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Near-invariant Tori on Exponentially Long Time for Poisson systems*

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Abstract This paper deals with the near-invariant tori for Poisson systems. It is shown that the orbits with the initial points near the Diophantine torus approach some quasi-periodic orbits over an extremely long time. In particular, the results hold for the classical Hamiltonian system, and in this case the drift of the motions is smaller than one in the past works.

Keywords Poisson system, near-invariant tori, rapidly Newton iteration.

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1. Problem, preliminaries and result

The problem of stability of Hamiltonian systems occupies a crucial place in the field of dynamic systems. As well known, KAM theory shows that most of quasi-periodic motions of the integrable Hamiltonian systems are persistent under a small perturbation. The name comes from the initials of Kolmogorov, Arnold and Moser who laid the foundation of the theory [1, 3, 6]. In 1977s, Nekhoroshev presented a global result. He showed that under a perturbation of order ε of an integrable Hamiltonian system with the steepness condition, the action variable of an arbitrary orbit vary only in the order of ε^b over a time interval of the order of $\exp(\varepsilon^{-a})$, where a and b are positive constants [7]. Now one refers to Nekhoroshev's theorem as effective stability. Later on, much mathematics are devoted to studying KAM theory and effective stability, and a great deal of significant results are obtained, see [2, 4, 8–10] and the references therein.

One remarkable problem is that the above works only localize on classical Hamiltonian systems which are defined on an even-dimensional manifold. Many systems in applications can not be written as Hamiltonian forms, for example, Lotka-Voterra model [11], the motion equation of a rigid body without any external forces, ABC flow and so on. The reason is that their phase spaces are of odd-dimensional. Note

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that these systems possess general Poisson structures. The problem considered in this paper is to generalize the stability theory of Hamiltonian systems to Poisson systems defined on odd-dimensional spaces.

We first introduce the concept of Poisson systems. Moreover, some fundamental properties are given without proofs. For details, see [5].

Let $B : D \times T^n \to R^{(m+n)\times(m+n)}$ be a smooth matrix-valued function, where $D \subset R^m$ be a bounded, connected and closed region, and $T^n = R^n/Z^n$. For all $z = (y, x) \in D \times T^n$, set

$$\{F, G\}(z) = \nabla F(z)^T B(z) \nabla G(z). \tag{1.1}$$

Lemma 1.1. *The bracket defined in (1.1) is bilinear, skew-symmetric and satisfies*

$$\{\{F, G\}, H\} + \{\{G, H\}, F\} + \{\{H, F\}, G\} = 0, \tag{1.2}$$

$$\{F \cdot G, H\} = F \cdot \{G, H\} + G \cdot \{F, H\} \tag{1.3}$$

if and only if $B^T = -B$ *and for all* i, j, k,

$$\sum_{l=1}^{m+n} \left(\frac{\partial b_{ij}(z)}{\partial z_l} b_{lk}(z) + \frac{\partial b_{jk}(z)}{\partial z_l} b_{li}(z) + \frac{\partial b_{ki}(z)}{\partial z_l} b_{lj}(z) \right) = 0. \tag{1.4}$$

Definition 1.1. If $B(z)$ satisfies $B^T = -B$ and (1.4), formula (1.1) is said to represent a general Poisson bracket. The corresponding system

$$\dot{z} = B(z) \nabla H(z) \tag{1.5}$$

is said to be a Poisson system with Hamiltonian H.

Definition 1.2. A transformation $\varphi : U \to R^{m+n}$ (where U is an open set in R^{m+n}) is called a Poisson change with respect to the bracket (1.1), if the structure matrix B satisfies

$$\varphi'(z) B(z) \varphi'(z)^T = B(\varphi(z)).$$

Lemma 1.2. *If* $B(z)$ *is the structure matrix of a Poisson bracket, the flow* $\phi^t(z)$ *of (1.5) is a Poisson change.*

Lemma 1.3. *Let* $\phi^t(z)$ *be a flow of (1.5). Acting on a function* $F : R^{m+n} \to R$, *the following formula holds:*

$$\frac{\mathrm{d}}{\mathrm{d}t} F(\phi^t(z)) = \{F, G\}(\phi^t(z)).$$

Definition 1.3. Let F and G be two smooth functions defined on some open subset of R^{m+n}. F and G are said to be in involution, if $\{F, G\} = 0$.

