of Theorem 1.1 we only provide the details of the positive results in the proof of Theorem 1.3. We separate its proof into its four statements.

Proof. [Proof of statement (a) of Theorem 1.3] For the case (I.4), i.e. for

$$m_1 - m_2 = 0, \quad n_1 + m_2 - m_1 = 0, \quad n_3 + 3m_2 - m_1 = 1, \quad n_2 - m_1 = 1;$$

we take

$$m_1 = m_2 = n_1 = 0, \quad n_2 = n_3 = 1. \tag{3.4}$$

Then system (3.3) becomes

$$
\begin{aligned}
\dot{X} &= -AY + \varepsilon\big(CY^3 + B\sin t\big),\\
\dot{Y} &= X.
\end{aligned} \tag{3.5}
$$

We shall apply the averaging Theorem 4.1 to system (3.5) and we shall obtain the statement (a) of Theorem 1.3. In what follows we shall use the notation of Theorem 4.1. Thus $\mathbf{x} = (X, Y)^T$ and

$$F_0(t, \mathbf{x}) = \begin{pmatrix} -AY \\ X \end{pmatrix}, \quad F_1(t, \mathbf{x}) = \begin{pmatrix} CY^3 + B\sin t \\ 0 \end{pmatrix}, \quad F_2(t, \mathbf{x}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The unperturbed differential system (4.2) only has periodic solutions $\mathbf{x}(t, \mathbf{z}, 0) = (X(t), Y(t))^T$ such that $\mathbf{x}(0, \mathbf{z}, 0) = \mathbf{z} = (X_0, Y_0)^T$ if $A > 0$ given by

$$X(t) = X_0 \cos\left(\sqrt{A}t\right) - Y_0\sqrt{A}\sin\left(\sqrt{A}t\right), \quad Y(t) = \frac{X_0}{\sqrt{A}}\sin\left(\sqrt{A}t\right) + Y_0\cos\left(\sqrt{A}t\right).$$

In order that $\mathbf{x}(t, \mathbf{z}, 0)$ be a periodic solution of period $T = 2\pi$, the period of the function $\sin t$ and we can apply the averaging theory, we must choose $A = 1$. So we get

$$(X(t), Y(t)) = (X_0 \cos t - Y_0 \sin t, X_0 \sin t + Y_0 \cos t), \tag{3.6}$$

where $(X_0, Y_0) \in \mathbb{R}^2 \setminus \{(0, 0)\}$.

Now we compute the fundamental matrix $M_{\mathbf{z}}(t)$ associated to the first variational system (4.3) such that $M_{\mathbf{z}}(0) = \mathrm{Id}$ of $\mathbb{R}^2$, and we obtain

$$M_{\mathbf{z}}(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

Again since $m = n = 2$ in Theorem 4.1 we do not need to check any condition for applying the averaging theory described in Theorem 4.1 for studying the periodic solutions (3.6), of system (3.5) with $\varepsilon = 0$ and $A = 1$, which can be prolonged to the perturbed system (3.5) with $\varepsilon \neq 0$ sufficiently small and $A = 1$. Note that here $\mathbf{z} = \alpha$. Moreover, since for our differential system we have $\xi(X, Y) = (X, Y)$, we must compute the function (4.4), i.e.

$$\mathcal{F}(X_0, Y_0) = \int_0^{2\pi} M_{\mathbf{z}}^{-1}(t) F_1(t, \mathbf{x}(t, \mathbf{z}, 0)) dt = 2\pi \begin{pmatrix} \frac{3}{8} C Y_0 \left(X_0^2 + Y_0^2\right) \\ \frac{1}{8}\left(-4B - 3CX_0\left(X_0^2 + Y_0^2\right)\right) \end{pmatrix}.$$

The system $\mathcal{F}(X_0, Y_0) = (0, 0)$ has three solutions, but only one is real, namely

$$(X_0^*, Y_0^*) = \left(-\sqrt[3]{\frac{4B}{3C}}, 0\right).$$

Since

$$\det\left(\left.\frac{\partial \mathcal{F}(X_0, Y_0)}{\partial (X_0, Y_0)}\right|_{(X_0, Y_0) = (X_0^*, Y_0^*)}\right) = \frac{3^{\frac{5}{3}} B^{\frac{4}{3}} C^{\frac{2}{3}}}{2^{\frac{10}{3}}} \neq 0,$$

