Journal of Nonlinear Modeling and Analysis http://jnma.ca; http://jnma.ijournal.cn
Volume 1, Number 3, September 2019, 335–354 DOI:10.12150/jnma.2019.335

Dynamics of the Stochastic Chemostat Model with Monod-Haldane Response Function*

Caibin Zeng[1], Binghui Liao[1] and Jianhua Huang[2,†]

Abstract This paper is devoted to the asymptotic dynamics of stochastic chemostat model with Monod-Haldane response function. We first prove the existence of random attractors by means of the conjugacy method and further construct a general condition for internal structure of the random attractor, implying extinction of the species even with small noise. Moreover, we show that the attractors of Wong-Zakai approximations converges to the attractor of the stochastic chemostat model in an appropriate sense.

Keywords Stochastic chemostat model, random attractors, Wong-Zakai approximation, conjugacy method.

MSC(2010) 60H10, 37L10, 35B40.

1. Introduction

Chemostat refers to a basic piece of laboratory apparatus used for the continuous culture of microorganisms. It occupies a central place in mathematical ecology and has played an important role in many fields [4, 12, 16, 22, 30–32, 34]. It can also model waste water treatment [13, 26] or study recombinant problems in genetically altered microorganisms [17, 18]. Derivation and analysis of chemostat models are well documented in [9, 29, 33] and references therein.

The classic chemostat model with single species and single limiting substrate takes the form

$$\frac{dS(t)}{dt} = (S^0 - S(t))D - \mu(S(t))x(t), \tag{1.1}$$

$$\frac{dx(t)}{dt} = -Dx(t) + \mu(S(t))x(t), \tag{1.2}$$

where $S(t)$ and $x(t)$ denote concentrations of the nutrient and the microbial biomass, respectively; S^0 denotes the volumetric dilution rate and D is the dilution rate. The

[†]the corresponding author.
Email address: macbzeng@scut.edu.cn(C. Zeng), jhhuang32@nudt.edu.cn(J. Huang)
[1]School of Mathematics, South China University of Technology, Guangzhou, Guangdong 510640, China
[2]College of Liberal Arts and Science, National University of Defense Technology, Changsha, Hunan 410073, China
[*]Zeng is supported by National Natural Science Foundation of China (No.11871225, 11501216) and the Fundamental Research Funds for the Central Universities (No. 2018MS58), while Huang was supported by the NSFC (No.11771449,11371367).

growth rate of the microbial population is represented by the function $\mu(S)$, which is generally assumed to be non-negative.

However, there are very strong restrictions as the real world is non-autonomous and stochastic, and this justifies the analysis of stochastic chemostat model. In general, there exist several alternatives to model randomness and stochasticity. For example, one can replace the dilution rate D by $D+\alpha\dot{W}(t)$ and thus the original system (1.1)-(1.2) is replaced by the following stochastic differential equations understood in the Itô sense

$$dS(t)=[(S^0-S(t))D-\mu(S(t))x(t)]dt+\alpha(S^0-S(t))dW(t), \tag{1.3}$$
$$dx(t)=[-Dx(t)+\mu(S(t))x(t)]dt-\alpha x(t)dW(t), \tag{1.4}$$

where $W(t)$ is a standard Brownian motion defined in a complete probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},P)$, and $\alpha>0$ is its intensity. Biologically the model does not seem completely realistic due to the fact that the substrate $S(t)$ in the corresponding stochastic chemostat model (1.3)-(1.4) can take negative. Alternatively, following the idea in [15, 19], one can obtain the stochastic chemostat model

$$dS(t)=[(S^0-S(t))D-\mu(S(t))x(t)]dt-\alpha S(t)dW(t), \tag{1.5}$$
$$dx(t)=[-Dx(t)+\mu(S(t))x(t)]dt-\alpha x(t)dW(t). \tag{1.6}$$

Recently, the existence of the random attractor associated to the random dynamical system generated by the solution of system (1.3)-(1.4) (or (1.5)-(1.6)) was studied in [5–7] by using a function Holling type-II, $\mu(S(t))=mS(t)/(a+S(t))$, where a is the half-saturation constant and m is the maximal consumption rate of the nutrient and also the maximal specific growth rate of microorganisms. In particular, authors in [6] proved the existence of the global random attractor of system (1.5)-(1.6) with Holling type-II respond function, and further shown that the internal structure of the attractor consists of singleton subsets as long as $\bar{D}=D+\alpha^2/2>m$, which means that the microorganisms become extinct. In fact, one can choose α, large enough, such that $\bar{D}>m$ (see Figure 2 in [6]). In case $\bar{D}<m$, one cannot ensure the persistence of the microorganism (see Figure 1 in [6]).

As far as we know, no report has been found on the existence of random attractors of stochastic chemostat model under small noise. This fact inspires us to further explore relevant dynamics of system (1.5)-(1.6) in this respect. Besides, some experiments and observations indicate that not only insufficient nutrient but also excessive nutrient may inhibit the growth of a microbial population in the chemostat [1, 3, 20]. This situation suggested a non-monotonic response function, so-called Monod-Haldane function, to model such growth. Thus system (1.5)-(1.6) becomes the following specified form

$$dS(t)=[(S^0-S(t))D-\frac{mS(t)x(t)}{a+S(t)+KS^2(t)}]dt-\alpha S(t)dW(t), \tag{1.7}$$
$$dx(t)=[-Dx(t)+\frac{mS(t)x(t)}{a+S(t)+KS^2(t)}]dt-\alpha x(t)dW(t), \tag{1.8}$$

where the term $KS^2(t)$ describes the inhibitory effect of the substrate at high concentrations. By using the well-known conversion between Itô and Stratonovich senses, we obtain the following stochastic chemostat with Monod-Haldane function

$$dS(t)=[-\bar{D}S(t)-\frac{mS(t)x(t)}{a+S(t)+KS^2(t)}+S^0D]dt-\alpha S(t)\circ dW(t), \tag{1.9}$$

$$dx(t) = [-\bar{D}x(t) + \frac{mS(t)x(t)}{a + S(t) + KS^2(t)}]dt - \alpha x(t) \circ dW(t), \tag{1.10}$$

where $\bar{D} = D + \alpha^2/2$.

By the numerical scheme [21], we present the numerical simulation for the small noise situation, say, $\alpha = 0.1$. At this point, it follows that

$$\bar{D} = 2 + 0.1^2/2 = 2.005 < 5 = m.$$

Then we first display the phase plane (S, x) of the dynamics of our chemostat model, where the blue dashed lines represent the solutions of the deterministic (i.e., with $\alpha = 0$) and the other ones are different realizations for the stochastic chemostat model (1.9)-(1.10). In addition, we set the parameters $S^0 = 1$, $D = 2$, $a = 0.6$, $m = 5$, $\alpha = 0.1$ and initial conditions $S_0 = 5$, $x_0 = 10$, and consider $K = 0.1$ (the persistence case) and $K = 1$ (the extinction case), shown in Figure 1.

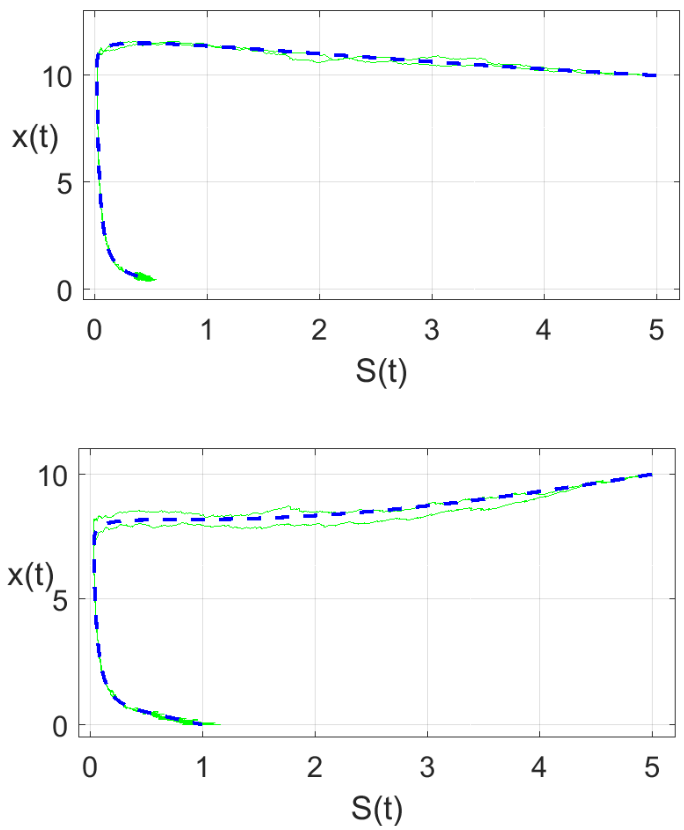

Figure 1. Stochastic chemostat system (1.9)-(1.10) with parameters $S^0 = 1$, $D = 2$, $a = 0.6$, $m = 5$, $S_0 = 5$, $x_0 = 10$, $\alpha = 0.1$, $K = 0.1$ (up) and $K = 1$ (down).

The novelty of this paper is to establish the existence of the random attractor associated to the random dynamical system generated by the solution to system (1.9)-(1.10) for both large and small α, which extends and improves some known results. In particular, we obtain the extinction conditions

$$\bar{D} \geq m \quad \text{or} \quad \bar{D} < m < (1 + 2\sqrt{aK})\bar{D},$$

which extend and improve the results in [5–7]. The (α, K) plane can be used to display the above inequalities, shown in Figure 2.