From now on, we begin to describe the main result of this paper. Consider a Poisson system

$$\dot{z} = B(y) \nabla H(z) \tag{1.6}$$

defined on some complex neighborhood of $D \times T^n$ in $C^m \times C^n$, where B is a structure matrix independent of x.

Through this paper, we assume that y_j, $j = 1, \cdots, m$, and x_k, $k = 1, \cdots, n$, respectively, satisfy the involution condition:

$$\{y_i, y_j\} = 0, \quad i, j = 1, \cdots, m, \tag{1.7}$$

$$\{x_k, x_l\} = 0, \quad k, l = 1, \cdots, n, \tag{1.8}$$

which imply that B can be simplified to the form

$$B = \begin{pmatrix} 0_m & B_{12} \\ -B_{12}^T & 0_n \end{pmatrix}, \tag{1.9}$$

where B_{12} is an $m \times n$ matrix, and 0_m and 0_n, respectively, are m and n order zero matrices. Hence,

$$\dot{y} = B_{12}\frac{\partial H}{\partial x}, \quad \dot{x} = -B_{12}^T\frac{\partial H}{\partial y}. \tag{1.10}$$

If $\dfrac{\partial H}{\partial x} = 0$, *that is*, H depends only on y, equation (1.10) suits

$$\dot{y} = 0, \quad \dot{x} = \omega(y) \tag{1.11}$$

with

$$-B_{12}^T\frac{\partial H}{\partial y}(y) = \omega(y). \tag{1.12}$$

We need the further assumption. Suppose that Poisson system (1.5) possesses one invariant torus. Thus, H can be writen as the form

$$H(y, x) = c_0 + a \cdot (y - y_*) + F(y - y_*, x) \tag{1.13}$$

with $F(y - y_*, x) = O((y - y_*)^2)$, namely, $\{y = y_*, x \in T^n\}$ is an invariant torus of (1.5) with frequency

$$\omega_* = -B_{12}^T(y_*)a,$$

where a is a fixed vector.

Let $|\cdot|$ denote the maximum norm of a vector in components, and $\|\cdot\|$ the usual supremum norm either for a function or for a matrix on the given set.

Theorem 1.1. *Let the above assumption hold. Let H be real analytic on the complex ρ_0-neighborhood of $B_{\delta_0}(y_*) \times T^m$ for some positive constants ρ_0 and δ_0, where $B_{\delta_0}(y_*)$ denotes a ball centered at y_* with radius δ_0. Suppose that ω_* suits Diophantine condition*

$$|k \cdot \omega_*| \geq \gamma |k|^{-\nu}, \quad 0 \neq k \in Z^n \tag{1.14}$$

for some positive constants γ and ν. Then there are positive constants ε_0, c_1, c_2 and c_3 such that for every $0 < \varepsilon \leq \varepsilon_0$, if $(y(t), x(t))$ is a solution of Poisson system with Hamiltonian (1.13) starting with $|y(0) - y_*| \leq \frac{1}{2}\varepsilon$, the following estimates hold:

$$|y(t) - y(0)| < c_1\varepsilon^{1+\frac{1}{\nu+n+2}},$$
$$|x(t) - \omega_{drift}(y(0), \varepsilon)t - x(0)| < c_2\varepsilon^{\frac{1}{\nu+n+2}}$$

for $|t| < \exp\left(c_3\varepsilon^{-\frac{1}{\nu+n+2}}\right)$. Moreover,

$$|\omega_{drift}(y(0), \varepsilon) - \omega_*| = O(\varepsilon).$$

2. Auxiliary Poisson system and small divisor problem

Let $D \subset R^l$. For small positive constants ρ and σ. We use $D+\rho$ and $D-\sigma$ to denote the ρ-complex neighborhood of D in C^l and the set of points contained in D together with σ-neighborhood, respectively. By $c_4, c_5, \cdots$, we denote the constants depending only on a, ρ, m, n, y_* and P in what follows.