Theorem 4.1 says that the periodic solution

$$(X(t), Y(t)) = \left(-\sqrt[3]{\frac{4B}{3C}} \cos t, -\sqrt[3]{\frac{4B}{3C}} \sin t\right)$$

of the differential system (3.5) with $\varepsilon = 0$ can be prolonged to a periodic solution of system (3.5) with $\varepsilon \neq 0$ sufficiently small. Therefore, going back through the change of variables (3.2) satisfying (3.4), i.e.

$$x = X, \quad y = Y, \quad a = A, \quad b = \varepsilon B, \quad c = \varepsilon C,$$

we get that the differential system (3.1) has the 2π–periodic solution given in the statement (a) of Theorem 1.3. □

Proof. [Proof of statement (*b*) of Theorem 1.3] For the case (III.3), i.e. for

$$m_1 - m_2 = 1, \quad n_1 + m_2 - m_1 = 0, \quad n_3 + 3m_2 - m_1 = 1, \quad n_2 - m_1 = 0,$$

we take

$$m_1 = n_1 = n_2 = 1, \quad m_2 = 0, \quad n_3 = 2. \tag{3.7}$$

Then system (3.3) becomes

$$\begin{aligned} \dot{X} &= -AY + \varepsilon C Y^3 + B \sin t, \\ \dot{Y} &= \varepsilon X, \end{aligned} \tag{3.8}$$

We shall apply the averaging Theorem 4.1 to system (3.8) and we shall obtain the statement (*b*) of Theorem 1.3. In what follows we shall use the notation of Theorem 4.1. Thus $\mathbf{x} = (X, Y)^T$ and

$$F_0(t, \mathbf{x}) = \begin{pmatrix} -AY + B \sin t \\ 0 \end{pmatrix}, \quad F_1(t, \mathbf{x}) = \begin{pmatrix} CY^3 \\ X \end{pmatrix}, \quad F_2(t, \mathbf{x}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The unperturbed differential system (4.2) has the solution

$$\mathbf{x}(t,\mathbf{z},0)=(X(t),Y(t))^T=(X_0+B-B\cos t-AY_0t,Y_0)\,.$$

such that $\mathbf{x}(0,\mathbf{z},0)=\mathbf{z}=(X_0,Y_0)^T$. In order that $\mathbf{x}(t,\mathbf{z},0)$ be a $2\pi-$periodic solution we must choose $B\neq 0$ and $Y_0=0$. Then we get

$$\mathbf{z}_\alpha=(\alpha,\beta_0(\alpha))=(X_0+B-B\cos t,0)\,,$$

Therefore, using the notation of Theorem 4.1, we have $n=2$ and $k=1$ for each one of these possible families of periodic solution. We compute the fundamental matrix $M_{\mathbf{z}_\alpha}(t)$ associated to the first variational system (4.3) such that $M_{\mathbf{z}_\alpha}(0)=$ Id of $\mathbb{R}^2$, and we obtain

$$M_{\mathbf{z}_\alpha}(t)=\begin{pmatrix}1 & -At\\ 0 & 1\end{pmatrix}.$$

Since the matrix

$$M^{-1}_{\mathbf{z}_\alpha}(0)-M^{-1}_{\mathbf{z}_\alpha}(2\pi)=\begin{pmatrix}0 & -2A\pi\\ 0 & 0\end{pmatrix}$$

has a non–zero 1×1 matrix in the upper right corner, and a zero 1×1 matrix in its lower right corner. Therefore we can apply Theorem 4.2 of averaging if $A\neq 0$, then by applying this theorem we study the periodic solutions which can be prolonged from the unperturbed differential system to the perturbed one. Now we must compute the function $\mathcal{F}(\alpha)=\mathcal{F}(X_0,0)$ given in (4.4), i.e.

$$\begin{aligned}\mathcal{F}(X_0,0)&=\int_0^{2\pi}M^{-1}_{\mathbf{z}_\alpha}(t)F_1(t,\mathbf{x}(t,\mathbf{z}_\alpha,0))dt\\&=\begin{pmatrix}\int_0^{2\pi}A\,(X_0+B-B\cos s)\,sds\\ \int_0^{2\pi}(X_0+B-B\cos s)\,ds\end{pmatrix}\\&=\begin{pmatrix}2\pi^2A\,(B+X_0)\\ 2\pi\,(B+X_0)\end{pmatrix}.\end{aligned}$$

