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Stability Analysis for the Numerical Simulation of Hybrid Stochastic Differential Equations*

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Abstract This paper is mainly concerned with the exponential stability of a class of hybrid stochastic differential equations–stochastic differential equations with Markovian switching (SDEwMSs). It first devotes to reveal that under the global Lipschitz condition, a SDEwMS is pth ($p \in (0,1)$) moment exponentially stable if and only if its corresponding improved Euler-Maruyama(IEM) method is pth moment exponentially stable for a suitable step size. It then shows that the SDEwMS is pth($p \in (0,1)$) moment exponentially stable or its corresponding IEM method with small enough step sizes implies the equation is almost surely exponentially stable or the corresponding IEM method, respectively. In that sense, one can infer that the SDEwMS is almost surely exponentially stable and the IEM method, no matter whether the SDEwMS is pth moment exponentially stable or the IEM method. An example is demonstrated to illustrate the obtained results.

Keywords Moment exponential stability, almost sure exponential stability, Markovian switching, improved Euler-Maruyama method.

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1. Introduction

Stochastic differential equations (SDEs) have been widely used in many branches of science and industry. Stability analysis as a particular interest of SDEs has aroused special attention of many scholars (see [2,8,16,17] and the literature cited therein). In the study of stochastic stability, Lyapunov functions technique is the classical and powerful technique. However, in general, it is not convenient for us to use this method for there is no an universal method can guarantee to find an appropriate Lyapunov function, which motivates us to employ numerical methods with sufficiently small step sizes to study the stochastic stability. Hence, for SDEs, many investigators have paid a deal of attention to stability analysis of numerical methods (e.g. [5,7,13,14,19]). Moreover, the following two questions are concerning:

(Q1) If a SDE is stochastically stable, will the numerical method be stochastically stable?

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(Q2) Conversely, if the numerical method is stochastically stable, will the SDE is stochastically stable?

Two natural and important notions of stochastical stability are moment exponentially stable and almost surely exponentially stable. Moment exponential stability, also known as the pth moment exponential stability. In the case of $p = 2$, moment exponential stability means exponential stability in mean square. When the stochastic stability is understood as the mean square exponential stability, answers to $(Q1)$ and $(Q2)$ can be found in [3, 13]. Higham et al. [4] showed that under the global Lipscitz condition, mean square exponential stability of SDEs and that of the numerical method with sufficiently small step sizes are equivalent, implying that answers to $(Q1)$ and $(Q2)$ are positive. For the stochastic stability means the almost sure exponential stability, there are a lot of results for $(Q1)$, but few answers to $(Q2)$. Higham et al. [5] is the first paper that discussed both $(Q1)$ and $(Q2)$ for a reasoned class of SDEs. For the linear scalar SDEs, they presented positive answers to $(Q1)$ and $(Q2)$ by the Euler-Maruyama (EM) method. For the nonlinear SDEs with the linear growth condition and some additional conditions, they also answered $(Q1)$ using the EM method. For the nonlinear SDEs without the linear growth condition, but required the drift coefficient obeys the one-sided Lipschitz condition, the answer to $(Q1)$ alone is positive through the backward Euler-Maruyama (BEM) method. Recently, Mao [9] proved that under the global Lipschitz condition, the almost sure exponential stability of the SDEs is shared with that the stochastic theta method, and therefore presented positive answers to $(Q1)$ and $(Q2)$.

As is known, hybrid stochastic differential equations have increasingly gained attention in biological systems, financial engineering, wireless communications and so forth (see [18, 20]). One of the important classes of the hybrid stochastic differential equations is the SDEwMSs. Generally, most of SDEwMSs can rarely be solved explicitly and hence numerical approximation becomes an important tool in studying them. When the SDEwMS is considered in the $(Q1)$ and $(Q2)$, Pang et al. [12] proved that under appropriate conditions, the EM method with sufficiently small step sizes can capture the almost sure and the pth moment exponential stability of the linear scalar SDEwMS, they therefore gave the positive answer to $(Q1)$, but not to $(Q2)$. For the nonlinear SDEwMSs, the authors in [10] showed that without the global Lipschitz condition, the BEM method may capture the almost sure exponential stability, they also positively answered $(Q1)$ but did not address $(Q2)$.

Although a lot of results on addressing $(Q1)$ and $(Q2)$ for SDEs have been obtained (see [6, 19] and the references therein), unfortunately, there are almost no answers to $(Q1)$ and $(Q2)$ for SDEwMSs due to the difficulty in dealing with the Markovian switching. Therefore, it is significant to investigate $(Q1)$ and $(Q2)$ for SDEwMSs. Motivated by Mao [9], this paper first shows that under the global Lipschitz condition, the SDEwMS is pth ($p \in (0, 1)$) moment exponentially stable if and only if the IEM method with a sufficiently small step size is pth moment exponentially stable. Based on such result, we can positively answer both $(Q1)$ and $(Q2)$ for the SDEwMS when the stochastic stability means exponential stability in the sense of pth ($p \in (0, 1)$) moment. This paper then proves that the SDEwMS is pth($p \in (0, 1)$) moment exponentially stable or the IEM method implies the SDEwMS is almost surely exponentially stable or the IEM method, respectively. Moreover, the obtained theory ensures that either the SDEwMS is pth moment exponentially stable or the IEM method, one can assert that the almost sure exponential stability of the SDEwMS and the IEM method. It is therefore that one can study the almost sure exponential stability by the IEM method without resort to using the Lyapunov function technique for the SDEwMSs.

The rest of the paper is organized as follows. Section 2 illustrates that the SDEwMS is pth ($p \in (0,1)$) moment exponentially stable if and only if the numerical method is pth moment exponentially stable for a sufficiently small step size. Section 3 presents positive answers to $(Q1)$ and $(Q2)$ when the stochastic stability is understood as the pth ($p \in (0,1)$) moment exponential stability. Section 4 reveals that if a SDEwMS or the IEM method is pth ($p \in (0,1)$) moment exponentially stable, then it is also almost surely exponentially stable. Section 5 provides a numerical example to illustrate the obtained results. Finally, this paper ends with brief conclusions.