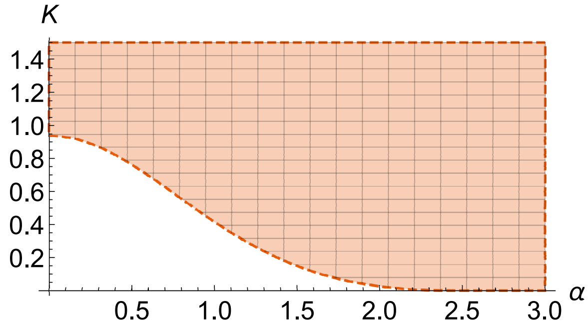

Figure 2. Extinction conditions for system (1.9)-(1.10) with parameters $D = 2$, $a = 0.6$, $m = 5$.

The rest of the paper is organized as follows. In Section 2 we recall some basic results on random dynamical systems, random attractors, Ornstein-Uhlenbeck (O-U) process and Wong-Zakai approximation. In Section 3 we prove the existence of the random attractors of the solutions and its internal structure explicitly. Finally, in Section 4 we prove the convergence of solutions of Wong-Zakai approximations and the upper semicontinuity of random attractors of the approximate random system as the size of approximation approaches zero.

2. Preliminaries

In this section, we recall some basic results [2] on random dynamical systems (RDSs), random attractors and Ornstein-Uhlenbeck (O-U) process in order to make our presentation as much self-contained as possible.

2.1. Random dynamical systems

We first recall some basic definitions on random dynamical system (RDS), two necessary lemmas on random attractors, and refer to the monograph [2] for more detailed information.

Let $(X; \|\cdot\|_X)$ be a separable Banach space.

Definition 2.1. A RDS on X consists of two ingredients: (a) a metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta\}_{t\in\mathbb{R}})$ where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and the family of mappings $\theta_t : \Omega \to \Omega$ satisfies

(1) $\theta_0 = \mathrm{Id}_\Omega$,

(2) $\theta_s \circ \theta_t = \theta_{s+t}$ for all $s, t \in \mathbb{R}$,

(3) the mapping $(t, \omega) \mapsto \theta_t\omega$ is measurable,

(4) the probability measure $\mathbb{P}$ is preserved by θ_t, i.e., $\theta_t\mathbb{P} = \mathbb{P}$,

and (b) a mapping $\varphi : [0, \infty) \times \Omega \times X \to X$ which is $(\mathcal{B}[0, \infty) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$-measurable, such that for each $\omega \in \Omega$,

(i) the mapping $\varphi(t,\omega) : X \to X$, $x \mapsto \varphi(t,\omega)x$ is continuous for every $t \geq 0$,
(ii) $\varphi(0,\omega)$ is the identity operator on X,
(iii) $\varphi(t+s,\omega) = \varphi(t,\theta_s\omega)\varphi(s,\omega)$ for all $s,t \geq 0$.

Definition 2.2. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space. A random set K is a measurable subset of $X \times \Omega$ with respect to the product σ-algebra $\mathcal{B}(X) \times \mathcal{F}$. Moreover K will be said a closed or a compact random set if $K(\omega) = \{x : (x,\omega) \in K\}$, $\omega \in \Omega$, is closed or compact for $\mathbb{P}$-almost all $\omega \in \Omega$, respectively.

Definition 2.3. A bounded random set $K(\omega) \subset X$ is said to be tempered with respect to $\{\theta_t\}_{t\in\mathbb{R}}$ if for a.e. $\omega \in \Omega$,

$$\lim_{t\to\infty} e^{-\beta t} \sup_{x\in K(\theta_{-t}\omega)} \|x\|_X = 0, \quad \text{for all } \beta > 0;$$

a random variable $\omega \mapsto r(\omega) \in \mathbb{R}$ is said to be tempered with respect to $\{\theta_t\}_{t\in\mathbb{R}}$ if for a.e. $\omega \in \Omega$,

$$\lim_{t\to\infty} e^{-\beta t} \sup_{t\in\mathbb{R}} |r(\theta_{-t}\omega)| = 0, \quad \text{for all } \beta > 0;$$

Definition 2.4. A random set $B(\omega) \subset X$ is called a random absorbing set in $\mathcal{E}(X)$ if for any $E \in \mathcal{E}(X)$ and a.e. $\omega \in \Omega$, there exists $T_E(\omega) > 0$ such that

$$\varphi(t,\theta_{-t}\omega)E(\theta_{-t}\omega) \subset B(\omega), \quad \text{for all } t \geq T_E(\omega).$$

Definition 2.5. Let $\{\varphi(t,\omega)\}_{t\geq 0,\omega\in\Omega}$ be an RDS over $(\Omega,\mathcal{F},\mathbb{P},\{\theta\}_{t\in\mathbb{R}})$ with state space X and let $A(\omega)(\subset X)$ be a random set. Then $\mathcal{A} = \{A(\omega)\}_{\omega\in\Omega}$ is called a global random $\mathcal{E}$-attractor (or pullback $\mathcal{E}$-attractor) for $\{\varphi(t,\omega)\}_{t\geq 0,\omega\in\Omega}$ if
(i) (compactness) $A(\omega)$ is a compact set of X for any $\omega \in \Omega$,
(ii) (invariance) for any $\omega \in \Omega$ and all $t \geq 0$, it holds

$$\varphi(t,\omega)A(\omega) = A(\theta_t\omega),$$

(iii) (attracting property) for any $E \in \mathcal{E}(X)$ and a.e. $\omega \in \Omega$,

$$\lim_{t\to\infty} \mathrm{dist}_X(\varphi(t,\theta_{-t}\omega)E(\theta_{-t}\omega), A(\omega)) = 0,$$

where $\mathrm{dist}_X(G,H) = \sup_{g\in G}\inf_{h\in H}\|g-h\|_X$ is the Hausdorff semi-metric for $G,H \subseteq X$.

Lemma 2.1 ([11,14]). *Let $B \in \mathcal{E}(X)$ be a closed absorbing set for the continuous RDS $\{\varphi(t,\omega)\}_{t\geq 0,\omega\in\Omega}$ that satisfies the asymptotic compactness condition for a.e. $\omega \in \Omega$, i.e., each sequence $x_n \in \varphi(t_n,\theta_{-t_n}\omega)B(\theta_{-t_n}\omega)$ has a convergent subsequence in X when $t_n \to \infty$. Then φ has a unique global random attractor $\mathcal{A} = \{A(\omega)\}_{\omega\in\Omega}$ with component subsets*

$$A(\omega) = \cap_{\tau\geq T_B(\omega)}\overline{\cup_{t\geq\tau}\varphi(t_n,\theta_{-t_n}\omega)B(\theta_{-t_n}\omega)}.$$

Lemma 2.2 ([8,10]). *Let φ_u be an RDS on X. Suppose that the mapping $\mathcal{T} : \Omega \times X \to X$ possesses the following properties: for fixed $\omega \in \Omega$, $\mathcal{T}(\omega,\cdot)$ is a homeomorphism on X, and for $x \in X$, the mappings $\mathcal{T}(\cdot,x)$, $\mathcal{T}^{-1}(\cdot,x)$ are measurable. Then the mapping*

$$(t,\omega,x) \to \varphi_v(t,\omega)x := \mathcal{T}^{-1}(\theta_t\omega,\varphi_u(t,\omega)\mathcal{T}(\omega,x))$$

is a (conjugated) RDS.

2.2. Ornstein-Uhlenbeck process

Let W be a two sided Wiener process. Kolmogorov's theorem [25] ensures that W has a continuous version, that we will denote by ω, whose canonical interpretation is as follows: let Ω be defined by

$$\Omega = \{\omega \in \mathcal{C}(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\} = \mathcal{C}_0(\mathbb{R}, \mathbb{R}),$$

$\mathcal{F}$ the Borel σ-algebra on Ω generated by the compact open topology and $\mathbb{P}$ the corresponding Wiener measure on $\mathcal{F}$. We consider the Wiener shift flow given by

$$\theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad t \in \mathbb{R},$$

then $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ is a metric dynamical system.

Now we introduce the following Ornstein-Uhlenbeck process on $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$

$$z^*(\theta_t \omega) = -\int_{-\infty}^{0} e^s \theta_t \omega(s) ds, \quad t \in \mathbb{R}, \omega \in \Omega,$$

which solves the following Langevin equation

$$dz + zdt = d\omega(t), \quad t \in \mathbb{R}. \tag{2.1}$$

Lemma 2.3 ([2,10]). *There exists a θ_t-invariant set $\tilde{\Omega} \in \mathcal{F}$ of Ω of full $\mathbb{P}$ measure such that for $\omega \in \tilde{\Omega}$, we have*

(i) the random variable $|z^*(\omega)|$ is tempered;

(ii) the mapping

$$(t, \omega) \to z^*(\theta_t \omega) = -\int_{-\infty}^{0} e^s \omega(t+s) ds + \omega(t)$$

is a stationary solution of (2.1) with continuous trajectories;

(iii) in addition, for any $\omega \in \tilde{\Omega}$:

$$\lim_{t\to\pm\infty} \frac{|z^*(\theta_t \omega)|}{t} = 0;$$
$$\lim_{t\to\pm\infty} \frac{1}{t} \int_0^t z^*(\theta_s \omega) ds = 0;$$
$$\lim_{t\to\pm\infty} \frac{1}{t} \int_0^t |z^*(\theta_s \omega)| ds = \mathbb{E}[z^*] < \infty.$$

In what follows we will consider the restriction of the Wiener shift θ to the set $\tilde{\Omega}$, and we restrict accordingly the metric dynamical system to this set, that is also a metric dynamical system, see [8]. For simplicity, we will still denote the restricted metric dynamical system by the old symbols $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta\}_{t\in\mathbb{R}})$. From now on, we denote $\mathcal{X} = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$.

