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New Proofs of Monotonicity of Period Function for Cubic Elliptic Hamiltonian*

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Abstract In [1] S.-N. Chow and J. A. Sanders proved that the period function is monotone for elliptic Hamiltonian of degree 3. In this paper we significantly simplify their proof, and give a new way to prove this fact, which may be used in other problems.

Keywords Periodic function, elliptic Hamiltonian, Abelian integrals.

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1. Introduction

Consider the cubic elliptic Hamiltonian function $H(x,y) = \frac{y^2}{2} + P_3(x)$, there P_3 is a polynomial of degree 3, the corresponding quadratic Hamiltonian system is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -P_3'(x).$$

Suppose that the origin is a non-degenerate center, so we can write $P_3(x) = \frac{1}{2}x^2 - \frac{a}{3}x^3$, where $a \neq 0$. If we write the closed orbit, surrounding the origin, by

$$\gamma_h \subset H^{-1}(h) = \{(x,y)|H(x,y) = h\},$$

then, from the first equation of the system, we can write the period function by

$$T(h) = \oint_{\gamma_h} \frac{1}{y}\,\mathrm{d}x, \tag{1.1}$$

where $y = y(x,h)$ is defined by $H(x,y) = h$. Note that by the scaling $(x,y) \mapsto (\frac{x}{a}, \frac{y}{a})$, the period function does not change, hence without loss of generality we can suppose that γ_h is defined by

$$H(x,y) = \frac{y^2}{2} + A(x) = h, \quad A(x) = \frac{x^2}{2} - \frac{x^3}{3}, \tag{1.2}$$

and the corresponding Hamiltonian system is

$$\begin{aligned} \tfrac{\mathrm{d}x}{\mathrm{d}t} &= y, \\ \tfrac{\mathrm{d}y}{\mathrm{d}t} &= -x + x^2 = x(x-1). \end{aligned} \tag{1.3}$$

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The continuous family of ovals is $\{\gamma_h \subset H^{-1}(h), 0 < h < \frac{1}{6}\}$, γ_h shrinks to the center at $(x,y) = (0,0)$ when $h \to 0^+$, and γ_h expand to the homoclinic loop Γ related to the saddle at $(x,y) = (1,0)$ when $h \to \frac{1}{6}^-$.

Theorem 1.1 (Theorem 3.8 of [1]). *The period function $T(h)$ is monotone for $0 < h < \frac{1}{6}$.*

For more information about the study of period functions, see Section 2.4 of [2], for example.

2. A simple proof of Theorem 1.1

We first give a very simple proof of Theorem 1.1 by using Picard-Fuchs equation.

Let

$$I_k(h) = \oint_{\gamma_h} x^k y \,\mathrm{d}x, \quad k = 0, 1, 2, \cdots, \tag{2.1}$$

then by using $yy_h = 1$ and (2.1) we have

$$I'_k(h) = \oint_{\gamma_h} \frac{x^k}{y} \,\mathrm{d}x, \quad k = 0, 1, 2, \cdots. \tag{2.2}$$

Lemma 2.1. *The following equalities hold:*

$$\begin{aligned} 5I_0 &= 6hI'_0 - I'_1, \\ 7I_1 &= I_0 + (6h-1)I'_1, \end{aligned} \tag{2.3}$$

where $I_k = I_k(h), I'_k = I'_k(h)$.

Proof. From (2.1), (1.2) and (2.2) we have

$$I_k = \oint_{\gamma_h} \frac{x^k y^2}{y} \,\mathrm{d}x = \oint_{\gamma_h} \frac{x^k(2h - x^2 + \frac{2}{3}x^3)}{y} \,\mathrm{d}x = 2hI'_k - I'_{k+2} + \frac{2}{3} I'_{k+3}.$$

On the other hand, by using integration by parts and the fact that $\mathrm{d}y = \frac{x^2-x}{y}\,\mathrm{d}x$ we have

$$I_k = \oint_{\gamma_h} x^k y \mathrm{d}x = -\frac{1}{k+1} \oint_{\gamma_h} x^{k+1} \mathrm{d}y = \frac{1}{k+1} \oint_{\gamma_h} \frac{x^{k+1}(x - x^2)}{y} \,\mathrm{d}x = \frac{I'_{k+2} - I'_{k+3}}{k+1}. \tag{2.4}$$