We consider (1.13) on $(\{y : |y-y_*| < \varepsilon\} + \varepsilon\rho) \times (T^n + \rho)$. Let

$$y - y_* = \varepsilon Y, \quad \hat{H}(Y,x) = \frac{c_0}{\varepsilon} + a \cdot Y + \varepsilon P(Y,x,\varepsilon). \tag{2.1}$$

Here

$$P(Y,x,\varepsilon) = \frac{1}{\varepsilon^2} F(\varepsilon Y, x).$$

Then (Y,x) is defined on $(\{(Y,x) : |Y| < 1\} \times T^n) + \rho$, and under the structure matrix

$$\hat{B}(Y,\varepsilon) = \begin{pmatrix} 0_m & B_{12} \\ -B_{12}^T & 0_n \end{pmatrix} (\varepsilon Y + y_*)$$

the Poisson system corresponding to (1.13) is changed to

$$\begin{pmatrix} \dot{Y} \\ \dot{x} \end{pmatrix} = \hat{B}(Y,\varepsilon) \nabla \hat{H}(Y,x). \tag{2.2}$$

Without loss of generality, taking $c_0 = 0$. We begin study Poisson system

$$\dot{Y} = B_{12} \frac{\partial \hat{H}}{\partial x}, \quad \dot{x} = -B_{12}^T \frac{\partial \hat{H}}{\partial Y} \tag{2.3}$$

with Hamiltonian

$$\hat{H} = a \cdot Y + \varepsilon P(Y,x,\varepsilon), \tag{2.4}$$

where we omit the variables $\varepsilon Y + y_*$ in B_{12}. For a function $l(y,x)$ defined on some subset of $C^m \times C^n$, let

$$\bar{l}(y) = \int_{T^n} l(y,x)\mathrm{d}x \quad \text{and} \quad \tilde{l}\,(y,x) = l(y,x) - \bar{l}(y).$$

Set $D_0 = (B_{\frac{1}{2}}(0) \times T^n) + \rho$. In order to prove Theorem A, we need the following small divisor lemma.

Lemma 2.1. *Assume that ω_* satisfies Diophantine condition (1.14). Let $P(y,x)$ be a real analytic function on D_0. Then the equation*

$$\omega_* \cdot \frac{\partial \phi}{\partial x} + \tilde{P} = 0 \tag{2.5}$$

has only one real analytic solution ϕ satisfying $\bar{\phi} = 0$. Moreover, for any σ with $0 < \sigma < \rho$,

$$\|\phi\|_{D_0 - \sigma} \le \frac{c_4}{\sigma^{\nu+n}} \|P\|_{D_0}.$$

Proof. We write $P(y,x)$ as the Fourier expansion

$$P(y,x) = \sum_{k\in Z^n} p_k(y) e^{2\pi\sqrt{-1}k\cdot x},$$

and formally, let

$$\phi(y,x) = \sum_{0\neq k\in Z^n} \phi_k(y) e^{2\pi\sqrt{-1}k\cdot x}.$$

Inserting this formula into (2.5) we obtain

$$\phi(y,x) = -\sum_{0\neq k\in Z^n} \frac{p_k(y)}{2\pi\sqrt{-1}k\cdot\omega_*} e^{2\pi\sqrt{-1}k\cdot x},$$

which is a unique solution of (2.5) suiting $\bar{\phi}=0$. From Cauchy's formula, it follows that

$$\begin{aligned}
\|\phi\|_{D_0-\sigma} &\leq \sum_{0\neq k\in Z^n} \frac{\|\phi_k\|_{D_0-\sigma}}{2\pi|k\cdot\omega_*|}|e^{2\pi\sqrt{-1}k\cdot x}| \\
&\leq \frac{\|P\|_{D_0}}{2\pi\gamma} \sum_{0\neq k\in Z^n} \frac{|k|^\nu}{e^{2\pi\sigma|k|}} \\
&\leq \frac{2^{n-1}\|P\|_{D_0}}{2\pi\gamma} \sum_{j=1}^{\infty} j^{\nu+n-1} e^{-2\pi\sigma j} \\
&\leq c_5 \left(\int_0^{+\infty} x^{\nu+n-1} e^{-\sigma x} \mathrm{d}x + l_*^{\nu+n-1} e^{-2\pi\sigma l_*} \right) \|P\|_{D_0}.
\end{aligned}$$