2. Pth moment exponential stability

Throughout this paper, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. its right continuous and $\mathcal{F}_0$ contains all P-null sets). Let $\omega(t) = (\omega^1(t), \omega^2(t), \dots, \omega^d(t))$ be a d-dimensional Brownian motion. Let $\{r(t)\}_{t\geq 0}$ be a right-continuous time-homogeneous Markov chain (independent of the Brownian motion $\omega(t)$) on the probability space taking values in a finite state space $S = \{1, 2, \cdots, N\}$ with the generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$P\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta), & \text{if } \ i \neq j, \\ 1 + \gamma_{ii}\Delta + o(\Delta), & \text{if } \ i = j, \end{cases}$$

where $\Delta > 0$, $\lim\limits_{\Delta\to 0} o(\Delta)/\Delta = 0$, $\gamma_{ij} \geq 0$ is the transition rate from i to j, if $i \neq j$, while $\gamma_{ii} = -\sum\limits_{j\neq i} \gamma_{ij}$. It is well known that almost every sample path of $r(\cdot)$ is a right-continuous step function with finite number of simple jumps in any finite subinterval of $R^+ = [0, +\infty)$ ([1]). Denote by $R = (-\infty, +\infty)$, let $|x| = \sqrt{x^T x}$, $x \in R^n$ be the Euclidean vector norm. For $a, b \in R$, $a \vee b$ and $a \wedge b$ represent $\max\{a, b\}$ and $\min\{a, b\}$, respectively.

In this paper, we consider the following nonlinear SDEwMS

$$dx(t) = f(x(t), r(t))dt + g(x(t), r(t))d\omega(t), \ t \geq 0, \tag{2.1}$$

with initial values $x(0) = x_0 \in R^n$ and $r(0) = i_0 \in S$, where $f : R^n \times S \to R^n$ and $g : R^n \times S \to R^{n\times d}$.

Throughout this paper, we impose the following well known condition–the global Lipschitz condition holds.

$(\boldsymbol{H}_0)$ For all $x, y \in R^n$ and $i \in S$, there is a constant $L > 0$ such that f, g satisfy

$$|f(x,i) - f(y,i)| \vee |g(x,i) - g(y,i)| \leq L|x - y|. \tag{2.2}$$

Moreover, for the aim of stability to the trivial solution $x(t; x_0, i_0) \equiv 0$ of Eq. (2.1), we also assume that $f(0, i) = 0$ and $g(0, i) = 0$ for $\forall i \in S$. For convenience, denote $x(t) = x(t; x_0, i_0)$. It is obvious that $(\boldsymbol{H}_0)$ implies the linear growth condition

$$|f(x,i)| \vee |g(x,i)| \leq L|x|, \ \forall (x,i) \in R^n \times S. \tag{2.3}$$

Under ($\boldsymbol{H}_0$), one can get the unique global solution $x(t)$ ($t \geq 0$) of SDEwMS (2.1) and the solution satisfies

$$E|x(t)|^p \leq H(t,p,L)|x_0|^p, \ \forall t \geq 0, \ 0 < p < 1 \tag{2.4}$$

with

$$H(t,p,L) = 3^{\frac{p}{2}} e^{1.5pL^2 t(t+1)}. \tag{2.5}$$

In the following we will show that (2.4) hold. From Eq. (2.1), it is easy to see that

$$x(t) = x(0) + \int_0^t f(x(s), r(s))ds + \int_0^t g(x(s), r(s))d\omega(s).$$

By the linear growth condition (2.3), one obtains

$$\begin{aligned} E|x(t)|^2 \leq & 3\left(|x_0|^2 + E\left|\int_0^t f(x(s), r(s))ds\right|^2 + E\left|\int_0^t g(x(s), r(s))d\omega(s)\right|^2\right) \\ \leq & 3|x_0|^2 + 3L^2 t \int_0^t E|x(s)|^2 ds + 3E\left(\int_0^t |g(x(s), r(s))|^2 ds\right) \\ \leq & 3|x_0|^2 + 3L^2(t+1)\int_0^t E|x(s)|^2 ds. \end{aligned}$$

Applying the Gronwall inequality, one yields

$$E|x(t)|^2 \leq 3|x_0|^2 e^{3L^2 t(t+1)}, \tag{2.6}$$

which together with Hölder inequality, it follows from $p \in (0,1)$ that

$$E|x(t)|^p \leq H(t,p,L)|x_0|^p.$$

Remark 2.1. When the initial value x_0 is an $\mathcal{F}_0$-measurable R^n-valued random variable such that $E|x_0|^p < \infty (\forall p > 0)$, then by the property of the condition expectation, one obtains that

$$E|x(t)|^p = E(E(|x(t)|^p|\mathcal{F}_0)) \leq E(H(t,p,L)|x_0|^p) = H(t,p,L)E|x_0|^p, t \geq 0. \tag{2.7}$$

Therefore, in the sequel, we can only consider $x_0 \in R^n$ for convenience, that is, we only need to consider the case of x_0 is deterministic.

We state the definition of pth($p > 0$) moment exponential stability of the trivial solution to Eq. (2.1) as follows.

Definition 2.1. The Eq. (2.1) with initial values $x(0) = x_0 \in R^n$ and $r(0) = i_0 \in S$ is said to be pth moment exponentially stable if there is a pair of positive constants λ and M such that

$$E|x(t)|^p \leq M|x_0|^p e^{-\lambda t}, \ t \geq 0. \tag{2.8}$$

Remark 2.2. For any $t_0 \geq 0$, we can consider $x(t)$ ($t \geq t_0$) as the solution of the SDEwMS (2.1) with initial values $x(t_0) = x_0 \in R^n$ and $r(t_0) = i_0 \in S$ at $t = t_0$. It is easy to see from the time-homogeneity of the (2.1) and Remark 2.1 that (2.4) is equivalent to the following more general form

$$E|x(t)|^p \leq H(t - t_0, p, L)E|x(t_0)|^p, \ t \geq t_0, \ 0 < p < 1 \tag{2.9}$$

and (2.8) is equivalent to the

$$E|x(t)|^p \leq ME|x(t_0)|^p e^{-\lambda(t-t_0)}, \ t \geq t_0, \ p > 0. \tag{2.10}$$

Next, we will introduce the numerical solutions generated by a numerical method and give a key lemma which shows that the Eq. (2.1) is pth moment exponentially stable if and only if the corresponding numerical method with a sufficiently small step size Δ is pth moment exponentially stable.