Eliminating I'_{k+3} from the above two equalities, we obtain

$$(2k+5)I_k = 6hI'_k - I'_{k+2}.$$

Taking $k = 0, 1$, we find

$$\begin{aligned} 5I_0 &= 6hI'_0 - I'_2, \\ 7I_1 &= 6hI'_1 - I'_3. \end{aligned} \tag{2.5}$$

By integrating $(x - x^2)y \,\mathrm{d}x = y^2 \,\mathrm{d}y$ along γ_h we get $I_1(h) \equiv I_2(h)$, hence the first equation of (2.5) gives the first equality of (2.3). Taking $k = 0$ in (2.4) we have

$I_0 = I_2' - I_3' = I_1' - I_3'$, eliminating I_3' from this equation and the second one of (2.5) we get the second equality of (2.3). □

Now it is ready to prove Theorem 1.1.

Proof. Making one more derivative on the two equations of (2.3), we have

$$\begin{aligned} 6hI_0'' - I_1'' &= -I_0', \\ (1-6h)I_1'' &= I_0' - I_1'. \end{aligned} \tag{2.6}$$

Multiplying the first equation of (2.6) by $(1-6h)$, then add it to the second one, and finally using the first equation of (2.3), we get

$$6h(1-6h)I_0'' = 6hI_0' - I_1' = 5I_0.$$

Hence, by using (1.1) and (2.2) ($k=0$) we obtain

$$T'(h) = I_0''(h) = \frac{5}{36h(\frac{1}{6}-h)}\, I_0(h) > 0, \quad 0 < h < \frac{1}{6}.$$

Note that the orientation of γ_h is clockwise, $I_0(h)$ is the area surrounded by γ_h, hence $I_0(h) > 0$ for $0 < h < \frac{1}{6}$. □

3. A new way of the proof of Theorem 1.1

In this section we need the following two lemmas:

Lemma 3.1 (A simplified form of Lemma 4.1 of [3])**.** *Let γ_h be an oval inside the level curve $\{\frac{y^2}{2} + A(x) = h\}$, and we consider a function F such that $\frac{F}{A'}$ is analytic at $x=0$. Then, for any positive integer k,*

$$\oint_{\gamma_h} F(x) y^{k-2} \mathrm{d}x = \oint_{\gamma_h} G(x) y^k \mathrm{d}x,$$

where $G = \frac{1}{k}\left(\frac{F}{A'}\right)'(x)$.

Lemma 3.2 (A simplified form of Theorem 1 of [4])**.** *Let γ_h be an oval inside the level curve $\{\frac{y^2}{2} + A(x) = h\}$ for $h \in (c,d)$, where $A(x)$ is analytic, satisfying $A'(x)x > 0$ (or < 0) for $x \in (\alpha,0) \cup (0,\beta)$, where $A(\alpha) = A(\beta)$. Hence for each γ_h a unique function $\tilde{x} = \tilde{x}(x)$ can be defined by $A(\tilde{x}) = A(x)$ for $\alpha < x < 0 < \tilde{x} < \beta$. Condider a ratio of two Abelian integrals*

$$P(h) = \frac{\oint_{\gamma_h} f(x) y\, dx}{\oint_{\gamma_h} y\, dx}, \tag{3.1}$$

where f is differentiable. Define a function

$$\xi(x) = \left.\frac{f(x)A'(\tilde{x}) - f(\tilde{x})A'(x)}{A'(\tilde{x}) - A'(x)}\right|_{\tilde{x}=\tilde{x}(x)}, \tag{3.2}$$

where $x \in (\alpha,0)$. Then $\xi'(x) < 0$ (> 0) for $x \in (\alpha,0)$ implies $P'(h) > 0$ (< 0) for $h \in (c,d)$.

As we mentioned in Section 1 that the continuous family $\{\gamma_h \subset H^{-1}(h), 0 < h < \frac{1}{6}\}$ is bounded by the center $(0,0)$ and the homoclinic loop Γ related to the saddle $(1,0)$. Since $H(x,0) - \frac{1}{6} = -\frac{1}{6}(x-1)^2(2x+1)$, for this family we have

$$-\frac{1}{2} < x < 0 < \tilde{x} < 1, \tag{3.3}$$

where $\tilde{x} = \tilde{x}(x)$ is the unique function satisfying (3.3) and

$$H(x,y) = H(\tilde{x},y), \quad \text{i. e.} \quad A(x) = A(\tilde{x}). \tag{3.4}$$

