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Existence of Solutions for Fractional Integro-differential Equations with Impulsive and Integral Conditions*

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Abstract This paper presents the existence of solutions for a class of Cauchy problems with integral condition for impulsive fractional integro-differential equations. Based on definition of solution for impulsive fractional integro-differential equations, the existence theorems of solutions of fractional differential equation are obtained by applying fixed point methods. Finally, three examples are given to demonstrate the feasibility of the obtained results.

Keywords Existence of solutions, impulsive fractional differential equations, fixed point theorems.

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1. Introduction

During the past decades, impulsive differential equations have been attracting increasing attention due to their applications in various sciences such as Physics, Chemistry, Mechanics, Engineering, Biomedical sciences, etc. Moreover, fractional differential equations have been proved to be valuable tools to model of phenomena in both physical and social sciences.

Fractional impulsive differential equations have been extensively studied by many researchers, in which, fractional calculus, an important branch of mathematics, has been attached great importance to. For details, see [1–10] and references therein. For example, Anguraj et al. [2] considered the following initial value problems for impulsive fractional differential equation given by

$$\begin{cases} {}^cD^{\alpha}(y)(t) = f(t, y(t), \int_0^t k(t, s, y(s))ds), \ t \in J^{'} := J\backslash\{t_1, ..., t_m\}, \\ y(t_k^+) = y(t_k^-) + y_k, \qquad y_k \in R, \\ y(0) = \int_0^1 g(s)y(s)ds, \end{cases} \tag{1.1}$$

where $0 < \alpha \leq 1$ and $J = [0, 1]$. They proved the existence results for the above equation by means of the contraction mapping principle and the Krasnoselskii fixed

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point theorem.

Liu et al. [7] studied the following problem

$$\begin{cases} {}^cD_t^\alpha u(t) = f(t,u(t),u'(t),u''(t)), \qquad t \in J^{'} := J\backslash\{t_1,...,t_m\}, \\ \Delta u(t_k) = A_k(u(t_k^-)), \quad \Delta u'(t_k) = B_k(u(t_k^-)), \quad \Delta u''(t_k) = C_k(u(t_k^-)), \\ u(0) = \lambda_1 u(T) + \xi_1 \displaystyle\int_0^T q_1(s,u(s),u'(s),u''(s))ds, \\ u'(0) = \lambda_2 u(T) + \xi_2 \displaystyle\int_0^T q_2(s,u(s),u'(s),u''(s))ds, \\ u''(0) = \lambda_3 u(T) + \xi_3 \displaystyle\int_0^T q_3(s,u(s),u'(s),u''(s))ds, \quad \lambda_i \neq 1 (i=1,2,3), \end{cases} \tag{1.2}$$

where $2 < \alpha \leq 3$. By the use of the well-known fixed point theorems, they obtained the uniqueness and existence of the solutions for the above equation.

Motivated by the above mentioned works, we investigated sufficient conditions for the existence of solutions to the following impulsive fractional differential equations with integral initial condition :

$$\begin{cases} {}^cD^\alpha(x)(t) = f(t,x(t),\displaystyle\int_0^t k(t,s,x(s))ds), \qquad t \in J^{'} := J\backslash\{t_1,...,t_m\}, \\ \Delta x(t_k) = A_k(x(t_k^-)), \quad \Delta x'(t_k) = B_k(x(t_k^-)) \\ x(0) = \displaystyle\int_0^T g(s)x(s)ds, \quad x'(0) = \displaystyle\int_0^T h(s)x'(s)ds, \end{cases} \tag{1.3}$$

where $k = 1,\cdots,m$, $J = [0,T]$, $1 < \alpha \leq 2$, ${}^cD^\alpha$ is the Caputo fractional derivative, X denote a Banach space, $f : J \times X \times X \to X$ is a given function, the functions $A_k, B_k : X \to X$ are continuous, $\Omega = \{(t,s) : 0 \leq s \leq t \leq T\}$, $k : \Omega \times X \to X$, $g,h \in C[0,T]$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $\Delta x'(t_k) = x'(t_k^+) - x'(t_k^-)$. For brevity, let us take $Ax(t) = \int_0^t k(t,s,x(s))ds$.

This thesis is composed of four sections. In section 2, we will introduce some definitions, lemmas and preliminary results. In section 3, we will apply some standard fixed point principles to yield existence result of problem (1.3). In section 4, three examples are given to illustrate our main results.

2. Preliminaries

In this section, we introduce definitions and preliminary results which are needed in this paper. Let X be a Banach space. Let $C(J,X)$ be the Banach space of continuous functions $x(t)$ with $x(t) \in X$ for $t \in J = [0,T]$ and $\|x\|_{C(J,X)} = \max_{t\in J}|x(t)|$. Also consider the Banach space $PC(J,X) = \{x : J \to X : x \in C((t_k,t_{k+1}],X),\ k = 0,\cdots,m$ and there exist $x(t_k^+)$ and $x(t_k^-)$, $k = 1,\cdots,m$ with $x(t_k^-) = x(t_k)\}$, with the norm $\|x\|_{PC} = \sup_{t\in J}|x(t)|$. Set $J' := [0,T]\backslash\{t_1,...,t_m\}$.

Definition 2.1 (definition 2.1, [3]). The Riemann-Liouville fractional integral of order $\alpha > 0$, of a function $f \in L_1(\mathbb{R}_+)$ is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds, \quad \text{for} \quad \alpha > 0 \quad \text{and} \quad t > 0,$$

where $\Gamma(.)$ is the Euler gamma function.

