Journal of Nonlinear Modeling and Analysis
Volume 1, Number 3, September 2019, 397–413
http://jnma.ca; http://jnma.ijournal.cn
DOI:10.12150/jnma.2019.397

Lyapunov-type Inequalities for Fractional (p, q)-Laplacian Systems*

Yanfeng Li[1] and Qiaoluan Li[1,†]

Abstract In this paper, we establish some new Lyapunov type inequalities for fractional (p, q)-Laplacian operators in an open bounded set $\Omega \subset \mathbb{R}^N$, under homogeneous Dirichlet boundary conditions. Next, we use the obtained inequalities to derive some geometric properties of the generalized spectrum associated to the considered problem.

Keywords Fractional Sobolev spaces, fractional (p, q)-Laplacian operators, Lyapunov inequality, generalized eigenvalues.

MSC(2010) 35P15, 35P30.

1. Introduction

The Lyapunov inequality and its various generalizations have found applications in the study of properties of solutions such as oscillation theory, asymptotic theory, eigenvalue problems of differential and difference equations. On the other hand, the fractional p-Laplacian operator is a class of non-local pseudo differential operators. The equations involving the fractional p-Laplacian operators are used to describe the diffusion phenomenon, which has been widely used in fluid mechanics, material memory, biology, plasma physics, finance, and so on. In the last few decades, many authors have established various Lyapunov type inequalities for fractional p-Laplacian operators, see, for example the Refs. [2–6] and the references therein.

In [4], Mohamed Jleli, Mokhtar Kirane and Bessem Samet considered the fractional p-Laplacian operator $(-\Delta_p)^s$, where $1 < p < \infty$, $s \in (0, 1)$, in an open bounded set $\Omega \subset \mathbb{R}^N$, $N \geq 2$, under homogeneous Dirichlet boundary conditions. More precisely, they considered the following problem

$$\begin{cases} (-\Delta_p)^s u = w|u|^{p-2}u, & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \mathbb{R}^N \backslash \Omega, \end{cases}$$

where the weight function $w \in L^\infty(\Omega)$. They discussed two cases, the case $sp > N$ and the case $sp < N$. For each case, they obtained a Lyapunov-type inequality involved the inner radius of the domain and L^θ norms of the weight w.

†the corresponding author.
Email address: 1281886085@qq.com(Y. Li), qll71125@163.com(Q. Li)
[1]College of Mathematics and Information Science, Hebei Normal University, Shijiazhuang, Hebei 050024, China
*Q. Li is supported by National Natural Science Foundation of China (No 11571090).

In [5], Mohamed Jleli and Bessem Samet considered the following system involving $(p_i,\, q_i)$-Laplacian operators $(i = 1,\, 2)$:

$$\begin{cases} -(|u'(x)|^{p_1-2}u'(x))' - (|u'(x)|^{q_1-2}u'(x))' = f(x)|u(x)|^{\alpha-2}|v(x)|^{\beta}u(x), \\ -(|v'(x)|^{p_2-2}v'(x))' - (|v'(x)|^{q_2-2}v'(x))' = g(x)|u(x)|^{\alpha}|v(x)|^{\beta-2}v(x), \end{cases} \tag{1.1}$$

on the interval $(a,\, b)$, under Dirichlet boundary conditions

$$u(a) = u(b) = v(a) = v(b) = 0.$$

System (1.1) is investigated under the assumptions

$$\alpha \geq 2,\ \ \beta \geq 2,\ \ p_i \geq 2,\ \ q_i \geq 2,\ \ (i = 1,\, 2),$$

and

$$\frac{2\alpha}{p_1 + q_1} + \frac{2\beta}{p_2 + q_2} = 1.$$

Where f and g are two nonnegative real-valued functions such that $(f,\, g) \in L^1(a,\, b) \times L^1(a,\, b)$. It was proved that if (1.1) has a nontrivial solution $(u,\, v) \in C^2[a,\, b] \times C^2[a,\, b]$, then

$$\Big[\min\{\frac{2^{p_1}}{(b-a)^{p_1-1}}, \frac{2^{q_1}}{(b-a)^{q_1-1}}\}\Big]^{\frac{2\alpha}{p_1+q_1}} \Big[\min\{\frac{2^{p_2}}{(b-a)^{p_2-1}}, \frac{2^{q_2}}{(b-a)^{q_2-1}}\}\Big]^{\frac{2\beta}{p_2+q_2}} \\ \leq \Big(\frac{1}{2}\int_a^b f(x)dx\Big)^{\frac{2\alpha}{p_1+q_1}} \Big(\frac{1}{2}\int_a^b g(x)dx\Big)^{\frac{2\beta}{p_2+q_2}}.$$

Some nice applications to generalized eigenvalues are also presented in [5].