A numerical method is assumed to be available, given a step size $\Delta > 0$, discrete approximations $y_k \approx x(k\Delta)$ for $k \in Z^+$ with initial values $y_0 = x_0 \in R^n$ and $i_0 \in S$. The process defined by the numerical method is required to have the property of Markov, that is, given $y_{\bar{k}}(\bar{k} \in \mathbb{Z}^+)$, the process $\{y_k\}_{k\geq\bar{k}}$ can be seen as the process which is produced by the numerical method applied to the SDEwMS (2.1) on $t \geq \bar{k}\Delta$ with initial values $x(\bar{k}\Delta) = y_{\bar{k}}$ and $r(\bar{k}\Delta) = r^{\Delta}_{\bar{k}}$. Hence, if $y_{\bar{k}}$ is given, the process $\{y_k\}_{k\geq\bar{k}}$ are fully determined, but how the process has reached $y_{\bar{k}}$ is of no further use. Moreover, $\{y_k\}_{k\in\mathbb{Z}^+}$ is time-homogeneous for the time-homogeneity of SDEwMS (2.1). Such a process will be illustrated by the IEM method, which is introduced in the next section. Following Definition 2.1, we now define the pth moment exponential stability for the numerical solutions $\{y_k\}_{k\in\mathbb{Z}^+}$.

Definition 2.2. For a given step size $\Delta > 0$, a numerical method is said to be pth moment exponentially stable on Eq. (2.1) with initial values $x(0) = x_0 \in R^n$ and $r(0) = i_0 \in S$ if there is a pair of positive constants γ and N such that

$$E|y_k|^p \leq N|x_0|^p e^{-\gamma k\Delta}, \ \forall k \in Z^+. \tag{2.11}$$

By the time-homogeneous Markov property and Remark 2.1, it is easy to see that (2.11) is equivalent to the following more general form:

$$E|y_k|^p \leq NE|y_{\bar{k}}|^p e^{-\gamma(k-\bar{k})\Delta}, \ \forall k \geq \bar{k} \geq 0. \tag{2.12}$$

From here we will let $p \in (0, 1)$. We wish to answer both $(Q1)$ and $(Q2)$ in the sense of pth moment exponential stability for the SDEwMS (2.1). In order to achieve this goal, a finite pth moment condition is assumed on the numerical methods.

Assumption 1. For all sufficiently small step sizes Δ, the numerical method applied to Eq. (2.1) with initial values $x(0) = x_0 \in R^n$ and $r(0) = i_0 \in S$ has the property

$$\sup_{0\leq k\Delta\leq T} E|y_k|^p \leq \bar{H}(T, p, L)|x_0|^p, \ \forall T \geq 0, \tag{2.13}$$

where $\bar{H}(T, p, L)$ is positive and only depended on T, p, L, but independent of $(x_0, i_0) \in R^n \times S$ and Δ.

It follows from the time-homogeneous Markov property of the numerical method and Remark 2.1 that (2.13) implies

$$E|y_k|^p \leq \bar{H}(k - \bar{k}, p, L)E|y_{\bar{k}}|^p, \ k \geq \bar{k} \geq 0. \tag{2.14}$$

A finite-time convergence condition is also assumed on the numerical methods.

Assumption 2. For all sufficiently small step sizes Δ, the numerical method applied to Eq. (2.1) with initial values $x(0) = x_0 \in R^n$ and $r(0) = i_0 \in S$ has the property

$$\sup_{0\leq k\Delta\leq T} E|y_k - x(k\Delta)|^p \leq C_T|x_0|^p u(\Delta), \ \forall T \geq 0, \tag{2.15}$$

where C_T depends on T but is independent of $(x_0, i_0) \in R^n \times S$ and Δ, and $u : R^+ \to R^+$ is a strictly increasing continuous function with $u(0) = 0$.

Denote C_T related to T for this notation is important in the subsequent analysis. Meanwhile, for any $\bar{k} \in Z^+$, let $\hat{x}(t)$ be the solution of SDEwMS (2.1) on $t \geq \bar{k}\Delta$ with initial values $\hat{x}(\bar{k}\Delta) = y_{\bar{k}}$ and $r(\bar{k}\Delta) = r^{\Delta}_{\bar{k}}$. Then, it follows from the time-homogeneity of SDEwMS (2.1) and Remark 2.1 that (2.15) implies

$$\sup_{\bar{k}\Delta \leq k\Delta \leq \bar{k}\Delta + T} E|y_k - \hat{x}(t)|^p \leq C_T E|y_{\bar{k}}|^p u(\Delta). \tag{2.16}$$

Similar to the proof of the Theorem 2.8 in [9], one can obtain the following important lemma, which will be illustrated by the IEM method in the next section.

Lemma 2.1. *Suppose that a numerical method satisfies Assumptions 1 and 2, then the SDEwMS (2.1) with initial values $x(0) = x_0 \in R^n$ and $r(0) = i_0 \in S$ is pth moment exponentially stable if and only if there exists a step size $\Delta > 0$ such that the numerical method is pth moment exponentially stable with rate constant γ, growth constant N, and global error constant C_T satisfying*

$$2^p C_T u(\Delta) + e^{-\frac{3}{4}\gamma T} \leq e^{-\frac{1}{2}\gamma T},$$

where $T = \bar{k}\Delta$ and $\bar{k}$ is the smallest integer with $\bar{k} \geq 4\log(2^p N)/(\gamma\Delta)$.

Lemma 2.1 shows that under Assumptions 1 and 2, the numerical method for the sufficiently small step sizes Δ is pth moment exponentially stable if and only if the SDEwMS (2.1) is pth moment exponentially stable.

3. The improved Euler-Maruyama method

If Lemma 2.1 holds for any numerical solutions, it must need that the numerical solutions have the Markov property and satisfy Assumptions 1 and 2. However, which numerical method can produce such numerical solutions? The IEM method introduced in this section will give a positive answer. Based on this, when the stochastic stability is understood as the pth ($p \in (0,1)$) moment exponential stability, one can give positive answers to ($Q1$) and ($Q2$).

In order to analyze the IEM method and simulate the approximate solutions of Eq. (2.1), the following lemma is useful ([15]).

Lemma 3.1. *Given $\Delta > 0$, let $r^{\Delta}_k = r(k\Delta)$ for $k \geq 0$. Then $\{r^{\Delta}_k, k = 0, 1, 2, \ldots\}$ is a discrete Markov chain with one-step transition probability matrix.*

$$p(\Delta) = (p_{ij}(\Delta))_{N\times N} = e^{\Delta\Gamma}. \tag{3.1}$$

Given a step size $\Delta > 0$, the discrete Markov chain $\{r^{\Delta}_k,\ k = 0, 1, 2, \ldots\}$ can be simulated as follows: Compute the one-step transition probability matrix by (3.1). Let $r^{\Delta}_0 = i_0$ and generate a random number ξ_1 which is uniformly distributed in $[0, 1]$. Define

$$r^{\Delta}_1 = \begin{cases} i_1, & \text{if } i_1 \in S - \{N\} \text{ such that } \sum\limits_{j=1}^{i_1-1} p_{i_0,j}(\Delta) \leq \xi_1 < \sum\limits_{j=1}^{i_1} p_{i_0,j}(\Delta), \\ N, & \text{if } \sum\limits_{j=1}^{N-1} p_{i_0,j}(\Delta) \leq \xi_1, \end{cases}$$

where we set $\sum_{i=1}^{0} p_{i_0,j}(\Delta) = 0$ as usual. Generate independently a new random number ξ_2 which is again uniformly distributed in $[0,1]$ and then define

$$r_2^\Delta = \begin{cases} i_2, & \text{if } i_2 \in S - \{N\} \text{ such that } \sum_{j=1}^{i_2-1} p_{r_1^\Delta,j}(\Delta) \le \xi_2 < \sum_{j=1}^{i_2} p_{r_1^\Delta,j}(\Delta), \\ N, & \text{if } \sum_{j=1}^{N-1} p_{r_1^\Delta,j}(\Delta) \le \xi_2. \end{cases}$$

