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Periodic Solutions of a Class of Duffing Differential Equations*

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Abstract In this work we study the existence of new periodic solutions for the well knwon class of Duffing differential equation of the form $x'' + cx' + a(t)x + b(t)x^3 = h(t)$, where c is a real parameter, $a(t)$, $b(t)$ and $h(t)$ are continuous T–periodic functions. Our results are proved using three different results on the averaging theory of first order.

Keywords Periodic solution, averaging method, Duffing differential equation, bifurcation, stability.

MSC(2010) 34C15, 34C25.

1. Introduction and statement of the main result

Several classes of Duffing differential equations have been investigated by many authors. They are mainly interested in the existence of periodic solutions, in their multiplicity, stability and bifurcation. See the survey of J. Mawhin [12] and for example the articles [2–4, 6, 9, 10, 13, 16, 18, 19].

In this work we shall study the class of Duffing differential equations of the form

$$x'' + cx' + a(t)x + b(t)x^3 = h(t), \tag{1.1}$$

where $c > 0$ is a constant, and $a(t)$, $b(t)$ and $h(t)$ are continuous T–periodic functions. These differential equations were studied by Chen and Li in the papers [2, 3]. These authors studied the periodic solutions of equation (1.1) with the following additional conditions, either $b(t) > 0$, $h(t) > 0$ and $a(t)$ satisfies

$$a(t) \leq \frac{\pi^2}{T^2} + \frac{c^2}{4}, \quad \text{and} \quad a_0 = \frac{1}{T}\int_0^T a(t)dt > 0; \tag{1.2}$$

or $a(t) = a > 0$, $b(t) = 1$ and $c > 0$, a, c constants.

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In [8] the authors studied the existence and the stability of periodic solutions of the Duffing differential equation (1.1) with $c = \varepsilon C > 0$, $a(t) = \varepsilon^2 A(t)$, $A_0 b_0 > 0$ where $A_0 = \dfrac{1}{T}\displaystyle\int_0^T A(t)dt$ and $b_0 = \dfrac{1}{T}\displaystyle\int_0^T b(t)dt$, and ε is sufficiently small.

Instead of working with the Duffing differential equation (1.1) we shall work with the equivalent differential system

$$\begin{aligned} x' &= y, \\ y' &= -cy - a(t)x - b(t)x^3 + h(t). \end{aligned} \tag{1.3}$$

We define the polynomial

$$\begin{aligned} p(x_0) = &-\left(\int_0^T e^{-ct}\int_0^t e^{cs}b(s)\,ds\,dt - \frac{e^{-cT}}{c}\int_0^T e^{cs}b(s)\,ds\right)x_0^3 \\ &-\left(\int_0^T e^{-ct}\int_0^t e^{cs}a(s)\,ds\,dt - \frac{e^{-cT}}{c}\int_0^T e^{cs}a(s)\,ds\right)x_0 \\ &+\frac{e^{-cT}}{c}\int_0^T e^{cs}h(s)\,ds + \int_0^T e^{-ct}\int_0^t e^{cs}h(s)\,ds\,dt. \end{aligned}$$

Our first result on the periodic solutions of the differential system (1.3) is the following.

Theorem 1.1. *For every simple real root of the polynomial $p(x_0)$ the differential system* (1.3) *has a periodic solution $(x(t), y(t))$ such that $(x(0), y(0)) = (x_0, 0)$.*

Theorem 1.1 will be proved in section 3 using Theorem 4.1 of the averaging theory.

Now we define the polynomial

$$q(x_0) = -\left(\int_0^T b(s)\,ds\right)x_0^3 - \left(\int_0^T a(s)\,ds\right)x_0 + \int_0^T h(s)\,ds.$$

Theorem 1.2. *For every simple real root of the polynomial $q(x_0)$ the differential system* (1.3) *has a periodic solution $(x(t), y(t))$ such that $(x(0), y(0)) = (0, 0)$.*

Theorem 1.2 will be proved in section 4 using Theorem 4.2 of the averaging theory.