In this paper, we establish some new Lyapunov type inequalities for fractional Laplacian systems. More precisely, we consider:

$$\begin{cases} (-\Delta_{p_1})^s u(x) + (-\Delta_{p_2})^s u(x) = f(x)|u(x)|^{\alpha-2}|v(x)|^{\beta}u(x), \\ (-\Delta_{q_1})^s v(x) + (-\Delta_{q_2})^s v(x) = g(x)|u(x)|^{\alpha}|v(x)|^{\beta-2}v(x), & \text{in} \quad \Omega, \\ u = v = 0, & \text{on} \quad \mathbb{R}^N \backslash \Omega. \end{cases} \tag{1.2}$$

System (1.2) is investigated under the assumptions

$$s \in (0,\, 1),\ \ \alpha \geq 2,\ \ \beta \geq 2,\ \ p_i \geq 2,\ \ q_i \geq 2,\ \ (i = 1,\, 2),$$

and

$$\frac{2\alpha}{p_1 + p_2} + \frac{2\beta}{q_1 + q_2} = 1. \tag{1.3}$$

We also consider the system:

$$\begin{cases} \displaystyle\sum_{i=1}^{3}[(-\Delta_{p_i})^s u(x)] = f(x)|u(x)|^{\alpha-2}|v(x)|^{\beta}|w(x)|^{\gamma}u(x), \\ \displaystyle\sum_{i=1}^{3}[(-\Delta_{q_i})^s v(x)] = g(x)|u(x)|^{\alpha}|v(x)|^{\beta-2}|w(x)|^{\gamma}v(x), \\ \displaystyle\sum_{i=1}^{3}[(-\Delta_{r_i})^s w(x)] = h(x)|u(x)|^{\alpha}|v(x)|^{\beta}|w(x)|^{\gamma-2}w(x), & \text{in} \quad \Omega, \\ u = v = w = 0, & \text{on} \quad \mathbb{R}^N \backslash \Omega. \end{cases} \tag{1.4}$$

System (1.4) is investigated under the assumptions

$$s \in (0,\, 1),\ \ \alpha \geq 2,\ \ \beta \geq 2,\ \ \gamma \geq 2,\ \ p_i \geq 2,\ \ q_i \geq 2,\ \ r_i \geq 2,\ \ (i = 1,\, 2,\, 3),$$

and

$$\frac{3\alpha}{p_1 + p_2 + p_3} + \frac{3\beta}{q_1 + q_2 + q_3} + \frac{3\gamma}{r_1 + r_2 + r_3} = 1. \tag{1.5}$$

In the next section, we establish Lyapunov-type inequalities for problem (1.2) and (1.4). Then, we use the obtained inequalities to derive some geometric properties of the generalized spectrum associated to the considered problem.

2. Main results

We assume readers are familiar with the fractional p-Laplacian operator. For more details, we refer to [4]. The following fractional Sobolev-type inequalities will be useful later.

Lemma 2.1 (theorem 6.5, [9])**.** *Let $D \subset \mathbb{R}^N$ be bounded and open, $sp < N$, $s \in (0,\, 1)$, and $1 < p < \infty$. Then there is a constant $C_H > 0$ such that*

$$\|u\|^p_{L^{p^*_s}(\mathbb{R}^N)} \leq C_H [u]^p_{s,\, p}, \quad u \in W^{s,\, p}_0(D),$$

where $p^*_s = \frac{Np}{N-sp}$.

Lemma 2.2 (corollary 1.4, [1])**.** *Let $0 < s < 1$ and $1 < p < \infty$ be such that $sp < N$. Assume that $D \subset \mathbb{R}^N$ is a (bounded) uniform domain with a (locally) $(s,\, p)$-uniformly flat boundary. Then D admits an $(s,\, p)$-Hardy inequality, that is, there is a constant $C_S > 0$ such that*

$$\int_D \frac{|u(x)|^p}{d(x, \partial D)^{sp}} dx \leq C_S [u]^p_{s,\, p}, \quad u \in W^{s,\, p}_0(D),$$

where $d(x, \partial D)$ is the distance from $x \in D$ to the boundary ∂D.