Repeating this procedure, a trajectory of $\{r_k^\Delta, k = 0,1,2,\ldots\}$ can be generated. This procedure can be carried out independently to obtain more trajectories. After explaining how to simulate the discrete Markov chain $\{r_k^\Delta, k = 0,1,2,\ldots\}$, one can now give the IEM numerical solutions of SDEwMS (2.1).

Let $t_k = k\Delta$ for $k \in Z^+$, the numerical solutions produced by the IEM method are defined by

$$\begin{cases} y_k^* = y_k + f(y_k, r_k^\Delta)\Delta + g(y_k, r_k^\Delta)\Delta\omega_k, \\ y_{k+1} = y_k + \frac{1}{2}[f(y_k, r_k^\Delta) + f(y_k^*, r_k^\Delta)]\Delta + g(y_k, r_k^\Delta)\Delta\omega_k. \end{cases} \tag{3.2}$$

with initial values $x(0) = x_0 \in R^n$ and $r(0) = i_0 \in S$, where $y_k \approx x(t_k)$, $\Delta\omega_k = \omega(t_{k+1}) - \omega(t_k)$ and $r_k^\Delta = r(t_k)$.

Given $y_{\bar{k}}(\bar{k} \in Z^+)$, the process $\{y_k\}_{k \ge \bar{k}}$ can be fully determined by (3.2), but how the process researches $y_{\bar{k}}$ has no further use, therefore, the discrete process $\{y_k\}_{k \in Z^+}$ is a time-homogeneous Markov process, that is, the IEM method (3.2) has the Markov property.

The main result is stated in the following.

Theorem 3.1. *Let ($\boldsymbol{H}_0$) hold. Then the SDEwMS (2.1) with initial values $x(0) = x_0 \in R^n$ and $r(0) = i_0 \in S$ is pth moment exponentially stable if and only if there exists a step size $\Delta > 0$ satisfying $(1 \vee 2L^2)\Delta < 1$ such that the IEM method is pth moment exponentially stable with rate constant γ, growth constant N, and global error constant C_T satisfying*

$$2^p C_T(\Delta)^{\frac{p}{2}} + e^{-\frac{3}{4}\gamma T} \le e^{-\frac{1}{2}\gamma T},$$

where $T = \bar{k}\Delta$ and $\bar{k}$ is the smallest integer with $\bar{k} \ge 4\log(2^p N)/(\gamma\Delta)$.

Remark 3.1. If this theorem holds, one can positively answer both ($Q1$) and ($Q2$) when the stochastic stability means the pth moment exponential stability.

In fact, as the IEM method (3.2) has the Markov property, if it also satisfies Assumptions 1–2, then from Lemma 2.1, one can get the conclusion in Theorem 3.1. Therefore, we just need to show that the IEM method (3.2) satisfies Assumptions 1–2. We shall prove Theorem 3.1 by several lemmas.

The following lemma is useful for proving that the IEM method satisfies Assumptions 1 and 2.

Lemma 3.2. *Let ($\boldsymbol{H}_0$) hold. Let Δ be sufficiently small for $\Delta < 1$, then the discrete process $\{y_k\}_{k \in Z^+}$ and $\{y_k^*\}_{k \in Z^+}$ defined by the IEM method (3.2) satisfy*

$$E|y_k^*| \le KE|y_k|, \tag{3.3}$$

where $K = 3(1 + 2L^2)$.

Proof. From (3.2), one can get

$$|y_k^*|^2 \le 3|y_k|^2 + 3|f(y_k, r_k^\Delta)\Delta|^2 + 3|g(y_k, r_k^\Delta)\Delta\omega_k|^2. \tag{3.4}$$

By the linear growth condition (2.3), it follows

$$\begin{aligned}E|y_k^*|^2 \le& 3E|y_k|^2 + 3E|f(y_k, r_k^\Delta)\Delta|^2 + 3E|g(y_k, r_k^\Delta)\Delta\omega_k|^2\\ \le& 3E|y_k|^2+3E|f(y_k, r_k^\Delta)\Delta|^2+3E|g(y_k, r_k^\Delta)|^2\Delta+3E\left[|g(y_k, r_k^\Delta)|^2(|\Delta\omega_k|^2-\Delta)\right]\\ \le& 3(1 + L^2\Delta^2 + L^2\Delta)E|y_k|^2,\end{aligned}$$

this and $\Delta < 1$ yield

$$E|y_k^*| \le 3(1 + 2L^2)E|y_k|^2. \tag{3.5}$$

Hence, (3.3) holds. The proof is complete.

□

The following lemma shows that the IEM method satisfies Assumption 1.

Lemma 3.3. *Let ($\boldsymbol{H}_0$) hold. Let Δ be sufficiently small for $\Delta < 1$, then the discrete process $\{y_k\}_{k\in Z^+}$ defined by the IEM method (3.2) satisfies*

$$\sup_{0\le t_k\le T} E|y_k|^p \le \bar{H}(T, p, L)|x_0|^p, \ \forall T \ge 0, \tag{3.6}$$

where $\bar{H}(T, p, L) = 2^p e^{\frac{p}{2}L^2[(K+1)T+4]T}$.