2.3. Wong-Zakai approximations for white noise

In this subsection, we first review the well-known Wong-Zakai approximation for Gaussian white noise [23,24,27,28]. In fact, for each $\delta > 0$, we define the random variable $\mathcal{G}_\delta : \Omega \to \mathbb{R}$ by

$$\mathcal{G}_\delta(\omega) = \frac{1}{\delta} \omega(\delta).$$

Then we have

$$\mathcal{G}_\delta(\theta_t\omega) = \frac{1}{\delta}(\omega(\delta + t) - \omega(t)). \tag{2.2}$$

It was shown that $\mathcal{G}_\delta(\theta_t\omega)$ is a stationary stochastic process with a normal distribution, but it is unbounded in t for almost all ω. We next recall a useful claim that $\mathcal{G}_\delta(\theta_t\omega)$ can be viewed as an approximation of white noise in the Wong-Zakai sense.

Lemma 2.4 ([24,27]). *For each $\omega \in \Omega$, $\tau \in \mathbb{R}$, $T > 0$. Then for all $\epsilon > 0$, there exists a constant $\delta_0 = \delta_0(\epsilon, \omega, \tau, T) > 0$ such that for $0 < \delta < \delta_0$ and $t \in [\tau, \tau + T]$, we have*

$$\left|\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds - \omega(t)\right| < \epsilon.$$

Next we consider the Wong-Zakai approximation of the Langevin equation (2.1)

$$\dot{z}_\delta + z_\delta = \mathcal{G}_\delta(\theta_t\omega) \tag{2.3}$$

and present the pointwise convergence between them as follow:

Lemma 2.5 ([28]). *There exists a θ_t-invariant set $\tilde{\Omega} \in \mathcal{F}$ of Ω of full $\mathbb{P}$ measure such that for $\omega \in \tilde{\Omega}$, we have*

(i) the random variable

$$z^*_\delta(\omega) = \int_{-\infty}^0 e^s \mathcal{G}_\delta(\theta_s\omega)ds$$

exists and $|z^*_\delta(\omega)|$ is tempered;

(ii) the mapping

$$(t,\omega) \to z^*_\delta(\theta_t\omega) = \int_{-\infty}^0 e^s \mathcal{G}_\delta(\theta_{s+t}\omega)ds$$

is a stationary solution of (2.3) with continuous trajectories;

(iii) in addition, for any $\omega \in \tilde{\Omega}$:

$$\lim_{t\to\pm\infty} \frac{|z^*_\delta(\theta_t\omega)|}{t} = 0;$$
$$\lim_{t\to\pm\infty} \frac{1}{t}\int_0^t z^*_\delta(\theta_s\omega)ds = 0;$$
$$\lim_{t\to\pm\infty} \frac{1}{t}\int_0^t |z^*_\delta(\theta_s\omega)|ds = \mathbb{E}[z^*_\delta] < \infty;$$

(iv) in particular, for any $\omega \in \tilde{\Omega}$, any $\tau \in \mathbb{R}$ and $T > 0$, we have

$$\lim_{\delta\to 0} \|z^*_\delta(\theta_t\omega) - z^*(\theta_t\omega)\|_{C([\tau,\tau+T])} = 0. \tag{2.4}$$

3. The existence of random attractors

In this section, we will first transform our stochastic chemostat model (1.9)-(1.10) into random one by using the Ornstein-Uhlenbeck process. After proving that this random system possesses a unique nonnegative solution for any initial value, we analyze the existence of random attractors associated to the random dynamical system generated by the solution. By means of the conjugacy, we also prove the existence of random attractors for original stochastic model.

To do this, we introduce two new variables σ and κ as follows

$$\sigma(t) = S(t)e^{\alpha z^*(\theta_t\omega)}, \tag{3.1}$$

$$\kappa(t) = x(t)e^{\alpha z^*(\theta_t\omega)}. \tag{3.2}$$

For the sake of simplicity, we will rewrite S, x, z^*, σ, and κ instead of $S(t)$, $x(t)$, $z^*(\theta_t\omega)$, $\sigma(t)$, and $\kappa(t)$, respectively. Hence, direct calculation implies that

$$\frac{d\sigma}{dt} = -(\bar{D} + \alpha z^*)\sigma - \frac{m\sigma\kappa e^{-\alpha z^*}}{a + \sigma e^{-\alpha z^*} + K\sigma^2 e^{-2\alpha z^*}} + S^0 D e^{\alpha z^*}, \tag{3.3}$$

$$\frac{d\kappa}{dt} = -(\bar{D} + \alpha z^*)\kappa + \frac{m\sigma\kappa e^{-\alpha z^*}}{a + \sigma e^{-\alpha z^*} + K\sigma^2 e^{-2\alpha z^*}}. \tag{3.4}$$

Lemma 3.1. *For any $\omega \in \Omega$ and any initial value $u_0 := (\sigma_0, \kappa_0) \in \mathcal{X}$, where σ_0 and κ_0 stand for $\sigma(0)$ and $\kappa(0)$ respectively, the system (3.3)-(3.4) possesses a unique global solution $u(\cdot;\omega,u_0) := (\sigma(\cdot;\omega,u_0), \kappa(\cdot;\omega,u_0)) \in \mathcal{C}^1([0,\infty),\mathcal{X})$ with $u(0;\omega,u_0) = u_0$. Moreover the solution mapping generates a RDS $\varphi_u : \mathbb{R}^+ \times \Omega \times \mathcal{X} \to \mathcal{X}$ defined as $\varphi_u(t,\omega)u_0 = u(t;\omega,u_0)$ for all $t \in \mathbb{R}^+$, $\omega \in \Omega$, $u_0 \in \mathcal{X}$.*

Proof. Obviously, the system (3.3)-(3.4) is rewritten as

$$\frac{d\sigma}{dt} = -(\bar{D} + \alpha z^*)\sigma - m\kappa + \frac{(ma + mK\sigma^2 e^{-2\alpha z^*})\kappa}{a + \sigma e^{-\alpha z^*} + K\sigma^2 e^{-2\alpha z^*}} + S^0 D e^{\alpha z^*}, \tag{3.5}$$

$$\frac{d\kappa}{dt} = -(\bar{D} - m + \alpha z^*)\kappa - \frac{(ma + mK\sigma^2 e^{-2\alpha z^*})\kappa}{a + \sigma e^{-\alpha z^*} + K\sigma^2 e^{-2\alpha z^*}}. \tag{3.6}$$

Letting

$$L(\theta_t\omega) = \begin{pmatrix} -(\bar{D} + \alpha z^*) & -m \\ 0 & -(\bar{D} - m + \alpha z^*) \end{pmatrix}$$

and $F : \mathcal{X} \times \Omega \to \mathbb{R}^2$ of form

$$F(\eta,\omega) = \begin{pmatrix} \frac{(ma + mK\eta_1^2 e^{-2\alpha z^*(\omega)})\eta_2}{a + \eta_1 e^{-\alpha z^*(\omega)} + K\eta_1^2 e^{-2\alpha z^*(\omega)}} + S^0 D e^{\alpha z^*(\omega)} \\ -\frac{(ma + mK\eta_1^2 e^{-2\alpha z^*(\omega)})\eta_2}{a + \eta_1 e^{-\alpha z^*(\omega)} + K\eta_1^2 e^{-2\alpha z^*(\omega)}} \end{pmatrix}$$

where $\eta = (\eta_1, \eta_2) \in \mathcal{X}$, system (3.5)-(3.6) turns into

$$\frac{du}{dt} = L(\theta_t\omega)u + F(u, \theta_t\omega). \tag{3.7}$$

On the one hand, the operator L generates an evolution system on $\mathbb{R}^2$ because $t \to z^*(\theta_t\omega)$ is continuous. On the other hand, $F(\cdot, \theta_t\omega) \in \mathcal{C}(\mathcal{X} \times [0,\infty); \mathbb{R}^2)$ and is continuously differentiable with respect to the variables (η_1, η_2) since

$$\frac{\partial}{\partial\eta_1}[\pm\frac{(ma + mK\eta_1^2 e^{-2\alpha z^*})\eta_2}{a + \eta_1 e^{-\alpha z^*} + K\eta_1^2 e^{-2\alpha z^*}} + \tilde{C}] = \mp\frac{(mae^{-\alpha z^*} + mK\eta_1^2 e^{-3\alpha z^*})\eta_2}{(a + \eta_1 e^{-\alpha z^*} + K\eta_1^2 e^{-2\alpha z^*})^2}$$

and

$$\frac{\partial}{\partial \eta_2}[\pm \frac{(ma + mK\eta_1^2 e^{-2\alpha z^*})\eta_2}{a + \eta_1 e^{-\alpha z^*} + K\eta_1^2 e^{-2\alpha z^*}} + \tilde{C}] = \pm \frac{(ma + mK\eta_1^2 e^{-2\alpha z^*})}{a + \eta_1 e^{-\alpha z^*} + K\eta_1^2 e^{-2\alpha z^*}}$$

where $\tilde{C}$ is some constant which does not depend on $(\eta_1, \eta_2) \in \mathcal{X}$. This further implies that $F(\cdot, \theta_t\omega)$ is locally Lipschitz with respect to $(\eta_1, \eta_2) \in \mathcal{X}$. Therefore, system (3.7) possesses a unique local solution according to classical results from the theory of ordinary differential equations.

Next we show that this solution is in fact a global one. To this end, we define $V(t) = \sigma(t) + \kappa(t)$ and obtain that

$$\frac{dV}{dt} = -(\bar{D} + \alpha z^*)V + S^0 D e^{\alpha z^*}.$$

By variation of constants, the corresponding solution is given by

$$V(t) = V(0)e^{-(\bar{D}t + \alpha\int_0^t z^* ds)} + S^0 D \int_0^t e^{\alpha z^*} e^{-(\bar{D}(t-s) + \alpha\int_s^t z^* dr)} ds \tag{3.8}$$

which is clearly bounded since $\bar{D} > 0$. Moreover, it follows from (3.3) that

$$\frac{d\sigma}{dt} \leq -(\bar{D} + \alpha z^*)\sigma + S^0 D e^{\alpha z^*}.$$

By comparison principle and variation of constants, we have

$$\sigma(t) \leq \sigma(0)e^{-(\bar{D}t + \alpha\int_0^t z^* ds)} + S^0 D \int_0^t e^{\alpha z^*} e^{-(\bar{D}(t-s) + \alpha\int_s^t z^* dr)} ds$$

which ensures that σ is bounded either. Note that $\kappa(t) = V(t) - \sigma(t)$, thus κ is also bounded since V and σ are bounded in both cases. Therefore, the unique local solution to system (3.7) can be extended to a unique global one.