The system $\mathcal{F}(X_0,Y_0)=(0,0)$ has a solution

$$(X_0,0)=(-B,0).$$

Hence Theorem 4.2 says that if $B\neq 0$ and for every simple real root $(X_0,0)$ of the system $\mathcal{F}(X_0,0)=(0,0)$, the differential system (3.8) with $\varepsilon\neq 0$ sufficiently small has one $2\pi-$periodic solution

$$(X(t),Y(t))=(-B\cos t,0)$$

which can be prolonged for to a periodic solution

$$(X(t,\varepsilon),Y(t,\varepsilon))=(-B\cos t+O(\varepsilon),O(\varepsilon))\,.$$

Therefore, going back through the change of variables (3.2) satisfying (3.7), i.e.

$$\frac{x}{\varepsilon}=X,\quad y=Y,\quad C=\frac{c}{\varepsilon^2},\quad B=\frac{b}{\varepsilon},\quad A=\frac{a}{\varepsilon}.$$

we get that the differential system (3.1) has the 2π–periodic solution given in the statement (b) of Theorem 1.3. □

Proof. [Proof of statement (c) of Theorem 1.3] For the case (IV.3), i.e. for

$$m_1 - m_2 = 1, \quad n_1 + m_2 - m_1 = 1, \quad n_3 + 3m_2 - m_1 = 1, \quad n_2 - m_1 = 1$$

we take

$$n_2 = m_1 = 1, m_2 = 0, \qquad n_1 = n_3 = 2. \tag{3.9}$$

Then system (3.3) becomes

$$\begin{aligned} \dot{X} &= -\varepsilon AY + \varepsilon CY^3 + B\sin t, \\ \dot{Y} &= \varepsilon X, \end{aligned} \tag{3.10}$$

We shall apply the averaging Theorem 4.1 to system (3.10) and we shall obtain the statement (c) of Theorem 1.3. In what follows we shall use the notation of Theorem 4.1. Thus $\mathbf{x} = (X, Y)^T$ and

$$F_0(t, \mathbf{x}) = \begin{pmatrix} B\sin t \\ 0 \end{pmatrix}, \quad F_1(t, \mathbf{x}) = \begin{pmatrix} -AY + CY^3 \\ X \end{pmatrix}, \quad F_2(t, \mathbf{x}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The unperturbed differential system (4.2) only has periodic solutions $\mathbf{x}(t, \mathbf{z}, 0) = (X(t), Y(t))^T$ such that $\mathbf{x}(0, \mathbf{z}, 0) = \mathbf{z} = (X_0, Y_0)^T$ given by

$$X(t) = X_0 + B - B\cos t, \quad Y(t) = Y_0.$$

All these solutions are $2\pi-$periodic if and only if $B \neq 0$.

Since $k = n = 2$ in the Theorem 4.1, we do not need to check any condition for applying the Theorem 4.1.

We compute the fundamental matrix $M_{\mathbf{z}_\alpha}(t)$ associated to the first variational system (4.3) associated to the vector field $(\dot{Y}, \dot{X})$ given by (3.10) with $\varepsilon = 0$, and such that $M_{\mathbf{z}_\alpha}(0) = \mathrm{Id}$ of $\mathbb{R}^2$. We obtain

$$M_{\mathbf{z}_\alpha}(t) = \begin{pmatrix} 1 \; 0 \\ 0 \; 1 \end{pmatrix}.$$

Then by applying this theorem we study the periodic solutions which can be prolonged from the unperturbed differential system to the perturbed one, then we must compute the function $\mathcal{F}(\alpha) = \mathcal{F}(X_0, Y_0)$ given in (4.4), i.e.

$$\begin{aligned} \mathcal{F}(X_0, Y_0) &= \xi^{\perp}\left(\int_0^{2\pi} M_{\mathbf{z}_\alpha}^{-1}(t) F_1(t, \mathbf{x}(t, \mathbf{z}_\alpha, 0)) dt\right) \\ &= \begin{pmatrix} -AY_0 + CY_0^3 \\ B + X_0 \end{pmatrix} \end{aligned}$$

The system $\mathcal{F}(X_0, Y_0) = (0, 0)$ for $AC > 0$ has three solutions $(-B, Y_0)$ with $Y_0 \in \left\{0, \ \pm\sqrt{\frac{A}{C}}\right\}$,we want to study the only solutions $\left(-B, \pm\sqrt{\frac{A}{C}}\right)$ because the solution $(-B, 0)$ appears in the statement (b) of Theorem 1.3.