Lemma 2.3 (theorem 3, [7])**.** *Let $D \subset \mathbb{R}^N$ be bounded and open, $sp > N$ and $s \in (0,\, 1)$. Then there is a constant $C_M > 0$ such that for all $u \in W^{s,\, p}_0(D)$,*

$$|u(x) - u(y)| \leq C_M |x - y|^\beta [u]_{s,\, p}, \quad x, y \in \mathbb{R}^N,$$

where $\beta = \frac{sp-N}{p}$.

In the following, we suppose that $\Omega \subset \mathbb{R}^N$ is a bounded domain satisfying the regularities required by the fractional Sobolev inequalities given by Lemmas 2.1, 2.2 and 2.3.

First, we define the weak solutions for problem (1.2) and (1.4).

A (weak) solution of problem (1.2) is $(u, v) \in W_0^{s,p}(\Omega) \times W_0^{s,q}(\Omega)$ satisfying

$$\begin{cases} \sum_{i=1}^{2} \iint\limits_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p_i-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp_i}} dxdy = \\ \int_\Omega f(x)|u(x)|^{\alpha-2}|v(x)|^{\beta} u(x)\varphi(x)dx, \\ \\ \sum_{i=1}^{2} \iint\limits_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^{q_i-2}(v(x)-v(y))(\psi(x)-\psi(y))}{|x-y|^{N+sq_i}} dxdy = \\ \int_\Omega g(x)|u(x)|^{\alpha}|v(x)|^{\beta-2} v(x)\psi(x)dx, \end{cases} \tag{2.1}$$

for all $(\varphi(x), \psi(x)) \in W_0^{s,p}(\Omega) \times W_0^{s,q}(\Omega)$, where

$$\begin{cases} p = (p_1, \ p_2), \\ q = (q_1, \ q_2). \end{cases}$$

A (weak) solution of problem (1.4)) is $(u, v, w) \in W_0^{s,p}(\Omega) \times W_0^{s,q}(\Omega) \times W_0^{s,r}(\Omega)$ satisfying

$$\begin{cases} \sum_{i=1}^{3} \iint\limits_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p_i-2}(u(x)-u(y))(\varphi_1(x)-\varphi_1(y))}{|x-y|^{N+sp_i}} dxdy = \\ \int_\Omega f(x)|u(x)|^{\alpha-2}|v(x)|^{\beta}|w(x)|^{\gamma} u(x)\varphi_1(x)dx, \\ \\ \sum_{i=1}^{3} \iint\limits_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^{q_i-2}(v(x)-v(y))(\varphi_2(x)-\varphi_2(y))}{|x-y|^{N+sq_i}} dxdy = \\ \int_\Omega g(x)|u(x)|^{\alpha}|v(x)|^{\beta-2}|w(x)|^{\gamma} v(x)\varphi_2(x)dx, \\ \\ \sum_{i=1}^{3} \iint\limits_{\mathbb{R}^{2N}} \frac{|w(x)-w(y)|^{r_i-2}(w(x)-w(y))(\varphi_3(x)-\varphi_3(y))}{|x-y|^{N+sr_i}} dxdy = \\ \int_\Omega h(x)|u(x)|^{\alpha}|v(x)|^{\beta}|w(x)|^{\gamma-2} w(x)\varphi_3(x)dx, \end{cases} \tag{2.2}$$

for all $(\varphi_1(x), \varphi_2(x), \varphi_3(x)) \in W_0^{s,p}(\Omega) \times W_0^{s,q}(\Omega) \times W_0^{s,r}(\Omega)$, where

$$\begin{cases} p = (p_1, \ p_2, \ p_3), \\ q = (q_1, \ q_2, \ q_3), \\ r = (r_1, \ r_2, \ r_3). \end{cases}$$

Our first result is the following Lyapunov inequality for problem (1.2) in the case $sp_i > N$, $sq_i > N, (i = 1, 2)$.

Theorem 2.1. *Let $f,\, g \in L^1(\Omega)$ be a pair of non-negative weights. Suppose that problem (1.2) with $sp_i > N$, $sq_i > N$, $(i = 1,\, 2)$ has a non-trivial weak solution $(u,\, v) \in W_0^{s,p}(\Omega) \times W_0^{s,q}(\Omega)$. Then*

$$\Big(\int_\Omega f(x)dx\Big)^{\frac{2\alpha}{p_1+p_2}}\Big(\int_\Omega g(x)dx\Big)^{\frac{2\beta}{q_1+q_2}} \ge \frac{2}{C_M^{\alpha+\beta} r_\Omega^{s(\alpha+\beta)-N}}, \tag{2.3}$$

where C_M (a universal constant) is given by Lemma 2.3.