Definition 2.2 (definition 2.2, [3])**.** The Caputo fractional derivative of order $\alpha > 0$, $n-1<\alpha<n$, is defined as

$$(D_{0^+}^{\alpha}f)(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}}ds,$$

where the function $f(t)$ has absolutely continuous derivatives up to order $(n-1)$.

Lemma 2.1. *If $g,h\in C[0,T]$ satisfies $g(s)$, $h(s)\geq 0$, $\max\limits_{t\in[0,T]} g(t)=M_g$, $\max\limits_{t\in[0,T]} h(t)=M_h$ and $Q(\tau,\alpha)=\int_\tau^T (s-\tau)^{\alpha-1}g(s)ds$, $P(\tau,\alpha)=\int_\tau^T (s-\tau)^{\alpha-2}h(s)ds$, for $\tau\in[0,T]$, then*

(i) $\frac{Q(\tau,\alpha)}{\Gamma(\alpha)}\leq M_g e^T$,

(ii) $\frac{P(\tau,\alpha)}{\Gamma(\alpha-1)}\leq M_h e^T$,

(iii) $\frac{\int_0^t (t-s)^{\alpha-1}ds}{\Gamma(\alpha)}\leq e^T$.

Proof. The proof is similar to that in Lemma 3 [2]. □

Lemma 2.2. *Let $f: J\times X\times X\to X$ be continuous, $\int_0^T g(s)ds=a_1\neq 1$, $\int_0^T h(s)ds=a_2\neq 1$, $\int_0^T g(s)sds=b$, $\int_0^T g(s)(s-t_k)ds=c_k (k=1,2,\cdots,m)$. If x is a solution of the following impulsive initial value problem*

$$\begin{cases} {}^cD_t^{\alpha}x(t)=f(t,x(t),Ax(t))dt, \qquad t\in J', \qquad 1<\alpha\leq 2,\\ \Delta x(t_k)=A_k(x(t_k^-)), \quad \Delta x'(t_k)=B_k(x(t_k^-)), \qquad k=1,2,\cdots,m,\\ x(0)=\displaystyle\int_0^T g(s)x(s)ds, \quad x'(0)=\int_0^T h(s)x'(s)ds, \end{cases} \tag{2.1}$$

then $x(t)$ satisfies the following impulsive fractional integral equation

$$\begin{cases} \lambda_0+\lambda_1 t+\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1}f(s,x(s),Ax(s))ds, \qquad t\in[0,t_1],\\ \lambda_0+\lambda_1 t+\displaystyle\sum_{k=1}^{j-1}A_k(x(t_k))+\sum_{k=1}^{j-1}B_k(x(t_k))(t-t_k)\\ +\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1}f(s,x(s),Ax(s))ds, \qquad t\in(t_{j-1},t_j], \end{cases} \tag{2.2}$$

where

$$\begin{aligned} \lambda_0=&\frac{1}{1-a_1}\Big(a_1\sum_{k=1}^m A_k(x(t_k))+\sum_{k=1}^m B_k(x(t_k))\frac{c_k(1-a_2)+a_2 b}{1-a_2}\\ &+\frac{b}{(1-a_2)\Gamma(\alpha-1)}\int_0^T P(\tau,\alpha)f(\tau,x(\tau),Ax(\tau))d\tau\\ &+\frac{1}{\Gamma(\alpha)}\int_0^T Q(\tau,\alpha)f(\tau,x(\tau),Ax(\tau))d\tau\Big), \end{aligned}$$

$$\lambda_1 = \frac{1}{1-a_2}\Big(a_2\sum_{k=1}^{m} B_k(x(t_k)) + \frac{1}{\Gamma(\alpha-1)}\int_0^T P(\tau,\alpha)f(\tau,x(\tau),Ax(\tau))d\tau\Big),$$

$$P(\tau,\alpha) = \int_\tau^T (s-\tau)^{\alpha-2}h(s)ds, \quad Q(\tau,\alpha) = \int_\tau^T (s-\tau)^{\alpha-1}g(s)ds.$$

Proof. The proof is similar to that in Lemma 2.6 [7]. □

Theorem 2.1 (theorem 1, [2])**.** *Let M be a closed convex and nonempty subset of a Banach space X. Let A and B be two operators such that*

(i) $Ax + By \in M$ whenever $x, y \in M$,

(ii) A is compact and continuous,

(iii) B is a contraction mapping.

Then there exists $z \in M$ such that $z = Az + Bz$.

Theorem 2.2 (theorem 4.5, [10])**.** *Let X be a Banach spaces and $F : X \to X$ be a completely continuous operator. If the set*

$$E(F) = \{y \in X : y = \lambda Fy \quad for\ some \quad \lambda \in [0,1]\}$$

is bounded, then F has at least a fixed point.

Theorem 2.3 (theorem 4.7, [10])**.** *Let C be a nonempty convex subset of X. Let U be a nonempty open subset of C with $0 \in U$ and $F : \overline{U} \to C$ be a compact and continuous operators. Then either*

(i) F has fixed points, or

(ii) there exist $y \in \partial U$ and $\lambda^* \in [0,1]$ with $y = \lambda^* F(y)$.

3. Main results

Before stating and proving the main results, we introduce the following hypotheses:

(1) $f : J \times X \times X \to X$ is jointly continuous.

(2) There exists $L_f > 0$ such that

$$|f(t,x_1,y_1) - f(t,x_2,y_2)| \le L_f\big(|x_1-x_2| + |y_1-y_2|\big),$$

for all $x_1, y_1, x_2, y_2 \in X$ and $t \in J$.

(3) The function $k : \Omega \times X \to X$ is continuous and there exist positive constants k_1, k_2 such that

$$|k(t,s,x_1) - k(t,s,x_2)| \le k_1|x_1 - x_2|,$$

for all $x_1, x_2 \in X$, $k_2 = \sup_{(t,s)\in\Omega} |k(t,s,0)|$.