On the other hand we have the Jacobian matrix for the function $\mathcal{F}$ on (X_0, Y_0) gives by

$$\det\left(\frac{\partial \mathcal{F}(X_0,Y_0)}{\partial(X_0,Y_0)}\bigg|_{(X_0,Y_0)=\left(-B,\pm\sqrt{\frac{A}{C}}\right)}\right) = -2A.$$

Theorem 4.2 says that if $A \neq 0$ and for two simple real roots $\left(-B, \pm\sqrt{\frac{A}{C}}\right)$ of the function $\mathcal{F}(X_0, Y_0)$, the differential system (3.10) has two $2\pi-$ periodic solution $(X(t), Y(t)) = (-B\cos t, \pm\sqrt{\frac{A}{C}})$ with $\varepsilon = 0$ can be prolonged to a periodic solution

$$(X(t,\varepsilon), Y(t,\varepsilon)) = \left(-B\cos t + O(\varepsilon), \pm\sqrt{\frac{A}{C}} - \varepsilon B\sin t + O(\varepsilon)\right).$$

of system (3.10) with $\varepsilon \neq 0$ sufficiently small. Therefore, going back through the change of variables (3.2) satisfying (3.9), i.e.

$$\frac{x}{\varepsilon} = X, \quad y = Y, \quad C = \frac{c}{\varepsilon^2}, \quad B = \frac{b}{\varepsilon}, \quad A = \frac{a}{\varepsilon^2}.$$

we get that the differential system (3.1) has the tow 2π–periodic solution given in the statement (c) of Theorem 1.3. □

Proof. [Proof of statement (d) of Theorem 1.3] For the case (IV.4)

$$m_1 - m_2 = 1, \quad n_1 + m_2 - m_1 = 1, \quad n_3 + 3m_2 - m_1 = 1, \quad n_2 - m_1 = 1,$$

we take

$$m_1 = 1, \quad m_2 = 0, \qquad n_3 = n_2 = n_1 = 2. \tag{3.11}$$

Then system (3.3) becomes

$$\begin{aligned} \dot{X} &= -\varepsilon AY(t) + \varepsilon CY^3(t) + \varepsilon B\sin t \\ \dot{Y} &= \varepsilon X(t) \end{aligned} \tag{3.12}$$

We apply directly the Theorem 4.2 to system (3.12) and we shall obtain the statement (d) of Theorem 1.3.

In what follows we shall use the notation of Theorem 4.1 and Theorem 4.2. Thus $\mathbf{x} = (X, Y)^T$ and

$$F_0(t,\chi) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad F_1(t,\chi) = \begin{pmatrix} -AY(t) + CY^3(t) + B\sin t \\ X(t) \end{pmatrix}.$$

We must compute the function $g(\mathbf{y}) = \mathcal{G}(X_0, Y_0)$ given in (4.7), i.e

$$g(\mathbf{y}) = \begin{pmatrix} -AY_0 + CY_0^3 \\ X_0 \end{pmatrix}.$$

g is periodic of period $T = 2\pi$. If $AC > 0$, the system $\mathcal{G}(X_0, Y_0) = (0, 0)$ has three solutions

$$(X_0, Y_0) = (0, Y_0) \text{ where } Y_0 \in \left\{0, \pm\sqrt{\frac{A}{C}}\right\}.$$

Since

$$\det\left(\left.\frac{\partial\mathcal{G}(X_0,Y_0)}{\partial(X_0,Y_0)}\right|_{(X_0,Y_0)=(0,0)}\right)=A,$$

and

$$\det\left(\left.\frac{\partial\mathcal{G}(X_0,Y_0)}{\partial(X_0,Y_0)}\right|_{(X_0,Y_0)=\left(0,\pm\sqrt{\frac{A}{C}}\right)}\right)=-2A,$$