Proof. Let $z_1(t) = \sum_{k=0}^{\infty} y_k \mathbf{1}_{[t_k,t_{k+1})}(t)$, $z_2(t) = \sum_{k=0}^{\infty} y_k^* \mathbf{1}_{[t_k,t_{k+1})}(t)$, where $\mathbf{1}_G$ is the indicator function for the set G. It is easy to see from (3.2) that

$$\begin{aligned}y_{k+1} =& x_0 + \frac{1}{2}\int_0^{t_{k+1}} f(z_1(s), \bar{r}(s))ds + \frac{1}{2}\int_0^{t_{k+1}} f(z_2(s), \bar{r}(s))ds\\ &+ \int_0^{t_{k+1}} g(z_1(s), \bar{r}(s))d\omega(s),\end{aligned}$$

where $z_1(t) = y_k$, $z_2(t) = y_k^*$ and $\bar{r}(t) = r_k^\Delta$ for $t \in [t_k, t_{k+1})$. Applying the Hölder inequality, for $0 \le t_{k+1} \le T(\forall k \in \mathbb{Z}^+)$, one can show that

$$\begin{aligned}|y_{k+1}|^2 \le& 4\left(|x_0|^2 + \frac{1}{4}\left|\int_0^{t_{k+1}} f(z_1(s), \bar{r}(s))ds\right|^2 + \frac{1}{4}\left|\int_0^{t_{k+1}} f(z_2(s), \bar{r}(s))ds\right|^2\right.\\ &\left.+\left|\int_0^{t_{k+1}} g(z_1(s), \bar{r}(s))d\omega(s)\right|^2\right)\\ \le& 4|x_0|^2 + T\int_0^{t_{k+1}} |f(z_1(s), \bar{r}(s))|^2 ds + T\int_0^{t_{k+1}} |f(z_2(s), \bar{r}(s))|^2 ds\\ &+ 4\left|\int_0^{t_{k+1}} g(z_1(s), \bar{r}(s))d\omega(s)\right|^2.\end{aligned}$$

According to the linear growth condition (2.3), it follows

$$E|y_{k+1}|^2 \le 4|x_0|^2 + L^2T\int_0^{t_{k+1}} E|z_1(s)|^2 ds + L^2T\int_0^{t_{k+1}} E|z_2(s)|^2 ds + 4L^2\int_0^{t_{k+1}} E|z_1(s)|^2 ds$$

$$=4|x_0|^2 + L^2(T+4)\Delta\sum_{j=0}^{k} E|y_j|^2 + L^2T\Delta\sum_{j=0}^{k} E|y_j^*|^2.$$

By employing (3.3), one can obtain

$$E|y_{k+1}|^2 \le 4|x_0|^2 + L^2[(K+1)T+4]\Delta\sum_{j=0}^{k} E|y_j|^2.$$

By the discrete Gronwall inequality (see Theorem 2.5 of [11]), one obtains

$$E|y_{k+1}|^2 \le 4e^{L^2[(K+1)T+4]T}|x_0|^2,$$

implies

$$\sup_{0\le t_{k+1}\le T} E|y_{k+1}|^2 \le 4e^{L^2[(K+1)T+4]T}|x_0|^2. \tag{3.7}$$

For $p\in(0,1)$, by the Hölder inequality, it is easy to see

$$\sup_{0\le t_{k+1}\le T} E|y_{k+1}|^p \le \left(\sup_{0\le t_{k+1}\le T} E|y_{k+1}|^2\right)^{\frac{p}{2}} \le 2^p e^{\frac{p}{2}L^2[(K+1)T+4]T}|x_0|^p. \tag{3.8}$$

Therefore, (3.6) holds. The proof is complete. □

In order to prove that the IEM method satisfies Assumption 2, we need the following lemma.

Lemma 3.4. *Let ($\boldsymbol{H}_0$) hold. Let Δ be sufficiently small for $2L^2\Delta<1$, then the solution of SDEwMS (2.1) has the property*

$$E|x(t)-x(t_k)|^2 \le \bar{C}_T\Delta|x_0|^2, \quad 0\le t_k\le t\le t_{k+1}\le T \tag{3.9}$$

where $\bar{C}_T = 3(1+2L^2)e^{3L^2T(T+1)}$.

Proof. It follows from Eq. (2.1) that

$$x(t)-x(t_k) = \int_{t_k}^{t} f(x(s),r(s))ds + \int_{t_k}^{t} g(x(s),r(s))d\omega(s),$$

one can show that

$$E|x(t)-x(t_k)|^2 \le 2L^2(\Delta+1)\int_{t_k}^{t} E|x(s)|^2ds \le (1+2L^2)\int_{t_k}^{t} E|x(s)|^2ds.$$

From (2.6), one gains

$$E|x(t)-x(t_k)|^2 \le 3(1+2L^2)e^{3L^2T(T+1)}\Delta|x_0|^2,$$

which implies (3.9) holds. The proof is complete. □

Lemma 3.5. *Let ($\boldsymbol{H}_0$) hold. Let $p\in(0,1)$ and Δ be sufficiently small for $(1\vee 2L^2)\Delta<1$, then the IEM numerical solution (3.2) and the true solution of SDEwMS (2.1) satisfy*

$$\sup_{0\le k\Delta\le T} E|y_k - x(t_k)|^p \le C_T(\Delta)^{\frac{p}{2}}|x_0|^p, \ \forall T>0, \tag{3.10}$$

where

$$C_T = B^{\frac{p}{2}} e^{6pL^2(T+1)T},$$

here

$$B = 6C_1 + \frac{3}{2}TC_1(1+K) + 6L^2T\bar{C}_T(T+2) + 72L^4T^2e^{L^2[(K+1)T+4]}$$

with

$$C_1 = 16L^2Te^{L^2[(K+1)T+4]T} \max_{1\leq i\leq N}(-\gamma_{ii}),$$

and $\bar{C}_T$ is defined in Lemma 3.4.

Proof. From SDEwMS (2.1) and

$$y_{k+1} = x_0 + \int_0^{t_{k+1}} \left[\frac{1}{2}f(z_1(s),\bar{r}(s)) + \frac{1}{2}f(z_2(s),\bar{r}(s))\right] ds + \int_0^{t_{k+1}} g(z_1(s),\bar{r}(s))d\omega(s),$$

it follows

$$\begin{aligned} y_{k+1} - x(t_{k+1}) = &\int_0^{t_{k+1}} \left[\frac{1}{2}f(z_1(s),\bar{r}(s)) + \frac{1}{2}f(z_2(s),\bar{r}(s)) - f(x(s),r(s))\right] ds \\ &+ \int_0^{t_{k+1}} \left(g(z_1(s),\bar{r}(s)) - g(x(s),r(s))\right) d\omega(s). \end{aligned}$$