As we shall see Theorem 4.3 of the averaging theory will provide results on the periodic solutions of system (1.3) which are already contained in Theorems 1.1 and 1.2.

In order to apply the three theorems of the averaging theory of first order for studying the periodic solutions of the differential system (1.3) in section 2 we rescale the variables, the parameters and the functions of system (1.3).

The results of averaging theory that we use in this paper are described in section 4.

2. Preliminary results

We start doing a rescaling of the variables (x, y), of the functions $a(t)$, $b(t)$ and $h(t)$ and of the parameter c as follows:

$$\begin{aligned} &x = \varepsilon^{m_1} X, \qquad y = \varepsilon^{m_2} Y, \\ &c = \varepsilon^{m_3} C, \qquad a(t) = \varepsilon^{n_1} A(t), \\ &b(t) = \varepsilon^{n_2} B(t),\ h(t) = \varepsilon^{n_3} H(t). \end{aligned} \tag{2.1}$$

In such a way that the differential equation (1.3) becomes

$$\begin{aligned} &X' = \varepsilon^{m_2-m_1} Y, \\ &Y' = -\varepsilon^{m_3} CY - \varepsilon^{n_1+m_1-m_2} A(t)X - \varepsilon^{n_2+3m_1-m_2} B(t)X^3 + \varepsilon^{n_3-m_2} H(t), \\ &\text{where} \quad 0 \le m_3, \quad 0 \le m_1 \le m_2 \le n_3, \quad m_2 \le n_1 + m_1, \quad m_2 \le n_2 + 3m_1, \\ &\text{and} \quad \{m_2 - m_1, m_3, n_1 + m_1 - m_2, n_2 + 3m_1 - m_2, n_3 - m_2\} \cap \{1\} \neq \emptyset. \end{aligned} \tag{2.2}$$

We distinguish the following seven cases with their corresponding subcases, recall that we want to apply the averaging theory of first order for studying the periodic solutions of the differential system (1.2), see a summary on this theory at the appendix.
Case I: $m_2 - m_1 = 0$ and $m_3 = 0$. Then we have the following subcases

$$\begin{aligned} &(I.1) \quad n_1 + m_1 - m_2 = 0, \quad n_2 + 3m_1 - m_2 = 0, \quad n_3 - m_2 = 1; \\ &(I.2) \quad n_1 + m_1 - m_2 = 0, \quad n_2 + 3m_1 - m_2 = 1, \quad n_3 - m_2 = 0; \\ &(I.3) \quad n_1 + m_1 - m_2 = 1, \quad n_2 + 3m_1 - m_2 = 0, \quad n_3 - m_2 = 0; \\ &(I.4) \quad n_1 + m_1 - m_2 = 0, \quad n_2 + 3m_1 - m_2 = 1, \quad n_3 - m_2 = 1; \\ &(I.5) \quad n_1 + m_1 - m_2 = 1, \quad n_2 + 3m_1 - m_2 = 0, \quad n_3 - m_2 = 1; \\ &(I.6) \quad n_1 + m_1 - m_2 = 1, \quad n_2 + 3m_1 - m_2 = 1, \quad n_3 - m_2 = 0; \\ &(I.7) \quad n_1 + m_1 - m_2 = 1, \quad n_2 + 3m_1 - m_2 = 1, \quad n_3 - m_2 = 1. \end{aligned}$$

Case II: $m_2 - m_1 = 0$ and $m_3 \ge 1$.
Case III: $m_2 - m_1 = 1$ and $m_3 = 0$.
Case IV: $m_2 - m_1 = 1$ and $m_3 = 1$.
Case V: $m_2 - m_1 > 1$ and $m_3 = 1$.
Case VI: $m_2 - m_1 = 1$ and $m_3 > 1$.
Case VII: $m_2 - m_1 > 1$ and $m_3 > 1$.