Theorem 4.2 says that if $A\neq 0$ and for every simple real root $(0,Y_0)$ of the function $g(\mathbf{y})$,the differential system (3.12) has three $2\pi-$ periodic solutions

$$(X(t),Y(t))=(0,Y_0)\qquad Y_0\in\left\{0,\ \pm\sqrt{\frac{A}{C}}\right\}.$$

with $\varepsilon=0$ can be prolonged to the three periodic solutions

$$(X(t,\varepsilon),Y(t,\varepsilon))=(-\varepsilon B\,(\cos t-1)+O(\varepsilon),O(\varepsilon))\qquad \text{for } (X_0,Y_0)=(0,0)\,.$$

and

$$(X(t,\varepsilon),Y(t,\varepsilon))=\left(-\varepsilon B\,(\cos t-1)+O(\varepsilon),\pm\sqrt{\frac{A}{C}}+O(\varepsilon)\right),$$

for $(X_0,Y_0)=\left(0,\pm\sqrt{A/C}\right)$. Therefore, going back through the change of variables (3.2) satisfying (3.11), i.e.

$$\frac{x}{\varepsilon}=X,\quad y=Y,\quad C=\frac{c}{\varepsilon^2},\quad B=\frac{b}{\varepsilon^2},\quad A=\frac{a}{\varepsilon^2}.$$

We get that the differential system (3.1) has the three 2π–periodic solutions given in the statement (d) of Theorem 1.3. □

4. Appendix: Periodic solutions via the averaging theory

In this section we present the basic results on the averaging theory of first order that we need for proving our results.

We consider the problem of bifurcation of T–periodic solutions from the differential systems of the form

$$\dot{\mathbf{x}}=F_0(t,\mathbf{x})+\varepsilon F_1(t,\mathbf{x})+\varepsilon^2F_2(t,\mathbf{x},\varepsilon),\tag{4.1}$$

with $\varepsilon=0$ to $\varepsilon\neq 0$ sufficiently small. The functions $F_0,F_1:\mathbb{R}\times\Omega\to\mathbb{R}^n$ and $F_2:\mathbb{R}\times\Omega\times(-\varepsilon_0,\varepsilon_0)\to\mathbb{R}^n$ are $\mathcal{C}^2$, T–periodic in the first variable and Ω is an open subset of $\mathbb{R}^n$. The main assumption is that the unperturbed system

$$\dot{\mathbf{x}}=F_0(t,\mathbf{x}),\tag{4.2}$$

has a submanifold of periodic solutions. A solution of this problem is given using the averaging theory.

Let $\mathbf{x}(t, \mathbf{z}, \varepsilon)$ be the solution of system (4.2) such that $\mathbf{x}(0, \mathbf{z}, \varepsilon) = \mathbf{z}$. We write the linearization of the unperturbed system along a periodic solution $\mathbf{x}(t, \mathbf{z}, 0)$ as

$$\dot{\mathbf{y}} = D_{\mathbf{x}} F_0(t, \mathbf{x}(t, \mathbf{z}, 0))\mathbf{y}. \tag{4.3}$$

In what follows we denote by $M_{\mathbf{z}}(t)$ a fundamental matrix of the linear differential system (4.3), by $\xi : \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^m$ and $\xi^\perp : \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^{n-m}$ the projections of $\mathbb{R}^n$ onto its first m and $n - m$ coordinates respectively; i.e. $\xi(x_1, \dots, x_n) = (x_1, \dots, x_m)$, and $\xi^\perp(x_1, \dots, x_n) = (x_{m+1}, \dots, x_n)$

Theorem 4.1. *Let $V \subset \mathbb{R}^m$ be open and bounded, let $\beta_0 : Cl(V) \to \mathbb{R}^{n-m}$ be a $\mathcal{C}^k$ function and $\mathcal{Z} = \{\mathbf{z}_\alpha = (\alpha, \beta_0(\alpha)) \,|\, \alpha \in Cl(V)\} \subset \Omega$ its graphic in $\mathbb{R}^n$. Assume that for each $\mathbf{z}_\alpha \in \mathcal{Z}$ the solution $\mathbf{x}(t, \mathbf{z}_\alpha, 0)$ of (4.2) is T-periodic and that there exists a fundamental matrix $M_{\mathbf{z}_\alpha}(t)$ of (4.3) such that the matrix $M^{-1}_{\mathbf{z}_\alpha}(0) - M^{-1}_{\mathbf{z}_\alpha}(T)$*