Here, letting $h(\theta)=\theta^{\nu+n-1}e^{-2\pi\sigma\theta}$, l_* satisfies $h(l_*)=\max_{l\in Z^+} h(l)$. By finding critical point of $h(\theta)$, we have either $l_* = \left[\frac{\nu+n-1}{2\pi\sigma}\right]$ or $l_* = \left[\frac{\nu+n-1}{2\pi\sigma}\right]+1$, where $[\,\cdot\,]$ denotes the integer part of a given real number. So,

$$l_*^{\nu+n-1} e^{-2\pi\sigma l_*} \leq c_6 \sigma^{-(\nu+n-1)}. \tag{2.6}$$

It is easy to prove

$$\int_0^{+\infty} x^{\nu+n-1} e^{-2\pi\sigma x} \mathrm{d}x \leq c_7 \sigma^{-(\nu+n)}. \tag{2.7}$$

By (2.6) and (2.7), we have

$$\|\phi\|_{D_0-\sigma} \leq \frac{c_4}{\sigma^{\nu+n}} \|P\|_{D_0}.$$

□

3. Proof of Theorem1.1

We continue to consider Hamiltonian (2.4). Let

$$D_j = D_0 - 6j\sigma, \quad \sigma = K\varepsilon^\alpha, \quad j=1,\cdots,L,$$

where $K > 0$ is a constant determined below, and

$$\alpha = \frac{1}{\nu + n + 2}, \quad L = \left[\frac{\rho}{12K\varepsilon^\alpha}\right] + 1.$$

Obviously, there exists a constant $M > 0$ such that

$$\max\{\|\hat{B}\|, \|P\|, \|\nabla P\|, |a \cdot y|\} \le M \tag{3.1}$$

on D_0.

Assume that under the jth step Hamiltonian (2.4) is changed to the form

$$H_j(y,x) = N_j(y,\varepsilon) + \varepsilon P_j(y,x,\varepsilon), \tag{3.2}$$

$$N_j(y,\varepsilon) = a \cdot y + \varepsilon \sum_{i=1}^{j-1} \bar{P}_i(y,\varepsilon), \quad N_0(y,\varepsilon) = a \cdot y, \tag{3.3}$$

$$\|P_j\| \le \frac{1}{2^j} M, \tag{3.4}$$

defined on D_j. We introduce a Poisson change of coordinate $\Phi_{j+1} : D_{j+1} \to D_j$ by defining $\Phi_{j+1} = \phi_{j+1}^1$. Here ϕ_{j+1}^t is the flow of Poisson system

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_{j+1}^t = \varepsilon \hat{B} \nabla S_j(\phi_{j+1}^t). \tag{3.5}$$

On the basis of Lemma1.3and Taylor's formula, we have

$$\begin{aligned}
H_{j+1}(y,x) &= H_j \circ \Phi_{j+1}(y,x) \\
&= N_j(y,x) + \varepsilon\{N_j, S_j\} + \varepsilon^2 \int_0^1 (1-t)\{\{N_j, S_j\}, S_j\} \circ \phi_{j+1}^t \mathrm{d}t \\
&\qquad + \varepsilon P_j(y,x,\varepsilon) + \varepsilon^2 \int_0^1 \{P_j, S_j\} \circ \phi_{j+1}^t \mathrm{d}t \\
&= N_j(y,\varepsilon) + \varepsilon \bar{P}_j(y,\varepsilon) \\
&\qquad + \varepsilon^2 \int_0^1 \{P_j + (1-t)\{N_j, S_j\}, S_j\} \circ \phi_{j+1}^t \mathrm{d}t \\
&\qquad + \varepsilon\{N_j - N_0, S_j\} \\
&\qquad + \varepsilon(\{N_0, S_j\} + \widetilde{P}_j\,(y,x,\varepsilon)).
\end{aligned} \tag{3.6}$$