(4) The functions $A_k, B_k : X \to X$ are continuous and there exist positive constants L_A, L_B such that

$$|A_k(x) - A_k(y)| \le L_A|x-y|, \quad |B_k(x) - B_k(y)| \le L_B|x-y|,$$

for each $t \in J$ and any $x, y \in X$.

(5) There exist positive constants M_A, M_B such that

$$|A_k(x)| \leq M_A, \quad |B_k(x)| \leq M_B,$$

for each $t \in J$ and any $x \in X$.

For the sake of convenience, we denote that

$$\begin{aligned}
M_f &= \sup_{t\in J} |f(t,0,0)|, \quad \rho = \frac{T^\alpha}{\Gamma(\alpha)}(\frac{1-\alpha_1}{\alpha-\alpha_1})^{1-\alpha_1} \quad (\alpha_1 \in (0,1)),\\
\theta_1 &= (|\frac{a_1}{1-a_1}|+1)m, \quad \theta_2 = \sum_{k=1}^{m}|\frac{c_k(1-a_2)+a_2 b}{(1-a_1)(1-a_2)}| + |\frac{a_2}{1-a_2}|mT + mT,\\
\theta_3 &= \big(|\frac{b}{(1-a_1)(1-a_2)}| + \frac{T}{|1-a_2|}\big)M_h e^T + \frac{1}{|1-a_1|}M_g e^T,\\
\eta_1 &= L_f(1+k_1T), \quad \eta_2 = L_f k_2 T + M_f,\\
\delta_1 &= \theta_3\eta_2 T + \frac{T^\alpha \eta_2}{\Gamma(\alpha+1)},\\
\delta_2 &= \theta_3\eta_1 T + L_f\Big(\frac{k_1 T^{\alpha+1}}{\Gamma(\alpha+1)} + \rho\Big),\\
\delta_3 &= \theta_1 M_A + \theta_2 M_B + \delta_1,\\
\omega &= \theta_1 L_A + \theta_2 L_B + \theta_3\eta_1 T + L_f\rho + \frac{L_f k_1 T^{\alpha+1}}{\Gamma(\alpha+1)}.\\
\Lambda_1 &= \theta_1 M_A + \theta_2 M_B + \theta_3 LT + \frac{1}{2}Lk_2\theta_3 T^2 + \frac{LT^\alpha}{\Gamma(\alpha+1)} + \frac{LT^{\alpha+1}k_2}{\Gamma(\alpha+2)}.\\
\Lambda_2 &= \theta_3 LT(1+\frac{1}{2}k_1 T) + L\rho + \frac{LT^{\alpha+1}k_1}{\Gamma(\alpha+1)}.
\end{aligned}$$

Theorem 3.1. *Let the assumptions* (1)−(5) *be satisfied. If* $\omega < 1$, *then the problem* (1.3) *has a unique solution on* J.

Proof. Define the operator $F : PC(J,X) \to PC(J,X)$ by

$$(Fx)(t) = \begin{cases} \lambda_0 + \lambda_1 t + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1} f(s,x(s),Ax(s))ds, & t \in [0,t_1],\\ \lambda_0 + \lambda_1 t + \displaystyle\sum_{k=1}^{j-1} A_k(x(t_k)) + \sum_{k=1}^{j-1} B_k(x(t_k))(t-t_k) & \\ + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1} f(s,x(s),Ax(s))ds, & t \in [t_{j-1},t_j], \end{cases} \tag{3.1}$$

where λ_0, λ_1 are given by Lemma 2.2.

Let us fix $r \geq \frac{\delta_3}{1-\delta_2}$. We shall show that $FB_r \subset B_r$, where

$$B_r = \{x \in PC(J,X) : \|x\|_{PC} \leq r,\ 0 \leq t \leq T\}.$$

For any $x \in B_r$, using (1) − (5) and Hölder's inequality, for each $t \in J$, we have

$$(Fx)(t) \leq |\lambda_0| + |\lambda_1 t| + \sum_{k=1}^{j-1}|A_k(x(t_k))| + \sum_{k=1}^{j-1}|B_k(x(t_k))(t-t_k)|$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,x(s),Ax(s))|ds\\
\le&|\frac{a_1}{1-a_1}|\sum_{k=1}^{m}|A_k(x(t_k))|+\sum_{k=1}^{m}|B_k(x(t_k))\frac{c_k(1-a_2)+a_2b}{(1-a_1)(1-a_2)}|\\
&+|\frac{a_2t}{1-a_2}|\sum_{k=1}^{m}|B_k(x(t_k))|\\
&+|\frac{b}{(1-a_1)(1-a_2)}|\int_0^T\frac{P(\tau,\alpha)}{\Gamma(\alpha-1)}|f(\tau,x(\tau),Ax(\tau))|d\tau\\
&+|\frac{1}{1-a_1}|\int_0^T\frac{Q(\tau,\alpha)}{\Gamma(\alpha)}|f(\tau,x(\tau),Ax(\tau))|d\tau\\
&+|\frac{t}{1-a_2}|\int_0^T\frac{P(\tau,\alpha)}{\Gamma(\alpha-1)}|f(\tau,x(\tau),Ax(\tau))|d\tau\\
&+\sum_{k=1}^{j-1}|A_k(x(t_k))|+\sum_{k=1}^{j-1}|B_k(x(t_k))(t-t_k)|\\
&+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,x(s),Ax(s))|ds\\
\le&\theta_1M_A+\theta_2M_B+\theta_3\int_0^T|f(\tau,x(\tau),Ax(\tau))-f(\tau,0,0)|+|f(\tau,0,0)|d\tau\\
&+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(|f(s,x(s),Ax(s))-f(s,0,0)|+|f(s,0,0)|)ds\\
\le&\theta_1M_A+\theta_2M_B+\theta_3\int_0^T\Big[L_f(|x|+k_1Tr+k_2T)+M_f\Big]d\tau\\
&+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(L_f(|x|+k_1Tr+k_2T)+M_f)ds\\
\le&\theta_1M_A+\theta_2M_B+\theta_3T(\eta_1r+\eta_2)+\frac{(\eta_2+L_fk_1Tr)t^\alpha}{\Gamma(\alpha+1)}\\
&+\frac{L_f}{\Gamma(\alpha)}\left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha_1}}ds\right)^{1-\alpha_1}\left(\int_0^t|x(s)|^{\frac{1}{\alpha_1}}ds\right)^{\alpha_1}\\
\le&\theta_1M_A+\theta_2M_B+\theta_3\eta_2T+\frac{\eta_2T^\alpha}{\Gamma(\alpha+1)}+\Big(\theta_3\eta_1T+\frac{L_fk_1T^{\alpha+1}}{\Gamma(\alpha+1)}+L_f\rho\Big)r\\
\le&\delta_3+\delta_2r\\
\le&r.
\end{aligned}
$$