By (1.1) and (1.2) we have for $h > 0$

$$T(h) = \oint_{\gamma_h} \frac{\mathrm{d}x}{y} = \frac{1}{h}\oint_{\gamma_h} (A(x) + \frac{y^2}{2})\frac{\mathrm{d}x}{y},$$

hence

$$h\,T(h) = \frac{1}{2}\oint_{\gamma_h} y\,\mathrm{d}x + \oint_{\gamma_h} \frac{A(x)}{y}\,\mathrm{d}x.$$

By Lemma 3.1 ($k=1$)

$$\oint_{\gamma_h} \frac{A(x)}{y}\,\mathrm{d}x = \oint_{\gamma_h} f(x)y\mathrm{d}x, \quad f(x) = \left(\frac{A(x)}{A'(x)}\right)' = \frac{2x^2-4x+3}{6(1-x)^2}.$$

We obtain

$$h\,T(h) = \oint_{\gamma_h} \left(\frac{1}{2} + f(x)\right) y\,\mathrm{d}x = \frac{1}{2}I_0(h) + I_1(h) = I_0(h)\left(\frac{1}{2} + P(h)\right), \tag{3.5}$$

where $I_0(h) = \oint_{\gamma_h} y\,\mathrm{d}x, I_1(h) = \oint_{\gamma_h} f(x)y\,\mathrm{d}x, P(h) = \frac{I_1(h)}{I_0(h)}$.

Lemma 3.3. *Let $x + \tilde{x} = u$, $x\tilde{x} = v$, where $\tilde{x} = \tilde{x}(x)$ is defined above. Then*
(1) Along any γ_h for $h \in (0, \frac{1}{6})$ we have $v = v(u) = u(u - \frac{3}{2})$;
(2) $u \in (0, \frac{1}{2})$ and $u_x < 0$ for $x \in (-\frac{1}{2}, 0)$.

Proof. The statement (1) can be easily obtained by using $A(x) = A(\tilde{x})$ and $x < 0 < \tilde{x}$. Note $v = x\tilde{x} < 0$ for all possible x. When $x \sim 0$ we have $u \sim 0$, hence from statement (1) we have $u > 0$ for $x \sim 0$, this implies $0 < u < \frac{3}{2}$ for all possible x. Note that $A'(\tilde{x}) = \tilde{x} - (\tilde{x})^2 > 0$, and

$$u_x = 1 + \tilde{x}'(x) = 1 + \frac{A'(x)}{A'(\tilde{x})} = \frac{u - u^2 + 2v}{A'(\tilde{x})} = \frac{u(u-2)}{A'(\tilde{x})} < 0.$$

This fact and (3.3) improve the estimation of $u \in (0, \frac{3}{2})$ to $u \in (0, \frac{1}{2})$. □

Lemma 3.4. *The following statements hold:*
(1) $\lim_{h\to 0^+} P(h) = \frac{1}{2}$;
(2) $P'(h) > 0$ for $h \in (0, \frac{1}{6})$.

Proof. The statement (1) can be easily obtained by using the mean-value theorem for integrations and the fact that $f(0) = \frac{1}{2}$. To prove Statement (2) we use Lemma 3.2. By using formula (3.2) and Lemma 3.3 (1) we can change $\{x, \tilde{x}\}$ to $\{u, v\}$, then use $v = v(u)$. Computations give

$$\eta(u) = \xi(x(u)) = \frac{4u^4 - 16u^3 + 19u^2 - 9u + 3}{3(u-1)(u-2)(2u-1)^2},$$

hence

$$\eta'(u) = \frac{8u^4 - 24u^3 + 12u^2 + 20u - 15}{3(u-1)^2(u-2)^2(2u-1)^3} > 0, \quad u \in \left(0, \frac{1}{2}\right).$$

By this estimation and Lemma 3.3 (2) we have $\xi'(x) = \eta'(u)u_x < 0$, hence by Lemma 3.2 we finally obtain $P'(h) > 0$ for $h \in (0, \frac{1}{6})$. □

Now it is ready to prove Theorem 1.1.

Proof. Making derivative on both sides of (3.5) and using $I_0'(h) = T(h)$ we have

$$h\,T'(h) = T(h)\left(P(h) - \frac{1}{2}\right) + I_0(h)P'(h).$$

From this equality and using Lemma 3.4 we immediately get $T'(h) > 0$ for $h \in (0, \frac{1}{6})$, since $T(h) > 0$ and $I_0(h) > 0$. □

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