We now turn to prove that the global solution of (3.7) belongs to the set $\mathcal{X}$ for any $t \in \mathbb{R}^2$. Suppose that if $\sigma(t) = 0$ for some $t = t^* \in \mathbb{R}^+$, it follows from (3.3) that

$$\frac{d\sigma}{dt}\Big|_{t*} = S^0 D e^{\alpha z^*} > 0.$$

In addition, given $(\sigma_0, 0)$ with $\sigma_0 > 0$, there exists a unique solution of system (3.7) satisfying $\sigma(t_0) = \sigma_0$ and $\kappa(t_0) = 0$ for some initial time $t_0 \geq 0$. Specifically, this unique solution is given by

$$\sigma(t) = \sigma(t_0)e^{-(\bar{D}t + \alpha\int_{t_0}^t z^* ds)} + S^0 D \int_{t_0}^t e^{\alpha z^*} e^{-(\bar{D}(t-s) + \alpha\int_s^t z^* dr)} ds, \tag{3.9}$$

$$\kappa(t) \equiv 0. \tag{3.10}$$

For $(\sigma_0, \kappa_0) \in \mathcal{X}$, there exists a unique solution $(\sigma(t), \kappa(t))$ such that $\sigma(0) = \sigma_0$ and $\kappa(0) = \kappa_0$. If there is some first $t^* > 0$ such that $\kappa(t^*) = 0$, then we obtain that $(\sigma(\cdot), \kappa(\cdot))$ is the unique solution of system (3.7) with $\sigma(t^*) = \sigma^*$ and $\kappa(t^*) = 0$. Meanwhile, $\kappa(t) > 0$ for all $0 < t < t^*$; however, we already have another solution $(\sigma(t), 0)$ given by (3.9)-(3.10) for all $t \geq t^* - \epsilon$ (for any $\epsilon > 0$ small enough) for this problem, so it leads to a contradiction. As a result, we deduce that for any initial data $u_0 \in \mathcal{X}$ the solution $u(t)$ remains in $\mathcal{X}$.

Finally, we define the solution mapping $\varphi_u : \mathbb{R} \times \Omega \times \mathcal{X} \to \mathcal{X}$ of the form

$$\varphi_u(t,\omega)u_0 := u(t;\omega,u_0), \quad \text{for all } t \geq 0, \omega \in \Omega, u_0 \in \mathcal{X}.$$

Since the function F is continuous in (u,t), and is measurable in ω, we obtain that the mapping φ_u is $(\mathcal{B}[0,\infty) \times \mathcal{F} \times \mathcal{B}(\mathcal{X}), \mathcal{B}(\mathcal{X}))$-measure. It then follows that (3.7) generate the continuous RDS $\varphi_u(t,\omega)(\cdot)$. □

Lemma 3.2. *The RDS* $\{\varphi_u(t,\omega)\}_{t\geq 0,\omega\in\Omega}$ *has a tempered compact random absorbing set* $B(\omega) \in \mathcal{E}(\mathcal{X})$.

Proof. Notice that $\varphi_u(t,\theta_{-t}\omega)u_0 = u(t;\theta_{-t}\omega,u_0)$ denotes the solution of the random chemostat model (3.3)-(3.4), satisfying that $u(0;\theta_{-t}\omega,u_0) = u_0$, where $u_0 = u_0(\theta_{-t}\omega) \in E(\theta_{-t}\omega)$. Thus we introduce the following norm

$$\begin{aligned}\|\varphi_u(t,\theta_{-t}\omega)u_0\| &= \|u(t;\theta_{-t}\omega,u_0(\theta_{-t}\omega))\| \\ &= \sigma(t;\theta_{-t}\omega,u_0(\theta_{-t}\omega)) + \kappa(t;\theta_{-t}\omega,u_0(\theta_{-t}\omega)).\end{aligned}$$

Noting that $V = \sigma + \kappa$ and replacing ω by $\theta_{-t}\omega$ in (3.8), we obtain

$$\begin{aligned}&\sigma(t;\theta_{-t}\omega,u_0(\theta_{-t}\omega)) + \kappa(t;\theta_{-t}\omega,u_0(\theta_{-t}\omega)) \\ &\quad = (\sigma(0)+\kappa(0))e^{-(\bar{D}t+\alpha\int_0^t z^*(\theta_{s-t}\omega)ds)} \\ &\qquad + S^0 D\int_0^t e^{\alpha z^*(\theta_{s-t}\omega)}e^{-(\bar{D}(t-s)+\alpha\int_s^t z^*(\theta_{r-t}\omega)dr)}ds \\ &\quad = (\sigma(0)+\kappa(0))e^{-(\bar{D}t+\alpha\int_{-t}^0 z^*(\theta_s\omega)ds)} \\ &\qquad + S^0 D\int_0^t e^{\alpha z^*(\theta_{s-t}\omega)}e^{-(\bar{D}(t-s)+\alpha\int_{s-t}^0 z^*(\theta_r\omega)dr)}ds \\ &\quad = (\sigma(0)+\kappa(0))e^{-(\bar{D}t+\alpha\int_{-t}^0 z^*(\theta_s\omega)ds)} \\ &\qquad + S^0 D\int_0^t e^{\alpha z^*(\theta_{-\tau}\omega)}e^{-(\bar{D}\tau+\alpha\int_{-\tau}^0 z^*(\theta_r\omega)dr)}d\tau \\ &\quad = (\sigma(0)+\kappa(0))e^{-\bar{D}t-\alpha\int_{-t}^0 z^*(\theta_s\omega)ds} \\ &\qquad + S^0 D\int_0^t e^{\tau[-\bar{D}+\frac{\alpha z^*(\theta_{-\tau}\omega)}{\tau}-\frac{\alpha}{\tau}\int_{-\tau}^0 z^*(\theta_r\omega)dr]}d\tau.\end{aligned}$$

Taking the limits on the previous equation, we further have

$$\lim_{t\to\infty}[\sigma(t;\theta_{-t}\omega,u_0(\theta_{-t}\omega)) + \kappa(t;\theta_{-t}\omega,u_0(\theta_{-t}\omega))] = S^0 D\rho^*(\omega),$$

where $\rho^*(\omega)$ is defined by

$$\rho^*(\omega) = \int_0^\infty e^{\tau[-\bar{D}+\frac{\alpha z^*(\theta_{-\tau}\omega)}{\tau}-\frac{\alpha}{\tau}\int_{-\tau}^0 z^*(\theta_r\omega)dr]}d\tau.$$

Moreover, $\rho^*(\omega)$ has sub-exponential growth due to the fact

$$e^{\tau[-\bar{D}+\frac{\alpha z^*(\theta_{-\tau}\omega)}{\tau}-\frac{\alpha}{\tau}\int_{-\tau}^0 z^*(\theta_r\omega)dr]} \to 0 \quad \text{if} \quad \tau\to\infty.$$

Therefore, for any given $\varepsilon > 0$, there exists $T_E(\omega,\varepsilon) > 0$ such that

$$|\|u(t;\theta_{-t}\omega,u_0(\theta_{-t}\omega))\| - S^0 D\rho^*(\omega)| \leq \varepsilon$$

for all $u_0 \in E(\theta_{-t}\omega)$ and $t \geq T_E(\omega, \varepsilon)$. We now define the set

$$B_\varepsilon(\omega) := \{(\sigma, \kappa) \in \mathcal{X} : S^0 D\rho^*(\omega) - \varepsilon \leq \sigma + \kappa \leq S^0 D\rho^*(\omega) + \varepsilon\},$$

thus $B_\varepsilon(\omega) \in \mathcal{E}$ is absorbing in $\mathcal{X}$ for any $\varepsilon > 0$. Hence, the proof of Lemma 3.2 is complete. □

Lemma 3.3. *If one of the following conditions holds*

(i) $\bar{D} \geq m$,

(ii) $\bar{D} < m < (1 + 2\sqrt{aK})\bar{D}$.

Then the random attractor associated to the RDS φ_u *has the following structure*

$$\mathcal{A} = \{A(\omega)\}_{\omega\in\Omega}, \quad A(\omega) = (S^0 D\rho^*(\omega), 0).$$

Proof. According to Lemma 2.1, the RDS φ_u possesses a unique random attractor given by $\mathcal{A} = \{A(\omega)\}_{\omega\in\Omega} \subset B_\varepsilon(\omega)$ for any $\varepsilon > 0$. Thus $\mathcal{A} = \{A(\omega)\}_{\omega\in\Omega} \subset B_0(\omega)$. In other word, we have the following expression for each component of our attractor

$$A(\omega) := (S^0 D\rho^*(\omega) - \kappa(\omega), \kappa(\omega)). \tag{3.11}$$

Denote by

$$g(\zeta) = \frac{m\zeta}{a + \zeta + K\zeta^2} - \bar{D}, \quad \zeta = \sigma e^{-\alpha z^*} > 0.$$

If (i) holds, there exists $c_1 > 0$ such that

$$g(\zeta) = \frac{m\zeta}{a + \zeta + K\zeta^2} - \bar{D} = \frac{-\bar{D}a - (\bar{D} - m)\zeta - \bar{D}K\zeta^2}{a + \zeta + K\zeta^2} \leq -c_1 < 0.$$

Then it follows from (3.4) that

$$\frac{d\kappa}{dt} = -(\bar{D} + \alpha z^*)\kappa + (g(\zeta) + \bar{D})\kappa \leq (-c_1 + \alpha z^*)\kappa. \tag{3.12}$$

If (ii) holds, there exists $c_2 > 0$ such that

$$g(\zeta) \leq \sup_{\zeta\in(0,\infty)} g(\zeta) = \frac{m - (1 + 2\sqrt{aK})\bar{D}}{1 + 2\sqrt{aK}} \leq -c_2 < 0.$$

Then it follows from (3.4) that

$$\frac{d\kappa}{dt} = -(\bar{D} + \alpha z^*)\kappa + (g(\zeta) + \bar{D})\kappa \leq (-c_2 + \alpha z^*)\kappa. \tag{3.13}$$

Thus, both solutions to (3.12) and (3.13) tend to zero after replacing ω by $\theta_{-t}\omega$ and making $t \to \infty$. Therefore, the internal structure of the attractor in both cases consists of singleton subsets $A(\omega) = (S^0 D\rho^*(\omega), 0)$. □

We are now in position to present the main result of this section.