Thus, F maps B_r into itself. Let $x,y\in PC(J,X)$. Then, for each $t\in J$, we have

$$
\begin{aligned}
&|Fx(t)-Fy(t)|\\
\le&|\frac{a_1}{1-a_1}|\sum_{k=1}^{m}|A_k(x(t_k))-A_k(y(t_k))|\\
&+\sum_{k=1}^{m}|\frac{c_k(1-a_2)+a_2b}{(1-a_1)(1-a_2)}\big(B_k(x(t_k))-B_k(y(t_k))\big)|
\end{aligned}
$$

$$
\begin{aligned}
&+|\frac{a_2 t}{1-a_2}|\sum_{k=1}^{m}|B_k(x(t_k))-B_k(y(t_k))|+\sum_{k=1}^{j-1}|A_k(x(t_k))-A_k(y(t_k))|\\
&+\sum_{k=1}^{j-1}|\big(B_k(x(t_k))-B_k(y(t_k))\big)(t-t_k)|\\
&+|\frac{b}{(1-a_1)(1-a_2)}|\int_0^T\frac{P(\tau,\alpha)}{\Gamma(\alpha-1)}|f(\tau,x(\tau),Ax(\tau))-f(\tau,y(\tau),Ay(\tau))\mid d\tau\\
&+|\frac{1}{1-a_1}|\int_0^T\frac{Q(\tau,\alpha)}{\Gamma(\alpha)}|f(\tau,x(\tau),Ax(\tau))-f(\tau,y(\tau),Ay(\tau))\mid d\tau\\
&+|\frac{t}{1-a_2}|\int_0^T\frac{P(\tau,\alpha)}{\Gamma(\alpha-1)}|f(\tau,x(\tau),Ax(\tau))-f(\tau,y(\tau),Ay(\tau))|d\tau\\
&+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}|f(s,x(s),Ax(s))-f(s,y(s),Ay(s))|ds\\
\le&\theta_1 L_A|x-y|+\theta_2 L_B|x-y|+\theta_3\int_0^T L_f(|x-y|+|Ax-Ay|)d\tau\\
&+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}L_f(|x-y|+k_1T\|x-y\|_{PC})ds\\
\le&(\theta_1L_A+\theta_2L_B)\|x-y\|_{PC}+\theta_3\eta_1T\|x-y\|_{PC}+L_f\rho\|x-y\|_{PC}+\frac{L_fk_1T^{\alpha+1}}{\Gamma(\alpha+1)}\|x-y\|_{PC}\\
\le&\Big(\theta_1L_A+\theta_2L_B+\theta_3\eta_1T+L_f\rho+\frac{L_fk_1T^{\alpha+1}}{\Gamma(\alpha+1)}\Big)\|x-y\|_{PC}\\
\le&\omega\|x-y\|_{PC}.
\end{aligned}
$$

And $\omega < 1$, which proves that the operator $F : PC(J,X) \to PC(J,X)$ is contraction. Applying Banach contraction fixed point, we deduce that the problem (1.3) has a unique solution on B_r. We complete the proof. $\square$

Theorem 3.2. *Suppose that* $(1)-(5)$ *are satisfied and* $\theta_1 L_A + \theta_2 L_B < 1$, $\delta_2 < 1$, *then the problem* (1.3) *has at least one solution.*

Proof. Choose $r \ge \frac{\delta_3}{1-\delta_2}$ and define on $B_r = \{x \in PC(J,X) : \|x\|_{PC} \le r\}$ the operators Φ, Ψ by

$$
(\Phi x)(t)=\begin{cases} e_0+e_1t, \quad t\in[0,t_1],\\ e_0+e_1t+\sum_{k=1}^{j-1}A_k(x(t_k))+\sum_{k=1}^{j-1}B_k(x(t_k))(t-t_k),\ t\in(t_{j-1},t_j],\end{cases} \tag{3.2}
$$

where

$$
\begin{aligned}
e_0&=\frac{1}{1-a_1}\big(a_1\sum_{k=1}^{m}A_k(x(t_k))+\sum_{k=1}^{m}B_k(x(t_k))\frac{c_k(1-a_2)+a_2b}{1-a_2}\big),\\
e_1&=\frac{a_2}{1-a_2}\sum_{k=1}^{m}B_k(x(t_k)),
\end{aligned}
$$

and

$$
(\Psi x)(t)=\frac{b}{(1-a_1)(1-a_2)\Gamma(\alpha-1)}\int_0^T P(\tau,\alpha)f(\tau,x(\tau),Ax(\tau))d\tau
$$

$$
\begin{aligned}
&+ \frac{1}{(1-a_1)\Gamma(\alpha)} \int_0^T Q(\tau,\alpha) f(\tau, x(\tau), Ax(\tau)) d\tau \\
&+ \frac{t}{(1-a_2)\Gamma(\alpha-1)} \int_0^T P(\tau,\alpha) f(\tau, x(\tau), Ax(\tau)) d\tau \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), Ax(s)) ds.
\end{aligned} \tag{3.3}
$$