Proof. Let $\varphi = u$ and $\psi = v$ in (2.1), we obtain

$$\begin{cases} \displaystyle\iint\limits_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p_1}}{|x-y|^{N+sp_1}}dxdy + \iint\limits_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p_2}}{|x-y|^{N+sp_2}}dxdy = \int_\Omega f(x)|u(x)|^\alpha |v(x)|^\beta dx, \\ \displaystyle\iint\limits_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^{q_1}}{|x-y|^{N+sq_1}}dxdy + \iint\limits_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^{q_2}}{|x-y|^{N+sq_2}}dxdy = \int_\Omega g(x)|u(x)|^\alpha |v(x)|^\beta dx, \end{cases}$$

that is,

$$\begin{cases} [u]_{s,\,p_1}^{p_1} + [u]_{s,\,p_2}^{p_2} = \displaystyle\int_\Omega f(x)|u(x)|^\alpha |v(x)|^\beta dx, \\ [v]_{s,\,q_1}^{q_1} + [v]_{s,\,q_2}^{q_2} = \displaystyle\int_\Omega g(x)|u(x)|^\alpha |v(x)|^\beta dx. \end{cases} \tag{2.4}$$

Using the inequality

$$A + B \ge 2\sqrt{A}\sqrt{B},$$

we get

$$\begin{cases} 2[u]_{s,\,p_1}^{\frac{p_1}{2}}[u]_{s,\,p_2}^{\frac{p_2}{2}} \le [u]_{s,\,p_1}^{p_1} + [u]_{s,\,p_2}^{p_2}, \\ 2[v]_{s,\,q_1}^{\frac{q_1}{2}}[v]_{s,\,q_2}^{\frac{q_2}{2}} \le [v]_{s,\,q_1}^{q_1} + [v]_{s,\,q_2}^{q_2}. \end{cases} \tag{2.5}$$

Since $sp_i > N$, $sq_i > N$, $(i = 1,\, 2)$, u, v are continuous in $\mathbb{R}^N$, in particular in $\overline{\Omega}$. But $(u,\, v) \in W_0^{s,p}(\Omega) \times W_0^{s,q}(\Omega)$ is non-trivial, then there exists $x_1,\, x_2 \in \Omega$ such that

$$|u(x_1)| = \max\{|u(x)| : x \in \mathbb{R}^N\} > 0,$$

$$|v(x_2)| = \max\{|v(x)| : x \in \mathbb{R}^N\} > 0.$$

From Lemma 2.3, we have

$$|u(x) - u(y)| \le C_M |x-y|^{\frac{sp_1 - N}{p_1}} [u]_{s,\,p_1}, \qquad x,\, y \in \mathbb{R}^N.$$

For u, taking $x = x_1$, we obtain

$$|u(x_1)| \le C_M |x_1 - y|^{\frac{sp_1 - N}{p_1}} [u]_{s,\,p_1}, \qquad y \in \partial\Omega,$$

which yields

$$|u(x_1)| \le C_M r_\Omega^{\frac{sp_1 - N}{p_1}} [u]_{s,\,p_1}. \tag{2.6}$$

Similarly, we obtain

$$|u(x_1)| \le C_M r_\Omega^{\frac{sp_2-N}{p_2}} [u]_{s,\,p_2}, \tag{2.7}$$

$$|v(x_2)| \le C_M r_\Omega^{\frac{sq_1-N}{q_1}} [v]_{s,\,q_1}, \tag{2.8}$$

$$|v(x_2)| \le C_M r_\Omega^{\frac{sq_2-N}{q_2}} [v]_{s,\,q_2}. \tag{2.9}$$

Combining (2.4), (2.5) with (2.6) and (2.7), where inequality (2.6) to a power $\frac{p_1}{2}$, inequality (2.7) to a power $\frac{p_2}{2}$ and multiplying the resulting inequalities, we obtain