Using the Hölder inequality, and then taking expectation, one obtains

$$\begin{aligned} E|y_{k+1} - x(t_{k+1})|^2 \leq &\frac{3}{4}T\int_0^{t_{k+1}} E|f(z_1(s),\bar{r}(s)) - f(x(s),r(s))|^2 ds \\ &+ \frac{3}{4}TE\int_0^{t_{k+1}} |f(z_2(s),\bar{r}(s)) - f(x(s),r(s))|^2 ds \\ &+ 3\int_0^{t_{k+1}} E|g(z_1(s),\bar{r}(s)) - g(x(s),r(s))|^2 ds \\ = &\frac{3}{4}TJ_k^1(t) + \frac{3}{4}TJ_k^2(t) + 3TJ_k^3(t), \end{aligned} \tag{3.11}$$

where

$$\begin{aligned} J_k^1(t) &= \int_0^{t_{k+1}} E|f(z_1(s),\bar{r}(s)) - f(x(s),r(s))|^2 ds, \\ J_k^2(t) &= E\int_0^{t_{k+1}} |f(z_2(s),\bar{r}(s)) - f(x(s),r(s))|^2 ds, \\ J_k^3(t) &= \int_0^{t_{k+1}} E|g(z_1(s),\bar{r}(s)) - g(x(s),r(s))|^2 ds. \end{aligned}$$

In order to get the inequality (3.10), one needs to estimate $J_k^i(t)(i=1,2,3)$. In the following, we estimate $J_k^i(t)(i=1,2,3)$ step by step.

Step 1. Estimate $J_k^1(t)$. Let $x_1(t) = \sum_{k=0}^{\infty} x(t_k)\mathbf{1}_{[t_k,t_{k+1})}(t)$. Then, for $0 \leq t_{k+1} \leq T$, by the Hölder inequality, (H_0) and Lemma 3.4, it is easy to see

$$J_k^1(t) \leq 2\int_0^{t_{k+1}} E\left|f(z_1(s),\bar{r}(s)) - f(z_1(s),r(s))\right|^2 ds$$

$$
\begin{aligned}
&+2\int_0^{t_{k+1}} E\left|f(z_1(s),r(s))-f(x(s),r(s))\right|^2 ds \\
\leq& 2\int_0^{t_{k+1}} E\left|f(z_1(s),\bar r(s))-f(z_1(s),r(s))\right|^2 ds+2L^2\int_0^{t_{k+1}} E|z_1(s)-x(s)|^2 ds \\
\leq& 2\int_0^{t_{k+1}} E\left|f(z_1(s),\bar r(s))-f(z_1(s),r(s))\right|^2 ds+4L^2\int_0^{t_{k+1}} E|z_1(s)-x_1(s)|^2 ds \\
&+4L^2\int_0^{t_{k+1}} E|x_1(s)-x(s)|^2 ds. \qquad (3.12)
\end{aligned}
$$

It follows (3.9) that

$$
\begin{aligned}
J_k^1(t)\leq& 2\int_0^{t_{k+1}} E|f(z_1(s),\bar r(s))-f(z_1(s),r(s))|^2 ds \\
&+4L^2\int_0^{t_{k+1}} E|z_1(s)-x_1(s)|^2 ds+4L^2\sum_{j=0}^{k}\int_{t_j}^{t_{j+1}} E|x(t_j)-x(s)|^2 ds \\
\leq& 2E\int_0^{t_{k+1}} |f(z_1(s),\bar r(s))-f(z_1(s),r(s))|^2 ds \\
&+4L^2\int_0^{t_{k+1}} E|z_1(s)-x_1(s)|^2 ds+4L^2T\bar C_T|x_0|^2\Delta. \\
\leq& 2E\int_0^{t_{k+1}} |f(z_1(s),\bar r(s))-f(z_1(s),r(s))|^2 ds \\
&+4L^2\Delta\sum_{j=0}^{k} E|y_j-x(t_j)|^2+4L^2T\bar C_T|x_0|^2\Delta. \qquad (3.13)
\end{aligned}
$$

According to the linear growth condition (2.3) and Markov property, one gains

$$
\begin{aligned}
\int_{t_j}^{t_{j+1}} E|f(y_j,r_j^\Delta)-f(y_j,r(s))|^2 ds \leq& 2\int_{t_j}^{t_{j+1}} E[(|f(y_j,r_j^\Delta)|^2+|f(y_j,r(s))|^2)\mathbf{1}_{\{r(s)\neq r_j^\Delta\}}]ds \\
\leq& 4L^2\int_{t_j}^{t_{j+1}} E(|y_j|^2\mathbf{1}_{\{r(s)\neq r_j^\Delta\}})ds \\
\leq& 4L^2\int_{t_j}^{t_{j+1}} E[E(|y_j|^2\mathbf{1}_{\{r(s)\neq r_j^\Delta\}}|r_j^\Delta)]ds \\
\leq& 4L^2\int_{t_j}^{t_{j+1}} E\left[E(|y_j|^2|r_j^\Delta)E(\mathbf{1}_{\{r(s)\neq r_j^\Delta\}}|r_j^\Delta)\right]ds, \qquad (3.14)
\end{aligned}
$$

where in the last step we use the fact that y_j and $\mathbf{1}_{\{r(s)\neq r_j^\Delta\}}$ are conditionally independent with respect to the σ-algebra generated by r_j^Δ. Similar to estimate (4.16) in Theorem 4.1 of [11], one can gain

$$
E[\mathbf{1}_{\{r(s)\neq r_j^\Delta\}}|r_j^\Delta]\leq C\Delta+o(\Delta) \qquad (3.15)
$$

with $C=\max_{1\leq i\leq N}(-\gamma_{ii})$. Substituting (3.15) into (3.14), it follows from (3.7) that

$$
\int_{t_j}^{t_{j+1}} E\left|f(y_j,r_j^\Delta)-f(y_j,r(s))\right|^2 ds\leq 4L^2(C\Delta+o(\Delta))\int_{t_j}^{t_{j+1}} E|y_j|^2 ds
$$

$$\leq 16L^2 e^{L^2[(K+1)T+4]T}(C\Delta + o(\Delta))\Delta|x_0|^2,$$

then

$$\begin{aligned}E\int_0^{t_{k+1}}|f(z_1(s),\bar{r}(s))-f(z_1(s),r(s))|^2ds &= \sum_{j=0}^{k}E\int_{t_j}^{t_{j+1}}|f(y_j,r_j^{\Delta})-f(y_j,r(s))|^2ds\\ &\leq 16L^2Te^{L^2[(K+1)T+4]T}(C\Delta+o(\Delta))|x_0|^2\\ &\triangleq (C_1\Delta+o(\Delta))|x_0|^2. \qquad (3.16)\end{aligned}$$

Applying (3.16) to (3.13), one has

$$J_k^1(t) \leq 2(C_1 + 2L^2T\bar{C}_T)\Delta|x_0|^2 + 4L^2\Delta\sum_{j=0}^{k}E|y_j - x(t_j)|^2. \qquad (3.17)$$