For any $x, y \in B_r$ and $t \in J$, we find that

$$
\begin{aligned}
&\|\Phi x + \Psi y\|_{PC} \\
\le& |e_0| + |e_1 t| + \sum_{k=1}^{j-1} |A_k(x(t_k))| + \sum_{k=1}^{j-1} |B_k(x(t_k))(t-t_k)| \\
&+ \left|\frac{b}{(1-a_1)(1-a_2)\Gamma(\alpha-1)}\right| \int_0^t P(\tau,\alpha) |f(\tau, y(\tau), Ay(\tau))| d\tau \\
&+ \left|\frac{1}{(1-a_1)\Gamma(\alpha)}\right| \int_0^T Q(\tau,\alpha) |f(\tau, y(\tau), Ay(\tau))| d\tau \\
&+ \left|\frac{t}{(1-a_2)\Gamma(\alpha-1)}\right| \int_0^T P(\tau,\alpha) |f(\tau, y(\tau), Ay(\tau))| d\tau \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, y(s), Ay(s))| ds \\
\le& \theta_1 M_A + \theta_2 M_B + \theta_3 \eta_2 T + \frac{\eta_2 T^\alpha}{\Gamma(\alpha+1)} + \Big(\theta_3 \eta_1 T + \frac{L_f k_1 T^{\alpha+1}}{\Gamma(\alpha+1)} + L_f \rho\Big) r \\
\le& \delta_3 + \delta_2 r \\
\le& r.
\end{aligned}
$$

Thus, $\Phi x + \Psi y \in B_r$. It is obviously that Φ is a contraction mapping on B_r since

$$
\|\Phi x - \Phi y\|_{PC} \le (\theta_1 L_A + \theta_2 L_B) \|x - y\|_{PC}.
$$

On the other hand, the operator Ψ is continuous by the continuity of f. Also, Ψ is uniformly bounded on B_r since

$$
\|\Psi(x)\|_{PC} \le \delta_1 + \delta_2 r.
$$

Now we will prove the compactness of the operator Ψ. For any $x \in PC(J, X)$, s_1, $s_2 \in J$, $t_{j-1} < s_1 < s_2 \le t_j$, we have

$$
\begin{aligned}
&|\Psi x(s_2) - \Psi x(s_1)| \\
\le& \frac{(s_2 - s_1)}{|1-a_2|\Gamma(\alpha-1)} \int_0^T P(\tau,\alpha) |f(\tau, x(\tau), Ax(\tau))| d\tau \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^{s_1} \big((s_2-s)^{\alpha-1} - (s_1-s)^{\alpha-1}\big) |f(s, x(s), Ax(s))| ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{s_1}^{s_2} (s_2-s)^{\alpha-1} |f(s, x(s), Ax(s))| ds
\end{aligned}
$$

$$\leq \frac{(s_2 - s_1)}{| 1 - a_2 |} M_h e^T (r\eta_1 T + \eta_2 T) + \frac{(s_2^\alpha - s_1^\alpha)}{\Gamma(\alpha+1)} (r\eta_1 + \eta_2),$$

which tends to zero as $s_2 \to s_1$. This means that Ψ is equicontinuous on interval $(t_{j-1}, t_j]$. By the means of the Arzela-Ascoli Theorem, we get that operator Ψ is completely continuous. Hence, by the conclusion of Theorem 2.1, the problem (1.3) has at least one solution on B_r. The proof is completed. □

In addition, we consider the existence of solution to fractional impulsive equation (1.3) with the following linear growth condition:
$(2')$ There exists a positive constant L such that

$$|f(t, x(t), Ax(t)| \leq L(1 + |x| + |Ax|),$$

for any $x \in X$ and $t \in J$.

Theorem 3.3. *Let the assumptions* $(1), (2')$ *and* $(3) - (5)$ *be satisfied and* $\Lambda_2 < 1$, *then the problem* (1.3) *has at least one solution.*

Proof. Consider the operator $F : PC(J, X) \to PC(J, X)$ defined as (3.1). Let $\{x_n\}$ be a sequence such that $x_n \to x$ in $PC(J, X)$. Then for any $t \in J$, we have