Every case α from II to VII can be split into the following eight subcases:

$$\begin{aligned} &(\alpha.1)\ n_1 + m_1 - m_2 = 0, \quad n_2 + 3m_1 - m_2 = 0, \quad n_3 - m_2 = 0, \\ &(\alpha.2)\ n_1 + m_1 - m_2 = 0, \quad n_2 + 3m_1 - m_2 = 0, \quad n_3 - m_2 = 1, \\ &(\alpha.3)\ n_1 + m_1 - m_2 = 0, \quad n_2 + 3m_1 - m_2 = 1, \quad n_3 - m_2 = 0, \\ &(\alpha.4)\ n_1 + m_1 - m_2 = 1, \quad n_2 + 3m_1 - m_2 = 0, \quad n_3 - m_2 = 0, \\ &(\alpha.5)\ n_1 + m_1 - m_2 = 0, \quad n_2 + 3m_1 - m_2 = 1, \quad n_3 - m_2 = 1, \end{aligned}$$

$$(\alpha.6)\ n_1+m_1-m_2=1,\quad n_2+3m_1-m_2=0,\quad n_3-m_2=1,$$
$$(\alpha.7)\ n_1+m_1-m_2=1,\quad n_2+3m_1-m_2=1,\quad n_3-m_2=0,$$
$$(\alpha.8)\ n_1+m_1-m_2=1,\quad n_2+3m_1-m_2=1,\quad n_3-m_2=1.$$

We have applied the three theorems of averaging (see section 4) for studying the existence of periodic solutions of the 55 previous subcases of differential systems (2.2). Theorem 1.1 comes from the subcase (III.1), and Theorem 1.2 follows from the subcase (IV.1).

All the subcases, different from (III.1) or (IV.1), either do not satisfy the hypotheses of one of the three theorems of averaging, or provide partial results of the ones stated in Theorems 1.1 and 1.2. So in what follows we shall consider only the subcases (III.1) or (IV.1).

Theorem 4.3 has been applied for studying the subcases $(\alpha, 8)$ for $\alpha = IV, \dots, VII$, and either do not provide periodic solutions, or provide particular cases of the results given in Theorems 1.1 and 1.2.

In short, in what follows we only provide the details of the positive results, i.e. we shall give the proofs of Theorems 1.1 and 1.2.

3. Proof of Theorem 1.1

For the case (III.1), i.e. for

$$m_2 = m_1+1,\quad m_3=0,\quad m_2=n_1=n_2=n_3=1 \text{ and } m_1=m_3=0; \tag{3.1}$$

system 2.2 becomes

$$\begin{aligned}\dot{X} &= \varepsilon Y,\\ \dot{Y} &= -CY - A(t)X - B(t)X^3 + H(t).\end{aligned} \tag{3.2}$$

We shall apply the averaging Theorem 4.1 to system (3.2) and we shall obtain Theorem 1.1. In what follows we shall use the notation of Theorem 4.1, see the appendix. Thus $\mathbf{x} = (X, Y)^T$ and

$$F_0(t,\mathbf{x}) = \begin{pmatrix} 0 \\ -CY - A(t)X - B(t)X^3 + H(t) \end{pmatrix},$$
$$F_1(t,\mathbf{x}) = \begin{pmatrix} Y \\ 0 \end{pmatrix},$$
$$F_2(t,\mathbf{x}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The unperturbed differential system (4.5) has the solution $\mathbf{x}(t,\mathbf{z},0) = (X(t), Y(t))^T$ such that $\mathbf{x}(0,\mathbf{z},0) = \mathbf{z} = (X_0, Y_0)^T$, where

$$X(t) = X_0,$$
$$Y(t) = e^{-Ct}\left(Y_0 + \int_0^t e^{Cs}\left(-B(s)X_0^3 - A(s)X_0 + H(s)\right) ds\right).$$

In order that $\mathbf{x}(t,\mathbf{z},0)$ be a periodic solution we must choose

$$Y_0 = \frac{1}{e^{CT}-1}\int_0^T e^{Cs}\left(-B(s)X_0^3 - A(s)X_0 + H(s)\right) ds.$$

So we get

$$\mathbf{z}_\alpha = (\alpha, \beta_0(\alpha)) = \left(X_0, \frac{1}{e^{CT}-1}\int_0^T e^{Cs}\left(-B(s)X_0^3 - A(s)X_0 + H(s)\right) ds\right).$$

Therefore, following the notation of Theorem 4.1, we have $n=2$ and $k=1$.