Theorem 3.1. *If one of the following conditions holds*

(i) $\bar{D} \geq m$,

(ii) $\bar{D} < m < (1 + 2\sqrt{aK})\bar{D}$.

Then the random attractor associated to the RDS generated by the original system (1.9)-(1.10) also has the structure composed of singleton subsets.

Proof. We define a mapping $\mathcal{T}:\Omega\times\mathcal{X}\to\mathcal{X}$ and its inverse, respectively, as

$$\mathcal{T}(\omega,\zeta)=(\zeta_1e^{\alpha z^*(\omega)},\zeta_2e^{\alpha z^*(\omega)})$$

and

$$\mathcal{T}^{-1}(\omega,\zeta)=(\zeta_1e^{-\alpha z^*(\omega)},\zeta_2e^{-\alpha z^*(\omega)}).$$

Denote $v(t)=(S(t),x(t))$ and using (3.1)-(3.2), we have that $v=ue^{-\alpha z^*(\omega)}$. Since $\mathcal{T}$ is a homeomorphism, thanks to Lemma 2.2 we obtain the following conjugated RDS

$$\begin{aligned}\varphi_v(t,\omega)v_0&=\mathcal{T}^{-1}(\theta_t\omega,\varphi_u(t,\omega)\mathcal{T}(\omega,v_0))\\&=\mathcal{T}^{-1}(\theta_t\omega,\varphi_u(t,\omega)u_0)\\&=\mathcal{T}^{-1}(\theta_t\omega,u(t;\omega,u_0))\\&=v(t;\omega,v_0)\end{aligned}$$

implying that φ_v is an RDS for our original stochastic system (1.9)-(1.10).

Then the global random attractor of the random chemostat system (3.3)-(3.4), $\mathcal{A}=\{A(\omega)\}_{\omega\in\Omega}\subset B_0(\omega)$, becomes into $\mathcal{A}^{\mathcal{T}}=\{A^{\mathcal{T}}(\omega)\}_{\omega\in\Omega}\subset B_0^{\mathcal{T}}(\omega)$, the global random attractor of the stochastic chemostat system (1.9)-(1.10), where

$$B_0^{\mathcal{T}}(\omega):=\{(S,x)\in\mathcal{X}:S+x=S^0D\rho^*(\omega)e^{-\alpha z^*(\omega)}\}.$$

Thus each component $A^{\mathcal{T}}(\omega)$, $\omega\in\Omega$ of the attractor can be written as

$$A^{\mathcal{T}}(\omega):=(S^0D\rho^*(\omega)e^{-\alpha z^*(\omega)}-Se^{-\alpha z^*(\omega)},Se^{-\alpha z^*(\omega)})$$

Moreover, we know that the internal structure of the attractor consists of singleton subsets

$$A^{\mathcal{T}}(\omega):=(S^0D\rho^*(\omega)e^{-\alpha z^*(\omega)},0)$$

as long as condition (i) or (ii) holds. □

4. Wong-Zakai approximations and convergency

In this section, we will first approximate the solutions of the stochastic chemostat model (1.9)-(1.10) by its Wong-Zakai approximation:

$$\frac{dS_\delta(t)}{dt}=-\bar{D}S_\delta(t)-\frac{mS_\delta(t)x_\delta(t)}{a+S_\delta(t)+KS_\delta^2(t)}+S^0D-\alpha S_\delta(t)\mathcal{G}_\delta(\theta_t\omega),\tag{4.1}$$

$$\frac{dx_\delta(t)}{dt}=-\bar{D}x_\delta(t)+\frac{mS_\delta(t)x_\delta(t)}{a+S_\delta(t)+KS_\delta^2(t)}-\alpha x_\delta(t)\mathcal{G}_\delta(\theta_t\omega),\tag{4.2}$$

where $\bar{D}=D+\alpha^2/2$.

To better compare the relations between the solutions of (1.9)-(1.10) and (4.1)-(4.2), as we did for (1.9)-(1.10), we introduce a similar transformation for (4.1)-(4.2). Let

$$\sigma_\delta(t)=S_\delta(t)e^{\alpha z_\delta^*(\theta_t\omega)},\tag{4.3}$$

$$\kappa_\delta(t)=x_\delta(t)e^{\alpha z_\delta^*(\theta_t\omega)}.\tag{4.4}$$

Similarly, we will rewrite S_δ, x_δ, z^*_δ, σ_δ, and κ_δ instead of $S_\delta(t)$, $x_\delta(t)$, $z^*_\delta(\theta_t\omega)$, $\sigma_\delta(t)$, and $\kappa_\delta(t)$, respectively. Then we get that

$$\frac{d\sigma_\delta}{dt} = -(\bar{D} + \alpha z^*_\delta)\sigma_\delta - \frac{m\sigma_\delta\kappa_\delta e^{-\alpha z^*_\delta}}{a + \sigma_\delta e^{-\alpha z^*_\delta} + K\sigma^2_\delta e^{-2\alpha z^*_\delta}} + S^0 De^{\alpha z^*_\delta}, \tag{4.5}$$

$$\frac{d\kappa_\delta}{dt} = -(\bar{D} + \alpha z^*_\delta)\kappa_\delta + \frac{m\sigma_\delta\kappa_\delta e^{-\alpha z^*_\delta}}{a + \sigma_\delta e^{-\alpha z^*_\delta} + K\sigma^2_\delta e^{-2\alpha z^*_\delta}}. \tag{4.6}$$

We first prove that this random dynamical system possesses a unique nonnegative solution for any initial value.

Lemma 4.1. *For any $\omega \in \Omega$ and any initial value $u_{\delta 0} := (\sigma_{\delta 0}, \kappa_{\delta 0}) \in \mathcal{X}$, where $\sigma_{\delta 0}$ and $\kappa_{\delta 0}$ stand for $\sigma_\delta(0)$ and $\kappa_\delta(0)$ respectively, the system (4.5)-(4.6) possesses a unique global solution $u_\delta(\cdot;\omega,u_{\delta 0}) := (\sigma_\delta(\cdot;\omega,u_{\delta 0}), \kappa_\delta(\cdot;\omega,u_{\delta 0})) \in \mathcal{C}^1([0,\infty),\mathcal{X})$ with $u_\delta(0;\omega,u_{\delta 0}) = u_{\delta 0}$. Moreover the solution mapping generates a RDS φ_{u_δ} : $\mathbb{R}^+ \times \Omega \times \mathcal{X} \to \mathcal{X}$ defined as $\varphi_{u_\delta}(t,\omega)u_{\delta 0} = u_\delta(t;\omega,u_{\delta 0})$ for all $t \in \mathbb{R}^+$, $\omega \in \Omega$, $u_0 \in \mathcal{X}$.*

Proof. Obviously, the system (4.5)-(4.6) is rewritten as

$$\frac{d\sigma_\delta}{dt} = -(\bar{D} + \alpha z^*_\delta)\sigma_\delta - m\kappa_\delta + \frac{(ma + mK\sigma^2_\delta e^{-2\alpha z^*_\delta})\kappa_\delta}{a + \sigma_\delta e^{-\alpha z^*_\delta} + K\sigma^2_\delta e^{-2\alpha z^*_\delta}} + S^0 De^{\alpha z^*_\delta}, \tag{4.7}$$

$$\frac{d\kappa_\delta}{dt} = -(\bar{D} - m + \alpha z^*_\delta)\kappa_\delta - \frac{(ma + mK\sigma^2_\delta e^{-2\alpha z^*_\delta})\kappa_\delta}{a + \sigma_\delta e^{-\alpha z^*_\delta} + K\sigma^2_\delta e^{-2\alpha z^*_\delta}}. \tag{4.8}$$

Letting

$$L_\delta(\theta_t\omega) = \begin{pmatrix} -(\bar{D} + \alpha z^*_\delta) & -m \\ 0 & -(\bar{D} - m + \alpha z^*_\delta) \end{pmatrix}$$

and $F_\delta : \mathcal{X} \times \Omega \to \mathbb{R}^2$ of form

$$F_\delta(\eta,\omega) = \begin{pmatrix} \frac{(ma + mK\eta_1^2 e^{-2\alpha z^*_\delta(\omega)})\eta_2}{a + \eta_1 e^{-\alpha z^*_\delta(\omega)} + K\eta_1^2 e^{-2\alpha z^*_\delta(\omega)}} + S^0 De^{\alpha z^*_\delta(\omega)} \\ -\frac{(ma + mK\eta_1^2 e^{-2\alpha z^*_\delta(\omega)})\eta_2}{a + \eta_1 e^{-\alpha z^*_\delta(\omega)} + K\eta_1^2 e^{-2\alpha z^*_\delta(\omega)}} \end{pmatrix}$$

where $\eta = (\eta_1, \eta_2) \in \mathcal{X}$, system (4.7)-(4.8) turns into

$$\frac{du_\delta}{dt} = L_\delta(\theta_t\omega)u_\delta + F_\delta(u_\delta, \theta_t\omega). \tag{4.9}$$