$$\begin{aligned}
&|(Fx_n)(t) - (Fx)(t)| \\
\leq & |\frac{a_1}{1-a_1}| \sum_{k=1}^{m} |A_k(x_n(t_k)) - A_k(x(t_k))| \\
&+ \sum_{k=1}^{m} |\frac{c_k(1-a_2) + a_2 b}{(1-a_1)(1-a_2)} \big(B_k(x_n(t_k)) - B_k(x(t_k))\big)| \\
&+ |\frac{a_2 t}{1-a_2}| \sum_{k=1}^{m} | B_k(x_n(t_k)) - B_k(x(t_k))| + \sum_{k=1}^{j-1} |A_k(x_n(t_k)) - A_k(x(t_k))| \\
&+ \sum_{k=1}^{j-1} |\big(B_k(x_n(t_k)) - B_k(x(t_k))\big)(t - t_k)| \\
&+ |\frac{b}{(1-a_1)(1-a_2)}| \int_0^T \frac{P(\tau,\alpha)}{\Gamma(\alpha-1)} |f(\tau, x_n(\tau), Ax_n(\tau)) - f(\tau, x(\tau), Ax(\tau))| d\tau \\
&+ |\frac{1}{1-a_1}| \int_0^T \frac{Q(\tau,\alpha)}{\Gamma(\alpha)} |f(\tau, x_n(\tau), Ax_n(\tau)) - f(\tau, x(\tau), Ax(\tau))| d\tau \\
&+ |\frac{t}{1-a_2}| \int_0^T \frac{P(\tau,\alpha)}{\Gamma(\alpha-1)} |f(\tau, x_n(\tau), Ax_n(\tau)) - f(\tau, x(\tau), Ax(\tau))| d\tau \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, x_n(s), Ax_n(s)) - f(s, x(s), Ax(s))| ds \\
\leq & (\theta_1 L_A + \theta_2 L_B) |x_n(t_k) - x(t_k)| \\
&+ \theta_3 T \int_0^T \frac{P(\tau,\alpha)}{\Gamma(\alpha-1)} |f(\tau, x_n(\tau), Ax_n(\tau)) - f(\tau, x(\tau), Ax(\tau))| d\tau \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, x_n(s), Ax_n(s)) - f(s, x(s), Ax(s))| ds,
\end{aligned}$$

which means that the operator F is continuous. There exists a positive constant l_1 such that for each $x \in B_{r^*} = \{x \in PC(J,X) : \|x\|_{PC} \leq r^*\}$ and $t \in J$, we have

$$\begin{aligned}
&|(Fx)(t)| \\
\leq&|\lambda_0| + |\lambda_1 t| + \sum_{k=1}^{j-1} |A_k(x(t_k))| + \sum_{k=1}^{j-1} |B_k(x(t_k))(t-t_k)| \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s,x(s),Ax(s)))ds \\
\leq&\theta_1 M_A + \theta_2 M_B + \theta_3 \int_0^T |f(\tau,x(\tau),Ax(\tau)|d\tau \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s,x(s),Ax(s))|ds \\
\leq&\theta_1 M_A + \theta_2 M_B + \theta_3 LT(1 + r^* + k_1 r^* T + k_2 T) \\
&+ \frac{LT^\alpha}{\Gamma(\alpha+1)}(1 + r^* + k_1 r^* T + k_2 T) \\
:=&l_1
\end{aligned}$$

which means that $\|(Fx)(t)\|_{PC} \leq l_1$. Hence, F is uniformly bounded on B_{r^*}. Now, we will prove the operator F is equicontinuous. For any $x \in PC(J,X)$, $s_1, s_2 \in J$, $t_{j-1} < s_1 < s_2 \leq t_j$, we have

$$\begin{aligned}
&|Fx(s_2) - Fx(s_1)| \\
\leq&|\frac{a_2}{1-a_2}| \sum_{k=1}^{m} |B_k(x(t_k))|(s_2 - s_1) + \sum_{k=1}^{j-1} |B_k(x(t_k))|(s_2 - s_1) \\
&+ |\frac{1}{1-a_2}|(s_2 - s_1) \int_0^T \frac{P(\tau,\alpha)}{\Gamma(\alpha-1)} |f(\tau,x(\tau),Ax(\tau)|d\tau \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^{s_2} (s_2 - s)^{\alpha-1} |f(s,x(s),Ax(s))|ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^{s_1} (s_1 - s)^{\alpha-1} |f(s,x(s),Ax(s))|ds \\
\leq&(|\frac{a_2}{1-a_2}| + 1)(s_2 - s_1) m M_B + \frac{M_h e^T LT}{|1-a_2|}(s_2 - s_1)(1 + r^* + k_1 r^* T + k_2 T) \\
&+ \frac{L(s_2^\alpha - s_1^\alpha)}{\Gamma(\alpha+1)}(1 + r^* + k_1 r^* T + k_2 T).
\end{aligned}$$

As $s_2 \to s_1$, the above inequality tends to zero. Thus, F is equicontinuous on B_{r^*}. By the Arzela-Ascoli Theorem, F is completely continuous. Let $x \in E(F) = \{x \in B_{r^*} : x = \lambda Fx \text{ for some } \lambda \in [0,1]\}$. For $t \in J$, we have

$$\begin{aligned}
&|x(t)| \leq |(Fx)(t)| \\
\leq&|\lambda_0| + |\lambda_1 t| + \sum_{k=1}^{j-1} |A_k(x(t_k))| + \sum_{k=1}^{j-1} |B_k(x(t_k))(t-t_k)| \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s,x(s),Ax(s))|ds
\end{aligned}$$