Now we compute the fundamental matrix $M_{\mathbf{z}_\alpha}(t)$ associated to the first variational system (4.6) such that $M_{\mathbf{z}_\alpha}(0) = \mathrm{Id}$ of $\mathbb{R}^2$, and we obtain

$$M_{\mathbf{z}_\alpha}(t) = \begin{pmatrix} 1 & 0 \\ -e^{-Ct}\left(\int_0^t e^{Cs}\left(3B(s)X_0 + A(s)\right) ds\right) & e^{-Ct} \end{pmatrix}.$$

Its inverse matrix is

$$M_{\mathbf{z}_\alpha}^{-1}(t) = \begin{pmatrix} 1 & 0 \\ \int_0^t e^{Cs}\left(3B(s)X_0^2 + A(s)\right) ds & e^{Ct} \end{pmatrix}.$$

Since the matrix

$$M_{\mathbf{z}_\alpha}^{-1}(0) - M_{\mathbf{z}_\alpha}^{-1}(T) = \begin{pmatrix} 0 & 0 \\ -\int_0^t e^{Cs}\left(3B(s)X_0^2 + A(s)\right) ds & 1-e^{CT} \end{pmatrix}$$

has a zero 1×1 matrix in the upper right corner and a non–zero 1×1 matrix in its lower right corner equal to $1-e^{CT}$, because $T \neq 0$. We can apply the averaging theory described in Theorem 4.1 for studying the periodic solutions which can be prolonged from the unperturbed differential system to the perturbed one. Therefore, since for our differential system we have $\xi(X,Y) = X$, then we must compute the function $\mathcal{F}(\alpha) = \mathcal{F}(X_0)$ given in (4.7), i.e.

$$\begin{aligned}
\mathcal{F}(X_0) &= \xi\left(\int_0^T M_{\mathbf{z}_\alpha}^{-1}(t) F_1(t, \mathbf{x}(t,\mathbf{z}_\alpha,0)) dt\right) \\
&= \int_0^T \Bigg[e^{-Ct}\Bigg(\frac{1}{-1+e^{CT}}\int_0^T e^{Cs}\left(-B(s)X_0^3 - X_0A(s) + H(s)\right) ds \\
&\qquad + \int_0^t e^{Cs}\left(-B(s)X_0^3 - X_0A(s) + H(s)\right) ds\Bigg)\Bigg] dt \\
&= \int_0^T Y(t)\, dt.
\end{aligned}$$

Theorem 4.1 says that for every simple real root X_0 of the polynomial $\mathcal{F}(X_0)$ the differential system (3.2) with $\varepsilon \neq 0$ sufficiently small has a periodic solution $(X(t), Y(t))$ such that $(X(0), Y(0))$ tends to $(X_0, \beta_0(X_0))$ when $\varepsilon \to 0$.

Now it is to check that the function $\mathcal{F}(X_0)$ after the change of variables (2.1) satisfying (3.1), i.e.

$$X = x, \quad Y = \frac{y}{\varepsilon}, \quad H(t) = \frac{h(t)}{\varepsilon}, \quad B(s) = \frac{b(s)}{\varepsilon}, \quad A(s) = \frac{a(s)}{\varepsilon},$$

becomes the polynomial $p(x_0)$ defined in section 1 just before the statement of Theorem 1.1. Hence Theorem 1.1 is proved.