(*a*) *has in the lower right corner the $(n-m)\times(n-m)$ matrix Δ_α with $\det(\Delta_\alpha) \neq 0$, and*

(*b*) *has in the upper right corner the $m \times (n - m)$ zero matrix.*

Consider the function $\mathcal{F} : Cl(V) \to \mathbb{R}^m$ defined by

$$\mathcal{F}(\alpha) = \xi \left(\int_0^T M^{-1}_{\mathbf{z}_\alpha}(t) F_1(t, \mathbf{x}(t, \mathbf{z}_\alpha, 0)) dt \right). \tag{4.4}$$

Suppose that there is $\alpha_0 \in V$ with $\mathcal{F}(\alpha_0) = 0$, then the following statements hold for $\varepsilon \neq 0$ sufficiently small.

(*i*) *If $\det((\partial\mathcal{F}/\partial\alpha)(\alpha_0)) \neq 0$, then there is a unique T-periodic solution $\mathbf{x}(t, \varepsilon)$ of system (4.1) such that $\mathbf{x}(t, \varepsilon) \to \mathbf{x}(t, \mathbf{z}_{\alpha_0}, 0)$ as $\varepsilon \to 0$.*

(*ii*) *If $m = 1$ and $\mathcal{F}'(\alpha_0) = \cdots = \mathcal{F}^{(s-1)}(\alpha_0) = 0$ and $\mathcal{F}^{(s)}(\alpha_0) \neq 0$ with $s \leq k$, then there are at most s T-periodic solutions $\mathbf{x}_1(t, \varepsilon), \dots, \mathbf{x}_s(t, \varepsilon)$ of system (4.1) such that $\mathbf{x}_i(t, \varepsilon) \to \mathbf{x}(t, \mathbf{z}_{\alpha_0}, 0)$ as $\varepsilon \to 0$ for $i = 1, \dots, s$.*

In any case now we shall recall the more classical result on averaging theory for studying periodic solutions. We consider the initial value problems

$$\dot{\mathbf{x}} = \varepsilon F_1(t, \mathbf{x}) + \varepsilon^2 F_2(t, \mathbf{x}, \varepsilon), \quad \mathbf{x}(0) = \mathbf{x}_0, \tag{4.5}$$

and

$$\dot{\mathbf{y}} = \varepsilon g(\mathbf{y}), \quad \mathbf{y}(0) = \mathbf{x}_0, \tag{4.6}$$

with $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{x}_0$ in some open Ω of $\mathbb{R}^n$, $t \in [0, \infty)$, $\varepsilon \in (0, \varepsilon_0]$. We assume that $\mathbf{F_1}$ and $\mathbf{F_2}$ are periodic of period T in the variable t, and we set

$$g(\mathbf{y}) = \frac{1}{T} \int_0^T F_1(t, \mathbf{y}) dt. \tag{4.7}$$

Theorem 4.2. *Assume that F_1, $D_{\mathbf{x}} F_1$,$D_{\mathbf{xx}} F_1$ and $D_{\mathbf{x}} F_2$ are continuous and bounded by a constant independent of ε in $[0, \infty) \times\ \Omega \times (0, \varepsilon_0]$, and that $y(t) \in \Omega$ for $t \in [0, 1/\varepsilon]$. Then the following statements holds:*

(*a*) *For $t \in [0, 1/\varepsilon]$ we have $\mathbf{x}(t) - \mathbf{y}(t) = O(\varepsilon)$ as $\varepsilon \to 0$.*

(b) If $p \neq 0$ *is a singular point of system* (4.6) *and*

$$\det D_{\mathbf{y}} g(p) \neq 0, \tag{4.8}$$

then there exists a periodic solution $\mathbf{x}(t, \varepsilon)$ *of period* T *for system* (4.5) *such that* $\mathbf{x}(0, \varepsilon) - p = O(\varepsilon)$ *as* $\varepsilon \to 0$*.*

(c) The stability of the periodic solution $\mathbf{x}(t, \varepsilon)$ *is given by the stability of the singular point.*

We have used the notation $D_{\mathbf{x}} g$ for all the first derivatives of g, and $D_{\mathbf{xx}} g$ for all the second derivatives of g.

For a proof of Theorem 4.2 see [19]. For more information on the averaging theory see the book [15].

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