Choose S_j such that

$$\omega_* \cdot \frac{\partial S_j}{\partial x} + \widetilde{P}_j\,(y,x,\varepsilon) = 0. \tag{3.7}$$

Then

$$H_{j+1}(y,x) = N_{j+1}(y,\varepsilon) + \varepsilon P_{j+1}(y,x,\varepsilon), \tag{3.8}$$

$$N_{j+1}(y,\varepsilon) = N_j(y,\varepsilon) + \varepsilon \bar{P}_j(y,\varepsilon), \tag{3.9}$$

$$\begin{aligned}
P_{j+1}(y,x,\varepsilon) = \varepsilon &\int_0^1 \{P_j + (1-t)\{N_j, S_j\}, S_j\} \circ \phi_{j+1}^t \mathrm{d}t \\
&+ \{N_j - N_0, S_j\} \\
&+ a \cdot (B_{12}(\varepsilon y + y_*) - B_{12}(y_*))\frac{\partial S_j}{\partial x}
\end{aligned}$$

$$= P^1_{j+1} + P^2_{j+1} + P^3_{j+1}. \tag{3.10}$$

Inductively, from (3.3), (3.4) and (3.1), it follows that

$$\|N_j - N_0\| \le \varepsilon \sum_{i=0}^{j-1} \frac{1}{2^i} M \le 2M\varepsilon \tag{3.11}$$

on D_j, provided ε is sufficiently small. Let $(Y_t, X_t) = \phi^t_{j+1}(y,x)$. From (3.7), Lemma 2.1 and Cauchy's formula, for all $(Y_t, X_t) \in D_j - 2\sigma$ with $0 \le t \le 1$, we have

$$\begin{aligned}
\Big|(y,x) - (Y_t, X_t)\Big| &\le M\varepsilon \Big\|\nabla S_j\Big(Y_t, X_t\Big)\Big\|_{D_j - 2\sigma} \\
&\le \frac{M\varepsilon}{\sigma} \Big\|S\Big(Y_t, X_t\Big)\Big\|_{D_j-\sigma} \\
&\le \frac{c_4 M\varepsilon}{\sigma^{\nu+n+1}} \|P_j\|_{D_j} \\
&\le \frac{1}{2^j} \frac{c_4 M^2}{K^{\nu+n+2}} \sigma \\
&< \frac{1}{2^j}\sigma < \sigma,
\end{aligned} \tag{3.12}$$

provided K satisfies

$$K^{\frac{1}{\nu+n+2}} > c_4 M^2. \tag{A}$$

By the geometric lemma in [1], $\phi^{-t}_{j+1}(D_j - 2\sigma) \supset D_j - 3\sigma$, and ϕ^{-t}_{j+1} is a diffeomorphism defined on $D_j - 6\sigma$. This shows that $\phi^t_{j+1}(D_{j+1}) \subset D_j$. If K satisfies the inequalities

$$\max\left\{\frac{2M^2 c_4}{5K^{\nu+n+2}}, \frac{c_8 M|a|}{K^{\nu+n+2}}\right\} < \frac{1}{6}, \tag{B}$$

by lemma2.1, (3.11) and Cauchy's formula, we derive

$$\begin{aligned}
\Big\|P^2_{j+1}\Big\|_{D_{j+1}} &\le \|\{N_j - N_0, S_j\}\|_{D_j - 5\sigma} \\
&\le \|\nabla(N_j - N_0)\|_{D_j-5\sigma} \|\hat{B}\|_{D_0} \|\nabla S_j\|_{D_j - 5\sigma} \\
&\le \frac{M}{5\sigma^2} \|N_j - N_0\|_{D_j} \|S_j\|_{D_j - 4\sigma} \\
&\le \frac{2M^2}{5\sigma^2}\varepsilon \cdot \frac{c_4}{\sigma^{\nu+n}} \|P_j\|_{D_j - 3\sigma} \\
&\le \frac{1}{6}\|P_j\|_{D_j};
\end{aligned} \tag{3.13}$$

$$\begin{aligned}
\Big\|P^3_{j+1}\Big\|_{D_{j+1}} &\le \Big\|P^3_{j+1}\Big\|_{D_j - 5\sigma} \\
&\le |a| \cdot \frac{\varepsilon M}{\sigma^2} \cdot \|S_j\|_{D_j - 4\sigma} \\
&\le \frac{c_4 M|a|\varepsilon}{\sigma^{\nu+n+2}} \|P_j\|_{D_j}
\end{aligned}$$

$$\begin{aligned} &\leq \frac{c_8 M|a|}{K^{\nu+n+2}}\|P_j\|_{D_j} \\ &< \frac{1}{6}\|P_j\|_{D_j}. \end{aligned} \tag{3.14}$$