On the one hand, the operator L_δ generates an evolution system on $\mathbb{R}^2$ because $t \to z^*_\delta(\theta_t\omega)$ is continuous. On the other hand, $F_\delta(\cdot,\theta_t\omega) \in \mathcal{C}(\mathcal{X} \times [0,\infty);\mathbb{R}^2)$ and is continuously differentiable with respect to the variables (η_1,η_2) since

$$\frac{\partial}{\partial\eta_1}[\pm\frac{(ma + mK\eta_1^2 e^{-2\alpha z^*_\delta})\eta_2}{a + \eta_1 e^{-\alpha z^*_\delta} + K\eta_1^2 e^{-2\alpha z^*_\delta}} + \tilde{C}] = \mp\frac{(mae^{-\alpha z^*_\delta} + mK\eta_1^2 e^{-3\alpha z^*_\delta})\eta_2}{(a + \eta_1 e^{-\alpha z^*_\delta} + K\eta_1^2 e^{-2\alpha z^*_\delta})^2}$$

and

$$\frac{\partial}{\partial\eta_2}[\pm\frac{(ma + mK\eta_1^2 e^{-2\alpha z^*_\delta})\eta_2}{a + \eta_1 e^{-\alpha z^*_\delta} + K\eta_1^2 e^{-2\alpha z^*_\delta}} + \tilde{C}] = \pm\frac{(ma + mK\eta_1^2 e^{-2\alpha z^*_\delta})}{a + \eta_1 e^{-\alpha z^*_\delta} + K\eta_1^2 e^{-2\alpha z^*_\delta}}$$

where $\tilde{C}$ is some constant which does not depend on $(\eta_1, \eta_2) \in \mathcal{X}$. This further implies that $F_\delta(\cdot, \theta_t\omega)$ is locally Lipschitz with respect to $(\eta_1, \eta_2) \in \mathcal{X}$. Therefore, system (4.9) possesses a unique local solution according to classical results from the theory of ordinary differential equations.

Next we show that this solution is in fact a global one. To this end, we define $V_\delta(t) = \sigma_\delta(t) + \kappa_\delta(t)$ and obtain that

$$\frac{dV_\delta}{dt} = -(\bar{D} + \alpha z_\delta^*)V_\delta + S^0 D e^{\alpha z_\delta^*}.$$

By variation of constants, the corresponding solution is given by

$$V_\delta(t) = V_\delta(0)e^{-(\bar{D}t + \alpha\int_0^t z_\delta^* ds)} + S^0 D \int_0^t e^{\alpha z_\delta^*} e^{-(\bar{D}(t-s) + \alpha\int_s^t z_\delta^* dr)} ds \tag{4.10}$$

which is clearly bounded since $\bar{D} > 0$. Moreover, it follows from (4.5) that

$$\frac{d\sigma_\delta}{dt} \leq -(\bar{D} + \alpha z_\delta^*)\sigma_\delta + S^0 D e^{\alpha z_\delta^*}.$$

By comparison principle and variation of constants, we have

$$\sigma_\delta(t) \leq \sigma_\delta(0)e^{-(\bar{D}t + \alpha\int_0^t z_\delta^* ds)} + S^0 D \int_0^t e^{\alpha z_\delta^*} e^{-(\bar{D}(t-s) + \alpha\int_s^t z_\delta^* dr)} ds$$

which ensures that σ_δ is bounded either. Note that $\kappa_\delta(t) = V_\delta(t) - \sigma_\delta(t)$, thus κ_δ is also bounded since V_δ and σ_δ are bounded in both cases. Therefore, the unique local solution to system (4.9) can be extended to a unique global one.

We now turn to prove that the global solution of (4.9) belongs to the set $\mathcal{X}$ for any $t \in \mathbb{R}^2$. Suppose that if $\sigma_\delta(t) = 0$ for some $t = t^* \in \mathbb{R}^+$, it follows from (4.5) that

$$\left.\frac{d\sigma_\delta}{dt}\right|_{t*} = S^0 D e^{\alpha z_\delta^*} > 0.$$

In addition, given $(\sigma_{\delta 0}, 0)$ with $\sigma_{\delta 0} > 0$, there exists a unique solution of system (4.9) satisfying $\sigma_\delta(t_0) = \sigma_{\delta 0}$ and $\kappa_\delta(t_0) = 0$ for some initial time $t_0 \geq 0$. Specifically, this unique solution is given by

$$\sigma_\delta(t) = \sigma_\delta(t_0)e^{-(\bar{D}t + \alpha\int_{t_0}^t z_\delta^* ds)} + S^0 D \int_{t_0}^t e^{\alpha z_\delta^*} e^{-(\bar{D}(t-s) + \alpha\int_s^t z_\delta^* dr)} ds, \tag{4.11}$$

$$\kappa_\delta(t) \equiv 0. \tag{4.12}$$

For $(\sigma_{\delta 0}, \kappa_{\delta 0}) \in \mathcal{X}$, there exists a unique solution $(\sigma_\delta(t), \kappa_\delta(t))$ such that $\sigma_\delta(0) = \sigma_{\delta 0}$ and $\kappa_\delta(0) = \kappa_{\delta 0}$. If there is some first $t^* > 0$ such that $\kappa_\delta(t^*) = 0$, then we obtain that $(\sigma_\delta(\cdot), \kappa_\delta(\cdot))$ is the unique solution of system (4.9) with $\sigma_\delta(t^*) = \sigma_\delta^*$ and $\kappa_\delta(t^*) = 0$. Meanwhile, $\kappa_\delta(t) > 0$ for all $0 < t < t^*$; however, we already have another solution $(\sigma_\delta(t), 0)$ given by (4.11)-(4.12) for all $t \geq t^* - \epsilon$ (for any $\epsilon > 0$ small enough) for this problem, so it leads to a contradiction. As a result, we deduce that for any initial data $u_{\delta 0} \in \mathcal{X}$ the solution $u_\delta(t)$ remains in $\mathcal{X}$.

Finally, we define the solution mapping $\varphi_{u_\delta} : \mathbb{R} \times \Omega \times \mathcal{X} \to \mathcal{X}$ of the form

$$\varphi_{u_\delta}(t, \omega)u_{\delta 0} := u_\delta(t; \omega, u_{\delta 0}), \quad \text{for all } t \geq 0, \omega \in \Omega, u_{\delta 0} \in \mathcal{X}.$$

Since the function F_δ is continuous in (u_δ, t), and is measurable in ω, we obtain that the mapping φ_{u_δ} is $(\mathcal{B}[0, \infty) \times \mathcal{F} \times \mathcal{B}(\mathcal{X}), \mathcal{B}(\mathcal{X}))$-measure. It then follows that (4.9) generate the continuous RDS $\varphi_{u_\delta}(t, \omega)(\cdot)$. $\square$

Lemma 4.2. *The RDS $\{\varphi_{u_\delta}(t,\omega)\}_{t\geq 0,\omega\in\Omega}$ has a tempered compact random absorbing set $B(\omega)\in\mathcal{E}(\mathcal{X})$.*

Proof. Notice that $\varphi_{u_\delta}(t,\theta_{-t}\omega)u_{\delta 0}=u_\delta(t;\theta_{-t}\omega,u_{\delta 0})$ denotes the solution of the random chemostat model (4.5)-(4.6), satisfying that $u_\delta(0;\theta_{-t}\omega,u_{\delta 0})=u_{\delta 0}$, where $u_{\delta 0}=u_{\delta 0}(\theta_{-t}\omega)\in E(\theta_{-t}\omega)$. Thus we introduce the following norm

$$\begin{aligned}\|\varphi_{u_\delta}(t,\theta_{-t}\omega)u_{\delta 0}\| &= \|u_\delta(t;\theta_{-t}\omega,u_{\delta 0}(\theta_{-t}\omega))\|\\ &= \sigma_\delta(t;\theta_{-t}\omega,u_{\delta 0}(\theta_{-t}\omega))+\kappa_\delta(t;\theta_{-t}\omega,u_{\delta 0}(\theta_{-t}\omega)).\end{aligned}$$

Noting that $V_\delta=\sigma_\delta+\kappa_\delta$ and replacing ω by $\theta_{-t}\omega$ in (4.10), we obtain

$$\begin{aligned}&\sigma_\delta(t;\theta_{-t}\omega,u_{\delta 0}(\theta_{-t}\omega))+\kappa_\delta(t;\theta_{-t}\omega,u_{\delta 0}(\theta_{-t}\omega))\\ &\quad=(\sigma_\delta(0)+\kappa_\delta(0))e^{-(\bar{D}t+\alpha\int_0^t z_\delta^*(\theta_{s-t}\omega)ds)}\\ &\qquad+S^0D\int_0^t e^{\alpha z_\delta^*(\theta_{s-t}\omega)}e^{-(\bar{D}(t-s)+\alpha\int_s^t z_\delta^*(\theta_{r-t}\omega)dr)}ds\\ &\quad=(\sigma_\delta(0)+\kappa_\delta(0))e^{-(\bar{D}t+\alpha\int_{-t}^0 z_\delta^*(\theta_{s}\omega)ds)}\\ &\qquad+S^0D\int_0^t e^{\alpha z_\delta^*(\theta_{s-t}\omega)}e^{-(\bar{D}(t-s)+\alpha\int_{s-t}^0 z_\delta^*(\theta_{r}\omega)dr)}ds\\ &\quad=(\sigma_\delta(0)+\kappa_\delta(0))e^{-(\bar{D}t+\alpha\int_{-t}^0 z_\delta^*(\theta_{s}\omega)ds)}\\ &\qquad+S^0D\int_0^t e^{\alpha z_\delta^*(\theta_{-\tau}\omega)}e^{-(\bar{D}\tau+\alpha\int_{-\tau}^0 z_\delta^*(\theta_{r}\omega)dr)}d\tau\\ &\quad=(\sigma_\delta(0)+\kappa_\delta(0))e^{-\bar{D}t-\alpha\int_{-t}^0 z_\delta^*(\theta_{s}\omega)ds}\\ &\qquad+S^0D\int_0^t e^{\tau[-\bar{D}+\frac{\alpha z_\delta^*(\theta_{-\tau}\omega)}{\tau}-\frac{\alpha}{\tau}\int_{-\tau}^0 z_\delta^*(\theta_r\omega)dr]}d\tau.\end{aligned}$$