4. Proof of Theorem 1.2

For the case (IV.1), i.e.

$$m_2 = m_1 + 1, \quad m_3 = 1, \quad m_1 = 1, \quad m_2 = 2, \quad n_1 = -n_2 = 1 \text{ and } n_3 = 2; \tag{4.1}$$

system (2.2) becomes

$$\begin{aligned} \dot{X} &= \varepsilon Y, \\ \dot{Y} &= -\varepsilon C Y - A(t)X - B(t)X^3 + H(t). \end{aligned} \tag{4.2}$$

We shall apply the averaging Theorem 4.2 to system (4.2) and we shall obtain Theorem 1.2. In what follows we shall use the notation of Theorem 4.2. Thus $\mathbf{x} = (X, Y)^T$ and

$$\begin{aligned} F_0(t, \mathbf{x}) &= \begin{pmatrix} 0 \\ -A(t)X - B(t)X^3 + H(t) \end{pmatrix}, \\ F_1(t, \mathbf{x}) &= \begin{pmatrix} Y \\ -CY \end{pmatrix}, \\ F_2(t, \mathbf{x}) &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned}$$

The unperturbed differential system (4.5) has the solution $\mathbf{x}(t, \mathbf{z}, 0) = (X(t), Y(t))^T$ such that $\mathbf{x}(0, \mathbf{z}, 0) = \mathbf{z} = (X_0, Y_0)^T$, where

$$\begin{aligned} X(t) &= X_0, \\ Y(t) &= Y_0 + \int_0^t \left(-B(s)X_0^3 - A(s)X_0 + H(s)\right) ds. \end{aligned}$$

In order that $\mathbf{x}(t, \mathbf{z}, 0)$ be a periodic solution X_0 must satisfy

$$\int_0^T \left(-B(s)X_0^3 - A(s)X_0 + H(s)\right) ds = 0, \tag{4.3}$$

and Y_0 is arbitrary. Therefore we get

$$\mathbf{z}_\alpha = (\alpha, \beta_0(\alpha)) = \left(Y_0, \bar{X}_0\right),$$

where $\bar{X}_0$ is a real root of the cubic polynomial (4.3). In short the unperturbed system (i.e. system (4.3) with $\varepsilon = 0$) has at most three families of periodic solutions because Y_0 is arbitrary and $\bar{X}_0$ is a real root of the cubic polynomial (4.3). Therefore, using the notation of Theorem 4.2, we have $n = 2$ and $k = 1$ for each one of these possible families of periodic solutions.

We compute the fundamental matrix $M_{\mathbf{z}_\alpha}(t)$ associated to the first variational system (4.6) associated to the vector field $(\dot{Y}, \dot{X})$ given by (4.2) with $\varepsilon = 0$, and such that $M_{\mathbf{z}_\alpha}(0) = \text{Id of } \mathbb{R}^2$, and we obtain

$$M_{\mathbf{z}_\alpha}(t) = \begin{pmatrix} 1 - \displaystyle\int_0^t \big(3B(s)X_0^2 + A(s)\big)\, ds \\ 0 \qquad\qquad 1 \end{pmatrix}.$$

Its inverse matrix is

$$M_{\mathbf{z}_\alpha}^{-1}(t) = \begin{pmatrix} 1 \displaystyle\int_0^t \big(3B(s)X_0^2 + A(s)\big)\, ds \\ 0 \qquad\qquad 1 \end{pmatrix}.$$

The matrix

$$M_{\mathbf{z}_\alpha}^{-1}(0) - M_{\mathbf{z}_\alpha}^{-1}(T) = \begin{pmatrix} 0 - \displaystyle\int_0^T \big(3B(s)X_0^2 + A(s)\big)\, ds \\ 0 \qquad\qquad 0 \end{pmatrix}$$

has a non–zero 1×1 matrix in the upper right corner if the real root $\bar{X}_0$ of the cubic polynomial (4.3) is simple, and a zero 1×1 matrix in its lower right corner. Therefore the assumptions of Theorem 4.2 hold, then by applying this theorem we study the periodic solutions which can be prolonged from the unperturbed differential system to the perturbed one. Since for our differential system we have $\xi^\perp(Y, X) = X$, then we must compute the function $\mathcal{G}(\alpha) = \mathcal{G}(Y_0)$ given in (4.7), i.e.