$$\begin{aligned} |u(x_1)|^{\frac{p_1+p_2}{2}} &\le C_M^{\frac{p_1+p_2}{2}} r_\Omega^{\frac{sp_1+sp_2-2N}{2}} [u]_{s,\,p_1}^{\frac{p_1}{2}} [u]_{s,\,p_2}^{\frac{p_2}{2}} \\ &\le \frac{1}{2} C_M^{\frac{p_1+p_2}{2}} r_\Omega^{\frac{sp_1+sp_2-2N}{2}} \int_\Omega f(x)|u(x)|^\alpha |v(x)|^\beta dx \\ &\le \frac{1}{2} C_M^{\frac{p_1+p_2}{2}} r_\Omega^{\frac{sp_1+sp_2-2N}{2}} \int_\Omega f(x)dx |u(x_1)|^\alpha |v(x_2)|^\beta, \end{aligned}$$

that is,

$$2 \le C_M^{\frac{p_1+p_2}{2}} r_\Omega^{\frac{sp_1+sp_2-2N}{2}} \int_\Omega f(x)dx |u(x_1)|^{\alpha-\frac{p_1+p_2}{2}} |v(x_2)|^\beta. \tag{2.10}$$

Similarly, combining (2.4), (2.5) with (2.8) and (2.9), where inequality (2.8) to a power $\frac{q_1}{2}$, inequality (2.9) to a power $\frac{q_2}{2}$ and multiplying the resulting inequalities, we obtain

$$2 \le C_M^{\frac{q_1+q_2}{2}} r_\Omega^{\frac{sq_1+sq_2-2N}{2}} \int_\Omega g(x)dx |u(x_1)|^\alpha |v(x_2)|^{\beta-\frac{q_1+q_2}{2}}. \tag{2.11}$$

Raising inequality (2.10) to a power $e_1 > 0$, inequality (2.11) to a power $e_2 > 0$ and multiplying the resulting inequalities, we choose e_1 and e_2 such that $|u(x_1)|$, $|v(x_2)|$ cancels out, i.e., e_1, e_2 solve the homogeneous linear system:

$$\begin{cases} (\alpha - \dfrac{p_1+p_2}{2})e_1 + \alpha e_2 = 0, \\ \beta e_1 + (\beta - \dfrac{q_1+q_2}{2})e_2 = 0. \end{cases}$$

Using (1.3), we may take

$$\begin{cases} e_1 = \dfrac{2\alpha}{p_1+p_2}, \\ e_2 = \dfrac{2\beta}{q_1+q_2}. \end{cases}$$

Therefore, we get

$$2 \le C_M^{\alpha+\beta} r_\Omega^{s(\alpha+\beta)-N} \Big(\int_\Omega f(x)dx\Big)^{\frac{2\alpha}{p_1+p_2}} \Big(\int_\Omega g(x)dx\Big)^{\frac{2\beta}{q_1+q_2}},$$

which yields

$$\Big(\int_\Omega f(x)dx\Big)^{\frac{2\alpha}{p_1+p_2}} \Big(\int_\Omega g(x)dx\Big)^{\frac{2\beta}{q_1+q_2}} \ge \frac{2}{C_M^{\alpha+\beta} r_\Omega^{s(\alpha+\beta)-N}}.$$

The proof is completed. □

Our second result is the following Lyapunov inequality for problem (1.4) in the case $sp_i > N$, $sq_i > N$, $sr_i > N$, $(i = 1, 2, 3)$.

Theorem 2.2. *Let $f, g, h \in L^1(\Omega)$ be a group of non-negative weights. Suppose that problem (1.4) with $sp_i > N$, $sq_i > N$, $sr_i > N$, $(i = 1, 2, 3)$ has a non-trivial weak solution $(u, v, w) \in W_0^{s,p}(\Omega) \times W_0^{s,q}(\Omega) \times W_0^{s,r}(\Omega)$. Then*

$$\Big(\int_\Omega f(x)dx\Big)^{\frac{3\alpha}{\sum_{i=1}^{3} p_i}}\Big(\int_\Omega g(x)dx\Big)^{\frac{3\beta}{\sum_{i=1}^{3} q_i}}\Big(\int_\Omega h(x)dx\Big)^{\frac{3\gamma}{\sum_{i=1}^{3} r_i}} \geq \frac{3}{C_M^{\alpha+\beta+\gamma} r_\Omega^{s(\alpha+\beta+\gamma)-N}},$$

where C_M is given by Lemma 2.3.