Step 2. Estimate $J_k^2(t)$. By the Hölder inequality, the linear growth condition (2.3), and Lemma 3.2, similar to estimate $J_k^1(t)$, one can obtain

$$J_k^2(t) \leq 2(KC_1 + 2L^2T\bar{C}_T)\Delta|x_0|^2 + 4L^2\Delta\sum_{j=0}^{k}E|y_j^* - x(t_j)|^2. \qquad (3.18)$$

In order to estimate the $J_k^2(t)$, here we need to further simplify $\sum_{j=0}^{k}E|y_j^* - x(t_j)|^2$. Using (3.2) and the linear growth condition (2.3), one yields

$$\begin{aligned}\sum_{j=0}^{k}E|y_j^*-x(t_j)|^2 &= \sum_{j=0}^{k}E|y_j + f(y_j,r_j^{\Delta})\Delta + g(y_j,r_j^{\Delta})\Delta\omega_j - x(t_j)|^2\\ &\leq 3\sum_{j=0}^{k}E|y_j-x(t_j)|^2 + 3(\Delta)^2\sum_{j=0}^{k}E|f(y_j,r_j^{\Delta})|^2\\ &\quad + 3\sum_{j=0}^{k}E|g(y_j,r_j^{\Delta})\Delta\omega_j|^2\\ &\leq 3\sum_{j=0}^{k}E|y_j-x(t_j)|^2 + 3L^2(\Delta)^2\sum_{j=0}^{k}E|y_j|^2 + 3L^2\Delta\sum_{j=0}^{k}E|y_j|^2,\end{aligned}$$

together with (3.7), for $0 \leq k\Delta \leq T$ and $\Delta < 1$, one obtains

$$\sum_{j=0}^{k}E|y_j^*-x(t_j)|^2 \leq 3\sum_{j=0}^{k}E|y_j-x(t_j)|^2 + 24L^2Te^{L^{2[(K+1)T+4]}T}|x_0|^2, \qquad (3.19)$$

this and (3.18) yield

$$\begin{aligned}J_k^2(t) \leq& 2(KC_1 + 2L^2T\bar{C}_T)\Delta|x_0|^2 + 12L^2\Delta\sum_{j=0}^{k}E|y_j-x(t_j)|^2\\ &+ 96L^4T\Delta e^{L^2[(K+1)T+4]T}|x_0|^2. \qquad (3.20)\end{aligned}$$

Step 3. Estimate $J_k^3(t)$. Similar to the estimate $J_k^1(t)$, one can derive

$$J_k^3(t) \le 2(C_1 + 2L^2T\bar{C}_T)\Delta E|x_0|^2 + 4L^2\Delta \sum_{j=0}^{k} E|y_j - x(t_j)|^2. \tag{3.21}$$

Based on Steps 1-3, it follows from (3.17) and (3.20)–(3.21) that

$$\begin{aligned} E|y_{k+1} - x(t_{k+1})|^2 \le & \left(\frac{3}{2}TC_1 + 3L^2T^2\bar{C}_T\right)\Delta|x_0|^2 + 3L^2T\Delta\sum_{j=0}^{k} E|y_j - x(t_j)|^2 \\ & + \left(\frac{3}{2}KTC_1 + 3L^2T^2\bar{C}_T\right)\Delta|x_0|^2 + 9L^2T\Delta\sum_{j=0}^{k} E|y_j - x(t_j)|^2 \\ & + 72L^4T^2e^{L^2[(K+1)T+4]T}\Delta|x_0|^2 + (6C_1 + 12L^2T\bar{C}_T)\Delta|x_0|^2 \\ & + 12L^2\Delta\sum_{j=0}^{k} E|y_j - x(t_j)|^2 \\ \triangleq & B\Delta|x_0|^2 + 12L^2(T+1)\sum_{j=0}^{k} E|y_j - x(t_j)|^2\Delta, \end{aligned}$$

where B is defined as before. By discrete Gronwall inequality, it is easy to see that

$$E|y_{k+1} - x(t_{k+1})|^2 \le Be^{12L^2T(T+1)}\Delta|x_0|^2.$$

Therefore, it follows from the Hölder inequality that for $p \in (0,1)$

$$\sup_{0\le t_k\le T} E|y_k - x(t_k)|^p \le B^{\frac{p}{2}}e^{6pL^2T(T+1)}(\Delta)^{\frac{p}{2}}|x_0|^p = C_T(\Delta)^{\frac{p}{2}}|x_0|^p \tag{3.22}$$

yields (3.10). The proof is therefore complete. □

Remark 3.2. Under the global Lipschitz condition, one can see that lemmas 3.3 and 3.5 imply the IEM method (3.2) satisfies Assumptions 1 and 2 for the sufficiently small step sizes Δ, respectively.

Proof of Theorem 3.1. It follows from Lemma 2.1, Lemma 3.3 and Lemma 3.5, one can obtain Theorem 3.1. □

4. Almost sure exponential stability

This section discusses the relationship between the pth moment exponential stability and the almost sure exponential stability of the SDEwMS (2.1). We show that the pth moment exponential stability of the true solution and the numerical solution for Eq.(2.1) imply the almost sure exponential stability of the true solution and the numerical solution, respectively.

The following theorem reveals that the pth moment exponential stability of the true solution implies the almost sure exponential stability. Readers can refer to Theorem 5.9 of [11].

Theorem 4.1. *Let ($\boldsymbol{H}_0$) hold. Assume that the SDEwMS (2.1) is pth moment exponentially stable and satisfies (2.8). Then the solution $x(t)$ of the SDEwMS (2.1) with initial values $x(0) = x_0 \in R^n$ and $r(0) = i_0 \in S$ satisfies*

$$\limsup_{t\to\infty} \frac{\log(|x(t)|)}{t} \leq -\frac{\lambda}{p} \quad a.s. \tag{4.1}$$

That is, SDEwMS (2.1) is almost surely exponentially stable.

The following result shows that the pth moment exponential stability of the numerical solution also implies the almost sure exponential stability. The proof of this result is similar to the proof of Theorem 4.2 in [9], here we omit it.

Theorem 4.2. *Assume that the numerical method is pth moment exponentially stable and satisfies (2.11). Then the numerical solution y_k of SDEwMS (2.1) with initial values $x(0) = x_0 \in R^n$ and $r(0) = i_0 \in S$ satisfies*

$$\limsup_{k\to\infty} \frac{1}{k\Delta} \log(|y_k|) \leq -\frac{\gamma}{p}\ a.s. \tag{4.2}$$

That is, the numerical method of SDEwMS (2.1) is also almost surely exponentially stable.