$$\mathcal{G}(Y_0) = \xi^\perp \left(\int_0^T M_{\mathbf{z}_\alpha}^{-1}(t) F_1(t, \mathbf{x}(t, \mathbf{z}_\alpha, 0)) dt \right) = -\int_0^T CY_0 = -CTY_0.$$

Theorem 4.2 says that for every simple real root $Y_0 = 0$ of the polynomial $\mathcal{G}(Y_0)$ the differential system (4.2) with $\varepsilon \neq 0$ sufficiently small has a periodic solution $(X(t), Y(t))$ such that $(X(0), Y(0))$ tends to $(\bar{X}_0, 0)$ when $\varepsilon \to 0$, being $\bar{X}_0$ a simple real root of the cubic polynomial (4.3).

Now it is easy to check that the cubic polynomial (4.3) after the change of variables (2.1) satisfying (4.1), i.e.

$$X = \frac{x}{\varepsilon}, \quad Y = \frac{y}{\varepsilon^2}, \quad H(t) = \frac{h(t)}{\varepsilon^2}, \quad B(s) = \frac{b(s)}{\varepsilon}, \quad A(s) = \frac{a(s)}{\varepsilon},$$

becomes the polynomial $q(x_0)$ defined in section 1 just before the statement of Theorem 1.2. Hence Theorem 1.2 is proved.

The appendix: Periodic solutions via the averaging theory

In this section we present the basic results on the averaging theory of first order that we need for proving our results.

We consider the problem of bifurcation of T–periodic solutions from the differential systems of the form

$$\dot{\mathbf{x}} = F_0(t,\mathbf{x}) + \varepsilon F_1(t,\mathbf{x}) + \varepsilon^2 F_2(t,\mathbf{x},\varepsilon), \tag{4.4}$$

with $\varepsilon = 0$ to $\varepsilon \neq 0$ sufficiently small. The functions $F_0, F_1 : \mathbb{R}\times\Omega \to \mathbb{R}^n$ and $F_2 : \mathbb{R}\times\Omega\times(-\varepsilon_0,\varepsilon_0) \to \mathbb{R}^n$ are $\mathcal{C}^2$, T–periodic in the first variable and Ω is an open subset of $\mathbb{R}^n$. The main assumption is that the unperturbed system

$$\dot{\mathbf{x}} = F_0(t,\mathbf{x}), \tag{4.5}$$

has a submanifold of periodic solutions. A solution of this problem is given using the averaging theory.

Let $\mathbf{x}(t,\mathbf{z},\varepsilon)$ be the solution of system (4.5) such that $\mathbf{x}(0,\mathbf{z},\varepsilon) = \mathbf{z}$. We write the linearization of the unperturbed system along a periodic solution $\mathbf{x}(t,\mathbf{z},0)$ as

$$\dot{\mathbf{y}} = D_{\mathbf{x}} F_0(t,\mathbf{x}(t,\mathbf{z},0))\mathbf{y}. \tag{4.6}$$

In what follows we denote by $M_{\mathbf{z}}(t)$ a fundamental matrix of the linear differential system (4.6), by $\xi : \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^m$ and $\xi^\perp : \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^{n-m}$ the projections of $\mathbb{R}^n$ onto its first m and $n-m$ coordinates respectively; i.e. $\xi(x_1,\ldots,x_n) = (x_1,\ldots,x_m)$, and $\xi^\perp(x_1,\ldots,x_n) = (x_{m+1},\ldots,x_n)$.