$$
\begin{aligned}
&\le \theta_1 M_A + \theta_2 M_B + \theta_3 L \int_0^T (1+|x|+|Ax|)d\tau \\
&\quad + \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}(1+|x|+|Ax|)ds \\
&\le \theta_1 M_A + \theta_2 M_B + \theta_3 LT + \theta_3 L \int_0^T |x(\tau)|d\tau + \theta_3 L \int_0^T \int_0^\tau k_1 |x(s)| ds d\tau \\
&\quad + \frac{1}{2} L k_2 \theta_3 T^2 + \frac{LT^\alpha}{\Gamma(\alpha+1)} + \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}|x(s)|ds \\
&\quad + \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \int_0^s k_1 |x(\tau)| d\tau ds + \frac{Lk_2}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s ds \\
&\le \theta_1 M_A + \theta_2 M_B + \theta_3 LT + \frac{1}{2} L k_2 \theta_3 T^2 + \frac{LT^\alpha}{\Gamma(\alpha+1)} + \frac{LT^{\alpha+1}k_2}{\Gamma(\alpha+2)} \\
&\quad + (\theta_3 LT + \frac{1}{2}\theta_3 LT^2 k_1)\|x\|_{pc} + \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}|x(s)|ds \\
&\quad + \frac{Lk_1}{\Gamma(\alpha)} \int_0^t \int_0^s (t-s)^{\alpha-1} |x(\tau)| d\tau ds \\
&\le \Lambda_1 + (\theta_3 LT + \frac{1}{2}\theta_3 LT^2 k_1)\|x\|_{pc} + \text{Ł}\rho\|x\|_{pc} + \frac{LT^{\alpha+1}k_1}{\Gamma(\alpha+1)}\|x\|_{pc} \\
&\le \Lambda_1 + \Big(\theta_3 LT(1+\frac{1}{2}k_1 T) + L\rho + \frac{LT^{\alpha+1}k_1}{\Gamma(\alpha+1)}\Big)\|x\|_{PC} \\
&\le \Lambda_1 + \Lambda_2 \|x\|_{PC},
\end{aligned}
$$

which implies that

$$
\|x\|_{pc} \le \frac{\Lambda_1}{1-\Lambda_2}.
$$

This proves that the set $E(F)$ is bounded. Therefore, by Theorem 2.2, we deduce that the problem (1.3) has at least one solution. The proof is completed. □

In addition, let us introduce two conditions for impulsive fractional differential equations (1.3) that will be useful in what follows:
$(2'')$ There exist a real valued function $\nu(t) \in C[0,T]$ and a nondecreasing function $\mu : [0,+\infty) \to (0,+\infty)$ such that

$$
|f(t, x(t), Ax(t)| \le \nu(t)\big(\mu(|x|) + |Ax|\big),
$$

for any $t \in J$ and $x \in X$.
(6) The following inequality

$$
\frac{r'}{\xi_1 + \xi_2\big(\mu(r') + k_1 T r' + k_2 T\big)} > 1,
$$

has at least a positive solution $r' > 0$, $\xi_1 = \theta_1 M_A + \theta_2 M_B$, $\xi_2 = \theta_3 T + \frac{T^\alpha}{\Gamma(\alpha+1)}$ and $M_\nu = \sup_{t \in J} \nu(t)$.

Theorem 3.4. *Assume that* $(1), (2'')$ *and* $(3)-(6)$ *are satisfied. Then the problem* (1.3) *has at least one solution.*

Proof. Consider the operator $F : PC \to PC$ defined as (3.1). It is easy to prove that F is continuous and completely continuous. Let $\lambda^* \in (0,1)$ and $x = \lambda^* Fx$, then for each $t \in J$, we have

$$
\begin{aligned}
&|x(t)| = |\lambda^* Fx(t)|\\
\le&|\lambda_0| + |\lambda_1 t| + \sum_{k=1}^{j-1} |A_k(x(t_k))| + \sum_{k=1}^{j-1} |B_k(x(t_k))(t-t_k)|\\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \nu(s)\big(\mu(|x|) + |Ax|\big) ds\\
\le&\theta_1 M_A + \theta_2 M_B + \theta_3 \int_0^T M_\nu \Big(\mu(\|x\|_{pc}) + Tk_1\|x\|_{pc} + Tk_2\Big) d\tau\\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} M_\nu \Big(\mu(\|x\|_{pc}) + Tk_1\|x\|_{pc} + Tk_2\Big) ds\\
\le&\theta_1 M_A + \theta_2 M_B + M_\nu \Big(\mu(\|x\|_{pc}) + Tk_1\|x\|_{pc} + Tk_2\Big)\Big(\theta_3 T + \frac{T^\alpha}{\Gamma(\alpha+1)}\Big)\\
\le&\xi_1 + \xi_2 M_\nu \Big(\mu(\|x\|_{pc}) + Tk_1\|x\|_{pc} + Tk_2\Big).
\end{aligned}
$$

Thus

$$
\frac{\|x\|_{pc}}{\xi_1 + \xi_2 M_\nu \Big(\mu(\|x\|_{pc}) + k_1 T\|x\|_{pc} + k_2 T\Big)} \le 1.
$$

By (6), there exists a positive constant r' such that $\|x\|_{PC} \ne r'$.

Let $U = \{x \in PC(J,X) : \|x\|_{PC} < r'\}$. The operator $F : \overline{U} \to PC(J,X)$ is completely continuous. From the choice U, there is no $x \in \partial U$ or $\lambda^* \in [0,1]$ with $x = \lambda^* F(x)$. As a consequence of Theorem 2.3, we can obtain that the operator F has a fixed point $x \in \overline{U}$, which means that the problem (1.3) has at least one solution. The proof is completed. □

4. Application

In this section, we will give three examples to demonstrate the feasibility of the obtained results.

Example 4.1. Let us consider the first fractional impulsive problem

$$
\begin{cases}
{}^cD^{\frac{3}{2}} x(t) = \dfrac{x(t)}{(t+4)^2(1+|x(t)|)} + \dfrac{1}{16}\displaystyle\int_0^t e^{-(t-s)}|x(s)|ds, \quad t \in [0,1]\backslash\{\dfrac{1}{3}\}\\
\Delta x(\dfrac{1}{3}) = \dfrac{|x(\frac{1}{3})|}{4+|x(\frac{1}{3})|} \quad \Delta x'(\dfrac{1}{3}) = 1 + \dfrac{1}{8}\sin^3(x(\dfrac{1}{3})),\\
x(0) = \dfrac{1}{20}\displaystyle\int_0^1 x(s)ds, \quad x'(0) = \dfrac{1}{40}\displaystyle\int_0^1 x'(s)ds.
\end{cases}
\tag{4.1}
$$