On the basis of (3.7), (3.13) and (3.14), it is concluded that

$$\|\{N_j, S\}\|_{D_j-5\sigma} \leq \left\|P_{j+1}^2\right\|_{D_j-5\sigma} + \left\|P_{j+1}^3\right\|_{D_j-5\sigma} + \left\|\ \widetilde{P}_j\ \right\|_{D_j-5\sigma} \leq 4\|P_j\|_{D_j}. \tag{3.15}$$

Hence, as K satisfies

$$\frac{5M}{K^{\nu+n+2}} < \frac{1}{6}, \tag{C}$$

we obtain

$$\begin{aligned} \|P_{j+1}^1\|_{D_{j+1}} &\leq \varepsilon\|\nabla(P_j + (1-t)\{N_j, S_j\})\|_{D_{j+1}}\|\hat{B}\|_{D_0}\|\nabla S_j\|_{D_{j+1}} \\ &\leq \frac{\varepsilon M}{\sigma}\|P_j + (1-t)\{N_j, S_j\}\|_{D_j-5\sigma}\cdot\frac{1}{\sigma}\|S_j\|_{D_j-5\sigma} \\ &\leq \frac{M\varepsilon}{\sigma^{\nu+n+2}}(\|P_j\|_{D_j} + \|\{N_j, S_j\}\|_{D_j-5\sigma}) \\ &\leq \frac{5M}{K^{\nu+n+2}}\|P_j\|_{D_j} \\ &< \frac{1}{6}\|P_j\|_{D_j}. \end{aligned} \tag{3.16}$$

From (3.13), (3.14) and (3.16), we have

$$\|P_{j+1}\|_{D_{j+1}} \leq \|P_{j+1}^1\|_{D_{j+1}} + \|P_{j+1}^2\|_{D_{j+1}} + \|P_{j+1}^3\|_{D_{j+1}} \leq \frac{1}{2}\|P_j\|_{D_j} < \frac{1}{2^{j+1}}M. \tag{3.17}$$

Putting $\Psi = \Phi_1 \circ \cdots \circ \Phi_L$ then $\Psi : D_L = (B_{\frac{1}{2}}(0) \times T^n) + \frac{\rho}{2} \to D_0$.

Let $\Psi(r, \theta) = (Y, x)$. Then

$$H_L(r,\theta) = \hat{H}\circ\Psi(r,\theta) = N_L(r,\varepsilon) + \varepsilon P_L(r,\theta,\varepsilon) \tag{3.18}$$

satisfying

$$\|N_L - N_0\|_{D_L} \leq 2M\varepsilon, \tag{3.19}$$

$$\|P_L\|_{D_L} \leq \frac{1}{2^L}M \leq c_{10}\exp\Big(-c_9\varepsilon^{-\alpha}\Big). \tag{3.20}$$

Corresponding to (3.18) Poisson system is

$$\dot{r} = \varepsilon B_{12}(y_* + \varepsilon r)\frac{\partial P_L}{\partial\theta}, \tag{3.21}$$

$$\begin{aligned} \dot{\theta} = \omega_* &- (B_{12}^T(y_* + \varepsilon r) - B_{12}^T(y_*))a \\ &- B_{12}^T(y_* + \varepsilon r)\frac{\partial}{\partial r}(N_L - N_0) - \varepsilon B_{12}^T(y_* + \varepsilon r)\frac{\partial P_L}{\partial r}. \end{aligned} \tag{3.22}$$