Theorem 4.1. *Let $V \subset \mathbb{R}^m$ be open and bounded, let $\beta_0 : \mathrm{Cl}(V) \to \mathbb{R}^{n-m}$ be a $\mathcal{C}^k$ function and $\mathcal{Z} = \{\mathbf{z}_\alpha = (\alpha,\beta_0(\alpha)) \mid \alpha \in \mathrm{Cl}(V)\} \subset \Omega$ its graphic in $\mathbb{R}^n$. Assume that for each $\mathbf{z}_\alpha \in \mathcal{Z}$ the solution $\mathbf{x}(t,\mathbf{z}_\alpha,0)$ of (4.5) is T-periodic and that there exists a fundamental matrix $M_{\mathbf{z}_\alpha}(t)$ of (4.6) such that the matrix $M^{-1}_{\mathbf{z}_\alpha}(0) - M^{-1}_{\mathbf{z}_\alpha}(T)$*

- (a) *has in the lower right corner the $(n-m)\times(n-m)$ matrix Δ_α with $\det(\Delta_\alpha) \neq 0$, and*
- (b) *has in the upper right corner the $m \times (n-m)$ zero matrix.*

Consider the function $\mathcal{F} : \mathrm{Cl}(V) \to \mathbb{R}^m$ defined by

$$\mathcal{F}(\alpha) = \xi\left(\int_0^T M^{-1}_{\mathbf{z}_\alpha}(t) F_1(t,\mathbf{x}(t,\mathbf{z}_\alpha,0))dt\right). \tag{4.7}$$

Suppose that there is $\alpha_0 \in V$ with $\mathcal{F}(\alpha_0) = 0$, then the following statements hold for $\varepsilon \neq 0$ sufficiently small.

- (i) *If $\det((\partial\mathcal{F}/\partial\alpha)(\alpha_0)) \neq 0$, then there is a unique T-periodic solution $\varphi_1(t,\varepsilon)$ of system (4.4) such that $\varphi_1(t,\varepsilon) \to \mathbf{x}(t,\mathbf{z}_{\alpha_0},0)$ as $\varepsilon \to 0$.*
- (ii) *If $m = 1$ and $\mathcal{F}'(\alpha_0) = \cdots = \mathcal{F}^{(s-1)}(\alpha_0) = 0$ and $\mathcal{F}^{(s)}(\alpha_0) \neq 0$ with $s \leq k$, then there are at most s T-periodic solutions $\varphi_1(t,\varepsilon),\ldots,\varphi_s(t,\varepsilon)$ of system (4.4) such that $\varphi_i(t,\varepsilon) \to \mathbf{x}(t,\mathbf{z}_{\alpha_0},0)$ as $\varepsilon \to 0$ for $i = 1,\ldots,s$.*

Theorem 4.1 is a classical result due to Malkin [11] and Roseau [14]. For a shorter proof of Theorem 4.1(a), see [1].

As we shall see in this paper we have cases where Theorem 4.1 cannot be applied for studying the existence of periodic solutions, because its assumptions are not satisfied. Then in [7] the following result on averaging has been proved.

Theorem 4.2. *Let $V \subset \mathbb{R}^m$ be open and bounded, let $\beta_0 : \mathrm{Cl}(V) \to \mathbb{R}^m$ be a $\mathcal{C}^k$ function and $\mathcal{Z} = \{\mathbf{z}_\alpha = (\alpha, \beta_0(\alpha)) \,|\, \alpha \in \mathrm{Cl}(V)\} \subset \Omega$ its graphic in $\mathbb{R}^{2m}$. Assume that for each $\mathbf{z}_\alpha \in \mathcal{Z}$ the solution $\mathbf{x}(t, \mathbf{z}_\alpha, 0)$ of (4.5) is T-periodic and that there exists a fundamental matrix $M_{\mathbf{z}_\alpha}(t)$ of (4.6) such that the matrix $M^{-1}_{\mathbf{z}_\alpha}(0) - M^{-1}_{\mathbf{z}_\alpha}(T)$*

(a) has in the upper right corner the $m \times m$ matrix Δ_α with $\det(\Delta_\alpha) \neq 0$, and

(b) has in the lower right corner the $m \times m$ zero matrix.