Taking the limits on the previous equation, we further have

$$\lim_{t\to\infty}[\sigma_\delta(t;\theta_{-t}\omega,u_{\delta 0}(\theta_{-t}\omega))+\kappa_\delta(t;\theta_{-t}\omega,u_{\delta 0}(\theta_{-t}\omega))]=S^0D\rho_\delta^*(\omega),$$

where $\rho_\delta^*(\omega)$ is defined by

$$\rho_\delta^*(\omega)=\int_0^\infty e^{\tau[-\bar{D}+\frac{\alpha z_\delta^*(\theta_{-\tau}\omega)}{\tau}-\frac{\alpha}{\tau}\int_{-\tau}^0 z_\delta^*(\theta_r\omega)dr]}d\tau.$$

Moreover, $\rho_\delta^*(\omega)$ has sub-exponential growth due to the fact

$$e^{\tau[-\bar{D}+\frac{\alpha z_\delta^*(\theta_{-\tau}\omega)}{\tau}-\frac{\alpha}{\tau}\int_{-\tau}^0 z_\delta^*(\theta_r\omega)dr]}\to 0 \quad \text{if} \quad \tau\to\infty.$$

Therefore, for any given $\varepsilon>0$, there exists $T_E(\omega,\varepsilon,\delta)>0$ such that

$$|\|u_\delta(t;\theta_{-t}\omega,u_{\delta 0}(\theta_{-t}\omega))\|-S^0D\rho_\delta^*(\omega)|\leq\varepsilon$$

for all $u_{\delta 0}\in E(\theta_{-t}\omega)$ and $t\geq T_E(\omega,\varepsilon,\delta)$. We now define the set

$$B_\varepsilon(\omega):=\{(\sigma_\delta,\kappa_\delta)\in\mathcal{X}:S^0D\rho_\delta^*(\omega)-\varepsilon\leq\sigma_\delta+\kappa_\delta\leq S^0D\rho_\delta^*(\omega)+\varepsilon\},$$

thus $B_\varepsilon(\omega)\in\mathcal{E}$ is absorbing in $\mathcal{X}$ for any $\varepsilon>0$. Hence, the proof of Lemma 4.2 is complete. □

Lemma 4.3. *If one of the following conditions holds*
(*i*) $\bar{D} \geq m$,
(*ii*) $\bar{D} < m < (1 + 2\sqrt{aK})\bar{D}$.
Then the random attractor associated to the RDS φ_{u_δ} *has the following structure*

$$\mathcal{A}_\delta = \{A_\delta(\omega)\}_{\omega\in\Omega}, \quad A_\delta(\omega) = (S^0 D\rho_\delta^*(\omega), 0).$$

Proof. According to Lemma 2.1, the RDS φ_{u_δ} possesses a unique random attractor given by $\mathcal{A}_\delta = \{A_\delta(\omega)\}_{\omega\in\Omega} \subset B_\varepsilon(\omega)$ for any $\varepsilon > 0$. Thus $\mathcal{A}_\delta = \{A_\delta(\omega)\}_{\omega\in\Omega} \subset B_0(\omega)$. In other word, we have the following expression for each component of our attractor

$$A_\delta(\omega) := (S^0 D\rho_\delta^*(\omega) - \kappa_\delta(\omega), \kappa_\delta(\omega)). \tag{4.13}$$

Denote by

$$g(\zeta) = \frac{m\zeta}{a + \zeta + K\zeta^2} - \bar{D}, \quad \zeta = \sigma e^{-\alpha z_\delta^*} > 0.$$

If (i) holds, there exists $c_1 > 0$ such that

$$g(\zeta) = \frac{m\zeta}{a + \zeta + K\zeta^2} - \bar{D} = \frac{-\bar{D}a - (\bar{D} - m)\zeta - \bar{D}K\zeta^2}{a + \zeta + K\zeta^2} \leq -c_1 < 0.$$

Then it follows from (4.6) that

$$\frac{d\kappa_\delta}{dt} = -(\bar{D} + \alpha z_\delta^*)\kappa_\delta + (g(\zeta) + \bar{D})\kappa_\delta \leq (-c_1 + \alpha z_\delta^*)\kappa_\delta. \tag{4.14}$$

If (ii) holds, there exists $c_2 > 0$ such that

$$g(\zeta) \leq \sup_{\zeta\in(0,\infty)} g(\zeta) = \frac{m - (1 + 2\sqrt{aK})\bar{D}}{1 + 2\sqrt{aK}} \leq -c_2 < 0.$$

Then it follows from (4.6) that

$$\frac{d\kappa_\delta}{dt} = -(\bar{D} + \alpha z_\delta^*)\kappa_\delta + (g(\zeta) + \bar{D})\kappa_\delta \leq (-c_2 + \alpha z_\delta^*)\kappa_\delta. \tag{4.15}$$

Thus, both solutions to (4.14) and (4.15) tend to zero after replacing ω by $\theta_{-t}\omega$ and making $t \to \infty$. Therefore, the internal structure of the attractor in both cases consists of singleton subsets $A_\delta(\omega) = (S^0 D\rho_\delta^*(\omega), 0)$. □

Theorem 4.1. *If one of the following conditions holds*
(*i*) $\bar{D} \geq m$,
(*ii*) $\bar{D} < m < (1 + 2\sqrt{aK})\bar{D}$.
Then the random attractor associated to the RDS generated by the system (4.1)-(4.2) also has the structure composed of singleton subsets.

Proof. We define a mapping $\mathcal{T}_\delta : \Omega \times \mathcal{X} \to \mathcal{X}$ and its inverse, respectively, as

$$\mathcal{T}_\delta(\omega, \zeta) = (\zeta_1 e^{\alpha z_\delta^*(\omega)}, \zeta_2 e^{\alpha z_\delta^*(\omega)})$$

and

$$\mathcal{T}_\delta^{-1}(\omega, \zeta) = (\zeta_1 e^{-\alpha z_\delta^*(\omega)}, \zeta_2 e^{-\alpha z_\delta^*(\omega)}).$$

Denote $v_\delta(t) = (S_\delta(t), x_\delta(t))$ and using (4.3)-(4.4), we have that $v_\delta = u_\delta e^{-\alpha z^*_\delta(\omega)}$. Since $\mathcal{T}_\delta$ is a homeomorphism, thanks to Lemma 2.2 we obtain the following conjugated RDS

$$\begin{aligned}\varphi_{v_\delta}(t,\omega)v_{\delta 0} &= \mathcal{T}_\delta^{-1}(\theta_t\omega, \varphi_{u_\delta}(t,\omega)\mathcal{T}_\delta(\omega, v_{\delta 0}))\\ &= \mathcal{T}_\delta^{-1}(\theta_t\omega, \varphi_{u_\delta}(t,\omega)u_{\delta 0})\\ &= \mathcal{T}_\delta^{-1}(\theta_t\omega, u_\delta(t;\omega, u_{\delta 0}))\\ &= v_\delta(t;\omega, v_{\delta 0})\end{aligned}$$

implying that φ_{v_δ} is an RDS for system (4.1)-(4.2).

Then the global random attractor of the system (4.5)-(4.6), $\mathcal{A}_\delta = \{A_\delta(\omega)\}_{\omega\in\Omega} \subset B_0(\omega)$, becomes into ${\mathcal{A}_\delta}^{\mathcal{T}_\delta} = \{{A_\delta}^{\mathcal{T}_\delta}(\omega)\}_{\omega\in\Omega} \subset B_0^{\mathcal{T}_\delta}(\omega)$, the global random attractor of the stochastic chemostat system (4.1)-(4.2), where

$$B_0^{\mathcal{T}_\delta}(\omega) := \{(S_\delta, x_\delta) \in \mathcal{X} : S_\delta + x_\delta = S^0 D\rho^*_\delta(\omega)e^{-\alpha z^*_\delta(\omega)}\}.$$

Thus each component ${A_\delta}^{\mathcal{T}_\delta}(\omega)$, $\omega \in \Omega$ of the attractor can be written as

$${A_\delta}^{\mathcal{T}_\delta}(\omega) := (S^0 D\rho^*_\delta(\omega)e^{-\alpha z^*_\delta(\omega)} - Se^{-\alpha z^*_\delta(\omega)}, Se^{-\alpha z^*_\delta(\omega)}).$$

Moreover, we know that the internal structure of the attractor consists of singleton subsets

$${A_\delta}^{\mathcal{T}_\delta}(\omega) := (S^0 D\rho^*_\delta(\omega)e^{-\alpha z^*_\delta(\omega)}, 0)$$

as long as condition (i) or (ii) holds. □

We finally present the upper semicontinuity of random attractors as $\delta \to 0$.

Theorem 4.2. *If one of the following conditions holds*

(i) $\bar{D} \geq m$,

(ii) $\bar{D} < m < (1 + 2\sqrt{aK})\bar{D}$.