Let $D_* = (B_{\frac{1}{2}}(0) \times T^n) + \frac{\rho}{4}$. By Cauchy's formula,

$$\max\left\{\left\|\frac{\partial P_L}{\partial\theta}\right\|_{D_*}, \left\|\frac{\partial P_L}{\partial r}\right\|_{D_*}\right\} < \frac{4}{\rho}\|P_L\|_{D_L}.$$

From (3.21), as $|t| \le \exp\left(\frac{1}{2}c_9\varepsilon^{-\alpha}\right)$, it follows that

$$|r(t) - r(0)| \le c_{11}\varepsilon\exp\left(-\frac{1}{2}c_{11}\varepsilon^{-\alpha}\right) \le c_{11}\varepsilon \tag{3.23}$$

on $(B_{\frac{1}{2}}(0) \times T^n) + \frac{\rho}{8}$. Writing

$$\omega_{**}(r,\varepsilon) = -(B_{12}^T(y_* + \varepsilon r) - B_{12}^T(y_*))a - B_{12}^T(y_* + \varepsilon T)\frac{\partial}{\partial r}(N_L - N_0). \tag{3.24}$$

Thus,

$$\begin{aligned}|\omega_{**}(r(t),\varepsilon) - \omega_{**}(r(0),\varepsilon)| &\le c_{12}\varepsilon|r(t) - r(0)|\\ &\le c_{13}\varepsilon\exp\left(-\frac{1}{2}c_9\varepsilon^{-\alpha}\right) \le c_{14}\varepsilon\end{aligned} \tag{3.25}$$

on $(B_{\frac{1}{2}}(0) \times T^n) + \frac{\rho}{16}$. It follows from (3.22), (3.20), (3.25) and Cauchy's formula that

$$|\theta(t) - (\omega_* + \omega_{**}(r(0),\varepsilon))t - \theta(0)| \le c_{14}\varepsilon\exp\left(-\frac{1}{4}c_9\varepsilon^{-\alpha}\right) \le c_{14}\varepsilon, \tag{3.26}$$

provided $|t| \le \exp\left(-\frac{1}{4}c_9\varepsilon^{-\alpha}\right)$.

We choose K to satisfy (A), (B) and (C). Let $(y(t), x(t))$ be a solution, with $|y(0) - y_*| < \frac{1}{2}\varepsilon$, of Poisson system with Hamiltonian (1.13). Then, by (2.1), $(Y(t), x(t))$ is a solution of (2.3) with $|Y(0)| < \frac{1}{2}$. Thus, if (Y, x) and (r, θ) are the corresponding expressions in the new and old coordinates, respectively, $(Y(t), x(t))$ and $(r(t), \theta(t))$ are also ones. By applying (3.12), we get

$$|(Y, x) - (r, \theta)| \le \sum_{j=0}^{L} \frac{1}{2^j}\sigma < 2\sigma. \tag{3.27}$$

From (3.23) and (3.27), as $|t| \le \exp\left(\frac{1}{2}c_9\varepsilon^{-\alpha}\right)$, it follows that

$$|Y(t) - Y(0)| \le |Y(t) - r(t)| + |r(t) - r(0)| + |Y(0) - r(0)| \le c_{15}\sigma. \tag{3.28}$$

Similarly, if $|t| \le \exp\left(\frac{1}{4}c_9\varepsilon^{-\alpha}\right)$, then

$$|x(t) - (\omega_* + \omega_{drift}(y(0),\varepsilon))t - x(0)| \le c_{16}\sigma, \tag{3.29}$$

where

$$\omega_{drift}(y(0),\varepsilon) = \omega_{**}\left(\frac{y(0) - y_*}{\varepsilon}, \varepsilon\right).$$

By (2.1) and (3.28), as $|t| \le \exp\left(\frac{1}{2}c_9\varepsilon^{-\alpha}\right)$, we have

$$|y(t) - y(0)| \le c_{15}\sigma\varepsilon. \tag{3.30}$$

According to the definition of σ, (3.29) and (3.30). The proof of Theorem A is finished.

4. Further results

Note that Lemma 2.1 plays an important role in the proof of Theorem A. Rüssmann has shown that the estimate of Lemma 2.1 holds with the optimal exponent ν replacing $\nu + n$ [9]. Hence, we can obtain an exact near-invariant torus theorem as follows.