Consider the function $\mathcal{G} : \mathrm{Cl}(V) \to \mathbb{R}^m$ defined by

$$\mathcal{G}(\alpha) = \xi^\perp \left(\int_0^T M^{-1}_{\mathbf{z}_\alpha}(t) F_1(t, \mathbf{x}(t, \mathbf{z}_\alpha, 0)) dt \right). \tag{4.8}$$

Suppose that there is $\alpha_0 \in V$ with $\mathcal{G}(\alpha_0) = 0$, then the following statements hold for $\varepsilon \neq 0$ sufficiently small.

(i) If $\det((\partial \mathcal{G}/\partial \alpha)(\alpha_0)) \neq 0$, then there is a unique T-periodic solution $\varphi_1(t, \varepsilon)$ of system (4.4) such that $\varphi_1(t, \varepsilon) \to \mathbf{x}(t, \mathbf{z}_{\alpha_0}, 0)$ as $\varepsilon \to 0$.

(ii) If $m = 1$ and $\mathcal{G}'(\alpha_0) = \cdots = \mathcal{G}^{(s-1)}(\alpha_0) = 0$ and $\mathcal{G}^{(s)}(\alpha_0) \neq 0$ with $s \leq k$, then there are at most s T-periodic solutions $\varphi_1(t, \varepsilon), \ldots, \varphi_s(t, \varepsilon)$ of system (4.4) such that $\varphi_i(t, \varepsilon) \to \mathbf{x}(t, \mathbf{z}_{\alpha_0}, 0)$ as $\varepsilon \to 0$ for $i = 1, \ldots, s$.

In any case now we shall recall the more classical result on averaging theory for studying periodic solutions. We consider the initial value problems

$$\dot{\mathbf{x}} = \varepsilon F_1(t, \mathbf{x}) + \varepsilon^2 F_2(t, \mathbf{x}, \varepsilon), \quad \mathbf{x}(0) = \mathbf{x}_0, \tag{4.9}$$

and

$$\dot{\mathbf{y}} = \varepsilon g(\mathbf{y}), \quad \mathbf{y}(0) = \mathbf{x}_0, \tag{4.10}$$

with $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{x}_0$ in some open Ω of $\mathbb{R}^n$, $t \in [0, \infty)$, $\varepsilon \in (0, \varepsilon_0]$. We assume that $\mathbf{F_1}$ and $\mathbf{F_2}$ are periodic of period T in the variable t, and we set

$$g(\mathbf{y}) = \frac{1}{T} \int_0^T F_1(t, \mathbf{y}) dt.$$

Theorem 4.3. *Assume that F_1, $D_\mathbf{x} F_1$,$D_{\mathbf{xx}} F_1$ and $D_\mathbf{x} F_2$ are continuous and bounded by a constant independent of ε in $[0, \infty) \times \ \Omega \times (0, \varepsilon_0]$, and that $y(t) \in \Omega$ for $t \in [0, 1/\varepsilon]$. Then the following statements holds:*

1. *For $t \in [0, 1/\varepsilon]$ we have $\mathbf{x}(t) - \mathbf{y}(t) = O(\varepsilon)$ as $\varepsilon \to 0$.*
2. *If $p \neq 0$ is a singular point of system (4.10) and $det D_\mathbf{y} g(p) \neq 0$, then there exists a periodic solution $\phi(t, \varepsilon)$ of period T for system (4.9) which is close to p and such that $\phi(0, \varepsilon) - p = O(\varepsilon)$ as $\varepsilon \to 0$.*
3. *The stability of the periodic solution $\phi(t, \varepsilon)$ is given by the stability of the singular point.*

We have used the notation $D_\mathbf{x} g$ for all the first derivatives of g, and $D_{\mathbf{xx}} g$ for all the second derivatives of g.

For a proof of Theorem 4.3 see [17]. For more information on the averaging theory see the book [15].

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