Set

$$
f(t, x(t), \int_0^t k(t,s,x(s))ds) = \frac{x(t)}{(t+4)^2(1+|x(t)|)} + \frac{1}{16}\int_0^t e^{-(t-s)}|x(s)|ds,
$$

$$
\begin{aligned}
&A_k(x(t_k^-)) = \frac{|x(\frac{1}{3})|}{4+|x(\frac{1}{3})|},\\
&B_k(x(t_k^-)) = 1+\frac{1}{8}\sin^3(x(\frac{1}{3})),\\
&g(s) = \frac{1}{20},\\
&h(s) = \frac{1}{40},\\
&\alpha_1 = \frac{1}{2},\\
&\delta_2 = \theta_3\eta_1 T + L_f\Big(\frac{k_1 T^{\alpha+1}}{\Gamma(\alpha+1)}+\rho\Big) \approx 0.124 < 1.
\end{aligned}
$$

We have

$$
\begin{aligned}
&|f(t,x(t),Ax(t)) - f(t,y(t),Ay(t))|\\
\le&\frac{|x-y|}{(t+4)^2(1+|x(t)|)(1+|y(t)|)} + \frac{1}{16}|Ax(t)-Ay(t)|\\
\le&\frac{1}{16}|x-y| + \frac{1}{16}|Ax-Ay|,
\end{aligned}
$$

and

$$
\omega = \theta_1 L_A + \theta_2 L_B + \theta_3\eta_1 T + L_f\rho + \frac{L_f k_1 T^{\alpha+1}}{\Gamma(\alpha+1)} \approx 0.775 < 1.
$$

Hence, according to Theorem 3.1, we figure out that the fractional impulsive problem (4.1) has a unique solution.

Example 4.2. Let us consider the second fractional impulsive problem

$$
\begin{cases}
{}^cD^{\frac{3}{2}}x(t) = \dfrac{e^{-t}|x(t)|}{(9+e^t)(2+|x(t)|)} + \dfrac{1}{20}\displaystyle\int_0^t e^{-(t-s)}|x(s)|ds, \quad t\in[0,1]\backslash\{\frac{1}{3}\},\\
\Delta x(\frac{1}{3}) = \dfrac{|x(\frac{1}{3})|}{3+|x(\frac{1}{3})|} \quad \Delta x'(\frac{1}{3}) = 1+\dfrac{1}{12}\cos^3(x(\frac{1}{3})),\\
x(0) = \dfrac{1}{30}\displaystyle\int_0^1 x(s)ds, \quad x'(0) = \dfrac{1}{60}\displaystyle\int_0^1 x'(s)ds.
\end{cases}
\tag{4.2}
$$

Set

$$
f(t,x(t),\int_0^t k(t,s,x(s))ds) = \frac{e^{-t}|x(t)|}{(9+e^t)(2+|x(t)|)} + \frac{1}{20}\int_0^t e^{-(t-s)}|x(s)|ds,
$$

and

$$
Ax(t) = \int_0^t e^{-(t-s)}|x(t)|ds, \quad \alpha_1 = \frac{1}{2}.
$$

Obviously, for each $t\in[0,1]\backslash\{\frac{1}{3}\}$, we have

$$
|f(t,x(t),Ax(t)) - f(t,y(t),Ay(t))| \le \frac{1}{20}|x-y| + \frac{1}{20}|Ax-Ay|,
$$

$$\delta_2 = \theta_3\eta_1 T + L_f\Big(\frac{k_1 T^{\alpha+1}}{\Gamma(\alpha+1)} + \rho\Big) \approx 0.092 < 1.$$

Hence, the conditions $(1)-(5)$ are satisfied with $\theta_1 L_A + \theta_2 L_B \approx 0.601 < 1$. According to Theorem 3.2, the problem (4.2) has at least one solution.

Example 4.3. Let us consider the third fractional impulsive problem

$$\begin{cases} {}^cD^{\frac{3}{2}}x(t) = \dfrac{e^{-t}|x(t)|}{(9+e^t)} + \dfrac{1}{10}\displaystyle\int_0^t e^{-(t-s)}|x(s)|ds + \dfrac{1}{11}t, \qquad t \in [0,1]\backslash\{\dfrac{1}{3}\}, \\ \Delta x(\dfrac{1}{3}) = \dfrac{|x(\frac{1}{3})|}{5+|x(\frac{1}{3})|} \quad \Delta x'(\dfrac{1}{3}) = 1 + \dfrac{1}{15}\cos^3(x(\dfrac{1}{3})), \\ x(0) = \dfrac{1}{8}\displaystyle\int_0^1 x(s)sds, \quad x'(0) = \dfrac{1}{12}\displaystyle\int_0^1 x'(s)sds. \end{cases} \tag{4.3}$$

Let $\alpha_1 = \frac{1}{2}$ and each $t \in [0,1]\backslash\{\frac{1}{3}\}$, we have

$$\begin{aligned} f(t, x(t), \int_0^t k(t,s,x(s))ds) &= \frac{e^{-t}|x(t)|}{(9+e^t)} + \frac{1}{10}\int_0^t e^{-(t-s)}|x(s)|ds + \frac{1}{11}t \\ &\leq \frac{1}{10}(1+|x(t)|+|Ax(t)|), \end{aligned}$$

and $\Lambda_2 = \theta_3 LT(1+\frac{1}{2}k_1 T) + L\rho + \frac{LT^{\alpha+1}k_1}{\Gamma(\alpha+1)} \approx 0.246 < 1$.

Hence, the conditions $(1), (2')$ and $(3)-(5)$ are satisfied. Veiwing from Theorem 3.3, the problem (4.3) has at least one solution.

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