5. Numerical simulation

This section presents a numerical example to illustrate the efficiency of the obtained results.

Denote $C^2(R^n \times S; R^+)$ the family of all non-negative functions $V(x,i)$ on $(R^n \times S)$ that are continuously twice differentiable in x. For each $V(x,i) \in C^2(R^n \times S; R^+)$ define an operator LV from $R^n \times S$ to R:

$$LV(x,i) = V_x^T(x,i)f(x,i) + \frac{1}{2}\text{trace}[g^T(x,i)V_{xx}(x,i)g(x,i)] + \sum_{j\in S}\gamma_{ij}V(x,j).$$

Let $\omega(t)$ be a scalar Brownian motion. Let $r(t)$ be a right-continuous time-homogeneous Markov chain taking values $S = \{1,2\}$ with generator

$$\Gamma = (\gamma_{ij})_{2\times 2} = \begin{bmatrix} -2 & 2 \\ 4 & -4 \end{bmatrix}.$$

It is easy to see that the Markov chain has a unique stationary distribution $\pi = (\frac{2}{3}, \frac{1}{3})$. We assume that $\omega(t)$ and $r(t)$ are independent.

Example 5.1. Consider the following nonlinear scalar SDEwMS:

$$dx(t) = f(x(t), r(t))dt + g(x(t), r(t))d\omega(t), \quad t > 0, \tag{5.1}$$

where $x(0) = 1$, $r(0) = 1$ and for $\forall (x,i) \in R \times S$,

$$\begin{aligned} f(x,i) = \sin x - 5x \ \text{ and } \ g(x,i) = 2x, \ \text{ if } \ i = 1, \\ f(x,i) = x \ \text{ and } \ g(x,i) = \frac{1}{2}x, \ \text{ if } \ i = 2. \end{aligned} \tag{5.2}$$

Note that the drift coefficient $f(x, i)$ and diffusion coefficient $g(x, i)$ of Eq. (5.1) satisfy ($\boldsymbol{H}_0$). One can choose $V(x, i) = |x|^2$. It is easy to show that $LV(x, 1) \leq -4x^2$, and $LV(x, 2) \leq 2.25x^2$. We hence have $\lambda = (-4, 2.25)$. By Theorem 3.1 in [20], one can obtain Eq. (5.1) is mean square exponentially stable. Applying the Hölder inequality, it is easy to see that Eq. (5.1) is pth ($p \in (0, 1)$) moment exponentially stable. According to the Theorems 3.1, 4.1 and 4.2, one can obtain Eq. (4.1) is almost surely exponentially stable (Lyapunov exponent is less than -0.1686) and the IEM method (3.2) with appropriate step sizes applied to (5.1) is also almost surely exponentially stable.

We can rewrite Eq. (4.1) into the following two equations

$$dx(t) = (\sin(x(t)) - 5x(t))dt + 2x(t)d\omega(t) \tag{5.3}$$

and

$$dx(t) = x(t)dt + \frac{1}{2}x(t)d\omega(t), \tag{5.4}$$

switching from one to the other according to the movement of the Markov chain $r(t)$. We can also prove that (5.3) is almost surely exponentially stable (Lyapunov exponent is less than -2) and the IEM method (3.2) with appropriate step sizes applied to (5.3) is also almost surely exponentially stable, while (5.4) is almost surely exponentially unstable (Lyapunov exponent is greater than 0.875) and the IEM method (3.2) with appropriate step sizes applied to (5.4) is also almost surely exponentially stable. However, as the result of the Markovian switching, the overall behavior of Eq. (5.1) is also almost surely exponentially stable.

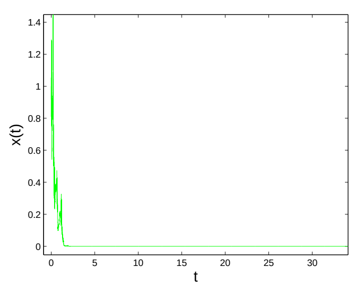

Figure 1. The sample paths of the solution to (5.1).

On the other hand, choose the step size $\Delta = 10^{-3}$ and carry out the numerical simulation of Eq. (5.1), Eq. (5.3) and Eq. (5.4) with initial values $x_0 = 1$ and $r_0 = 1$ based on the IEM method (3.2). The corresponding figures are shown in Figure 1 and Figure 2, respectively. From these figures, it can be seen that the numerical simulation with an appropriate sufficiently small step size is consistent with the theoretical results.

6. Conclusions

This paper investigates the exponential stability of SDEwMSs and its corresponding IEM method with sufficiently small step sizes. It illustrates that under the global Lipschitz condition, the SDEwMS is pth ($p \in (0, 1)$) moment exponentially stable if

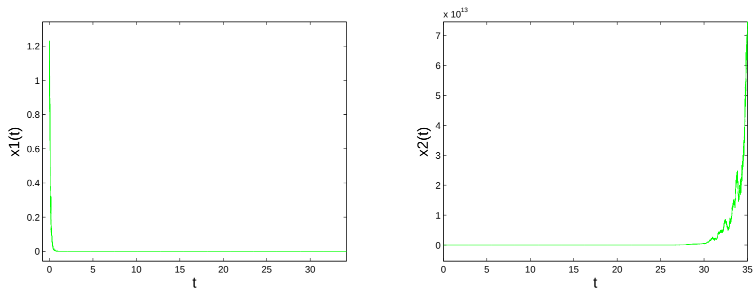

Figure 2. The sample paths of the solution to (5.3)(left panel), and the sample paths of the solution to (5.4)(right panel).

and only if the corresponding IEM method with sufficiently small step sizes is pth ($p \in (0, 1)$) moment exponentially stable. Based on such theory, when the stochastic stability is understood as the pth ($p \in (0, 1)$) moment exponential stability, we can give positive answers to ($Q1$) and ($Q2$) for the SDEwMS. It further derives that the pth moment exponential stability of such SDEwMS or the corresponding IEM method implies the almost sure exponential stability of the SDEwMS and the IEM method. Lyapunov functions method is the classical yet powerful technique in the study of stochastically stable, but in general, it is not convenient for us to use this technique for there is no an universal method can guarantee to find an appropriate Lyapunov function. When the Lyapunov functions approach is not available, the theory established in this paper enables us to study the exponential stability of SDEwMSs using the corresponding IEM method. Therefore, we can now carry out careful numerical simulation using the IEM method with a sufficiently small step size to simulate the solutions of SDEwMSs, so as to study the stochastic stability of SDEwMSs.

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