Then for any $\omega \in \Omega$, *we have*

$$\lim_{\delta\to 0} dist_X({\mathcal{A}_\delta}^{\mathcal{T}_\delta}, \mathcal{A}^{\mathcal{T}}) = 0. \tag{4.16}$$

Proof. We first need to prove the following equation holds

$$\lim_{\delta\to 0} \rho^*_\delta(\omega) = \rho^*(\omega). \tag{4.17}$$

Let $T > 0$ be large enough such that

$$\int_{-\infty}^{-T} \mathcal{G}_\delta(\theta_s\omega)ds < c,$$

where $0 < c < \bar{D}$. For convenience, we define

$$\mu^*_\delta(\tau,\omega) = e^{\tau[-\bar{D} + \frac{\alpha z^*_\delta(\theta_{-\tau}\omega)}{\tau} - \frac{\alpha}{\tau}\int_{-\tau}^0 z^*_\delta(\theta_r\omega)dr]},$$

$$\mu^*(\tau,\omega) = e^{\tau[-\bar{D} + \frac{\alpha z^*(\theta_{-\tau}\omega)}{\tau} - \frac{\alpha}{\tau}\int_{-\tau}^0 z^*(\theta_r\omega)dr]}.$$

Then

$$\rho^*_\delta(\omega) = \int_0^T \mu^*_\delta(\tau,\omega)d\tau + \int_T^\infty \mu^*_\delta(\tau,\omega)d\tau,$$

$$\rho^*(\omega) = \int_0^T \mu^*(\tau,\omega)d\tau + \int_T^\infty \mu^*(\tau,\omega)d\tau.$$

For $\int_0^T \mu_\delta^*(\tau,\omega)d\tau$, it follows from (2.4) that $z_\delta^*(\theta_t\omega) \to z^*(\theta_t\omega)$ uniformly on $[0,T]$ and $[-T,0]$ as $\delta \to 0$, therefore $\mu_\delta^*(t,\omega) \to \mu^*(t,\omega)$ uniformly on $[0,T]$ as $\delta \to 0$. Then

$$\lim_{\delta\to 0}\int_0^T \mu_\delta^*(\tau,\omega)d\tau = \int_0^T \mu^*(\tau,\omega)d\tau. \tag{4.18}$$

For $\int_T^\infty \mu_\delta^*(\tau,\omega)d\tau$,

$$\begin{aligned}
\mu_\delta^*(\tau,\omega) &= e^{\tau[-\bar{D}+\frac{\alpha z_\delta^*(\theta_{-\tau}\omega)}{\tau}-\frac{\alpha}{\tau}\int_{-\tau}^0 z_\delta^*(\theta_r\omega)dr]}\\
&= e^{\tau[-\bar{D}+\frac{\alpha}{\tau}(z_\delta^*(\theta_{-\tau}\omega)-\int_{-\tau}^0 z_\delta^*(\theta_r\omega)dr)]}\\
&= e^{\tau[-\bar{D}+\frac{\alpha}{\tau}(z_\delta^*(\omega)-\int_{-\tau}^0 \mathcal{G}_\delta(\theta_r\omega)dr)]}\\
&\le e^{\tau[-\bar{D}+\frac{\alpha}{\tau}(\int_{-\infty}^{-\tau} \mathcal{G}_\delta(\theta_r\omega)dr)]}\\
&\le e^{\tau(-\bar{D}+c)},
\end{aligned}$$

then $\int_T^\infty \mu_\delta^*(\tau,\omega)d\tau < \infty$. By Lebesgue dominated convergence theorem, we obtain

$$\lim_{\delta\to 0}\int_T^\infty \mu_\delta^*(\tau,\omega)d\tau = \int_T^\infty \mu^*(\tau,\omega)d\tau, \tag{4.19}$$

together with (4.18) we get (4.17). Then through (2.4) and (4.17), we complete the proof. □

References

[1] J.F. Andrews, *A mathematical model for the continuous culture of microorganisms utilizing inhibitory substrates*, Biotechnology and Bioengineering, 1968, 10(6), 707–723.

[2] L. Arnold, *Random Dynamical Systems*, Springer-Verlag, New York, 1998.

[3] B. Boon and H. Laudelout, *Kinetics of nitrite oxidation by nitrobacter winogradskyi*, Biochemical Journal, 1962, 85(3), 440–447.

[4] H.R. Bungay and M.L. Bungay, *Microbial interactions in continuous culture*, Advances in Applied Microbiology, 1968, 10, 269–290.

[5] T. Caraballo, M.J. Garrido-Atienza and J. López-de-la-Cruz, *Some aspects concerning the dynamics of stochastic chemostats*, in Advances in Dynamical Systems and Control, Springer International Publishing, Cham, 2016, 227–246.

[6] T. Caraballo, M.J. Garrido-Atienza and J. López-de-la-Cruz, *Dynamics of some stochastic chemostat models with multiplicative noise*, Communications on Pure and Applied Analysis, 2017, 16(5), 1893–1914.

[7] T. Caraballo, M.J. Garrido-Atienza, J. López-de-la-Cruz and A. Rapaport, *Corrigendum to the chapter "some aspects concerning the dynamics of stochastic chemostats"*, preprint, 2017, https://arxiv.org/abs/1710.00774v1.

[8] T. Caraballo, M. Garrido-Atienza, B. Schmalfuss and J. Valero, *Asymptotic behaviour of a stochastic semilinear dissipative functional equation without uniqueness of solutions*, Discrete and Continuous Dynamical Systems Series B, 2010, 14(2), 439–455.

[9] T. Caraballo and X. Han, *Applied Nonautonomous and Random Dynamical Systems, Applied Dynamical Systems*, Springer, 2016.

[10] T. Caraballo, P.E. Kloeden and B. Schmalfuss, *Exponentially stable stationary solutions for stochastic evolution equations and their perturbation*, Applied Mathematics & Optimization, 2004, 50(3), 183–207.

[11] T. Caraballo, G. Lukaszewicz and J. Real, *Pullback attractors for asymptotically compact non-autonomous dynamical systems*, Nonlinear Analysis, 2006, 64(3), 484–498.

[12] A. Cunningham and R.M. Nisbet, *Transients and oscillations in continuous cultures*, in Mathematics in Microbiology (ed. M. Bazin), Academic Press, London, 1983, 77–103.

[13] G. D'ans, P.V. Kokotovic and D. Gottlieb, *A nonlinear regulator problem for a model of biological waste treatment*, IEEE Transactions on Automatic Control, 1971, 16(4), 341–347.

[14] F. Flandoli and B. Schmalfuss, *Random attractors for the 3D stochastic Navier-Stokes equation with multiplicative noise*, Stochastics and Stochastics Reports, 1996, 59(1-2), 21–45.

[15] D. Foster and P. Young, *Stochastic evolutionary game dynamics*, Theoretical Population Biology, 1990, 38(2), 219–232.

[16] A.G. Fredrickson and G. Stephanopoulos, *Microbial competition*, Science, 1981, 213(4511), 972–979.

[17] R. Freter, *Mechanisms that control the microflora in the large intestine*, in Human Intestinal Microflora in Health and Disease (ed. J. Hentges), Academic Press, New York, 1983, 33–54.

[18] R. Freter, *An understanding of colonization of the large intestine requires mathematical analysis*, Microecology and Therapy, 1986, 16, 147–155.

[19] D. Fudenberg and C. Harris, *Evolutionary dynamics with aggregate shocks*, Journal of Economic Theory, 1992, 57(2), 420–441.

[20] J.B.S. Haldane, *Enzymes*, Longmans, Green & Co., London, 1930.

[21] D.J. Higham, *An algorithmic introduction to numerical simulation of stochastic differential equations*, SIAM Review, 2001, 43(3), 525–546.

[22] H.W. Jannash and R.T. Mateles, *Experimental bacterial ecology studies in continuous culture*, Advances in Microbial Physiology, 1974, 11, 165–212.

[23] K. Lu and Q. Wang,*Chaotic behavior in differential equations driven by a Brownian motion*, Journal of Differential Equations, 2011, 251(10), 2853–2895.

[24] K. Lu and B. Wang, *Wong-Zakai approximations and long term behavior of stochastic partial differential equations*, Journal of Dynamics and Differential Equations, 2017.
DOI: 10.1007/s10884-017-9626-y

[25] B. Øksendal, *Stochastic Differential Equations: An Introduction with Applications*, Springer, Berlin, 2003.

[26] J.W.M.La Rivière, *Microbial ecology of liquid waste treatment*, Advances in Microbial Ecology, 1977, 1, 215–259.

[27] J. Shen and K. Lu, *Wong-Zakai approximations and center manifolds of stochastic differential eqations*, Journal of Differential Equations, 2017, 263(8), 4929–4977.

[28] J. Shen, J. Zhao, K. Lu and B. Wang, *The Wong-Zakai approximaions of invariant manifolds and foliations for stochastic evolution equations*, Journal of Differential Equations, 2019, 266(8), 4568–4623.

[29] H.L. Smith and P. Waltman, *The Theory of the Chemostat: Dynamics of Microbial Competition*, Cambridge University Press, Cambridge, UK, 1995.

[30] P.A. Taylor and J.L. Williams, *Theoretical studies on the coexistence of competing species under continuous-flow conditions*, Canadian Journal of Microbiology, 1975, 21(1), 90–98.

[31] H. Veldcamp, *Ecological Studies with the Chemostat*, in Advances in Microbial Ecology, vol. 1, Springer, Boston, 1977, 59–95.

[32] P. Waltman (ed.), *Competition models in population biology*, vol. 45 of CBMS-NSF Regional Conference Series in Applied Mathematics, Society for Industrial and Applied Mathematics, Philadelphia, 1983.

[33] P. Waltman, *Coexistence in chemostat-like model*, Rocky Mountain Journal of Mathematics, 1990, 20(4), 777–807.

[34] P. Waltman, S.P. Hubbel and S.B. Hsu, *Theoretical and experimental investigations of microbial competition in continuous culture*, in Modeling and Differential Equations in Biology, vol. 58 of Lecture Notes in Pure and Appl. Math., Dekker, New York, 1980, 107–152, (Conf., southern Illinois Univ. Carbonadle, III., 1978).