Theorem 4.1. *Under the assumptions of Theorem1.1 there is positive constant* ε_0 *such that for every* $0 < \varepsilon \leq \varepsilon_0$*, if* $(y(t), x(t))$ *is a solution of Poisson system (1.13) with initial value* $(y(0), x(0))$ *satisfying* $|y(0) - y_*| < \frac{1}{2}\varepsilon$ *and* $x(0) \in T^n$*, then, as* $|t| < \exp\left(c_3 \varepsilon^{-\frac{1}{\nu+2}}\right)$,

$$|y(t) - y(0)| < c_1 \varepsilon^{1+\frac{1}{\nu+2}},$$
$$|x(t) - \omega_{drift}(y(0), \varepsilon)t - x(0)| < c_2 \varepsilon^{\frac{1}{\nu+2}}$$

for some positive constants c_1, c_2 *and* c_3*, which are independent of* $\varepsilon, y(0)$ *and* $x(0)$*. Moreover,*

$$|\omega_{drift}(y(0), \varepsilon) - \omega_*| = O(\varepsilon).$$

Remark 4.1. If B is independent of y and x, and $m = n$, *that is,* the dimension of action variables is equal to one of angle variables, and $B = J$ (the standard symplectic structure matrix), the Poisson system is an usual Hamiltonian system. In this case, Perry and Wiggins gave a theorem on near-invariant torus [8]. In their theorem the estimate of time is same as Theorem B, but the drift distance of the orbits with initial points near the torus is bigger than one in Theorem B, *that is,* our estimate $O\left(\varepsilon^{1+\frac{1}{\nu+2}}\right)$ is different from their estimate $O(\varepsilon)$.

By examining the proof of Theorem1.1, it is found that we can study a perturbed Poisson system

$$H(y, x) = c_0 + a \cdot (y - y_*) + F(y - y_*, x) + \varepsilon^2 G(y, x), \tag{4.1}$$

where ε^2 is a small parameter. When we introduce a change $y - y_* = \varepsilon Y$, the Poisson system with Hamiltonian (4.1) is equivalent to another Poisson system with the following generating function

$$\hat{H}(Y, x, \varepsilon) = \frac{c_0}{\varepsilon} + a \cdot Y + \varepsilon P(Y, x, \varepsilon), \tag{4.2}$$

where

$$P(Y, x, \varepsilon) = \frac{1}{\varepsilon^2} F(\varepsilon Y, x) + G(y_* + \varepsilon Y, x). \tag{4.3}$$

Moreover, the structure matrix B is changed to

$$\hat{B}(Y, \varepsilon) = \begin{pmatrix} 0_m & B_{12} \\ -B_{12}^T & 0_n \end{pmatrix} (\varepsilon Y + y_*).$$

Following the proof of Theorem1.1and combing Rüssmann's result [9], we have Theorem C as follows.

Theorem 4.2. *Assume that the conditions of Theorem1.1 hold. Let* G *be real analytic. Let* $(y(t), x(t))$ *be a solution of Poisson system with Hamiltonian (4.1)* *with initial value* $(y(0), x(0))$. *Then there exists a positive constant* c *such that for sufficiently small* $\varepsilon > 0$, *as* $|t| < \exp\left(c\varepsilon^{-\frac{1}{\nu+2}}\right)$,

$$|y(t) - y(0)| < c\varepsilon^{1+\frac{1}{\nu+2}},$$
$$|x(t) - \omega_{drift}(y(0), \varepsilon)t - x(0)| < c\varepsilon^{\frac{1}{\nu+2}},$$

provided $|y(0) - y_*| < \frac{1}{2}\varepsilon$ *and* $x(0) \in T^n$, *where* ω_{drift} *is a constant vector depending on* $y(0)$ *and* ε.

Remark 4.2. Theorem 4.2 shows that if Γ is an orbit of the small perturbed system of Poisson system possesses some Diophantine torus, and Γ starts with the initial points near this torus, then Γ approaches the quasi-periodic orbit on an extremely long time.

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