Proof. Let $\varphi_1 = u$, $\varphi_2 = v$ and $\varphi_3 = w$ in (2.2), we obtain

$$\sum_{i=1}^{3} \iint\limits_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p_i}}{|x - y|^{N+sp_i}} dxdy = \int_\Omega f(x)|u(x)|^\alpha |v(x)|^\beta |w(x)|^\gamma dx,$$

that is

$$\sum_{i=1}^{3} [u]_{s,\,p_i}^{p_i} = \int_\Omega f(x)|u(x)|^\alpha |v(x)|^\beta |w(x)|^\gamma dx.$$

Using the inequality

$$A + B + C \geq 3A^{\frac{1}{3}} B^{\frac{1}{3}} C^{\frac{1}{3}}, \qquad A,\, B,\, C > 0,$$

we have

$$3[u]_{s,\,p_1}^{\frac{p_1}{3}} [u]_{s,\,p_2}^{\frac{p_2}{3}} [u]_{s,\,p_3}^{\frac{p_3}{3}} \leq \sum_{i=1}^{3} [u]_{s,\,p_i}^{p_i} = \int_\Omega f(x)|u(x)|^\alpha |v(x)|^\beta |w(x)|^\gamma dx. \tag{2.12}$$

Similarly, we get

$$3[v]_{s,\,q_1}^{\frac{q_1}{3}} [v]_{s,\,q_2}^{\frac{q_2}{3}} [v]_{s,\,q_3}^{\frac{q_3}{3}} \leq \int_\Omega g(x)|u(x)|^\alpha |v(x)|^\beta |w(x)|^\gamma dx.$$

$$3[w]_{s,\,r_1}^{\frac{r_1}{3}} [w]_{s,\,r_2}^{\frac{r_2}{3}} [w]_{s,\,r_3}^{\frac{r_3}{3}} \leq \int_\Omega h(x)|u(x)|^\alpha |v(x)|^\beta |w(x)|^\gamma dx.$$

Since $sp_i > N$, $sq_i > N$, $sr_i > N$, $(i = 1, 2, 3)$, u, v, w are continuous in $\mathbb{R}^N$, in particular in $\overline{\Omega}$. But $(u, v, w) \in W_0^{s,p}(\Omega) \times W_0^{s,q}(\Omega) \times W_0^{s,r}(\Omega)$ is non-trivial, then there exists x_1, x_2, $x_3 \in \Omega$ such that

$$|u(x_1)| = \max\{|u(x)| : x \in \mathbb{R}^N\} > 0,$$

$$|v(x_2)| = \max\{|v(x)| : x \in \mathbb{R}^N\} > 0,$$

$$|w(x_3)| = \max\{|w(x)| : x \in \mathbb{R}^N\} > 0.$$

From the proof of Theorem 2.1, we have

$$|u(x_1)| \leq C_M r_\Omega^{\frac{sp_1 - N}{p_1}} [u]_{s,\,p_1}, \tag{2.13}$$

$$|u(x_1)| \le C_M r_\Omega^{\frac{sp_2-N}{p_2}}[u]_{s,\,p_2}, \tag{2.14}$$

$$|u(x_1)| \le C_M r_\Omega^{\frac{sp_3-N}{p_3}}[u]_{s,\,p_3}. \tag{2.15}$$

Combining (2.12) with (2.13), (2.14) and (2.15), where inequality (2.13) to a power $\frac{p_1}{3}$, inequality (2.14) to a power $\frac{p_2}{3}$, inequality (2.15) to a power $\frac{p_3}{3}$ and multiplying the resulting inequalities, we obtain

$$\begin{aligned}|u(x_1)|^{\sum\limits_{i=1}^{3}\frac{p_i}{3}} &\le C_M^{\sum\limits_{i=1}^{3}\frac{p_i}{3}} r_\Omega^{\sum\limits_{i=1}^{3}\frac{sp_i}{3}-N}[u]_{s,\,p_1}^{\frac{p_1}{3}}[u]_{s,\,p_2}^{\frac{p_2}{3}}[u]_{s,\,p_3}^{\frac{p_3}{3}}\\ &\le \frac{1}{3}C_F\int_\Omega f(x)|u(x)|^\alpha|v(x)|^\beta|w(x)|^\gamma dx\\ &\le \frac{1}{3}C_F\int_\Omega f(x)dx|u(x_1)|^\alpha|v(x_2)|^\beta|w(x_3)|^\gamma,\end{aligned}$$

where $C_F = C_M^{\sum\limits_{i=1}^{3}\frac{p_i}{3}} r_\Omega^{\sum\limits_{i=1}^{3}\frac{sp_i}{3}-N}$, that is,

$$3 \le C_F\int_\Omega f(x)dx|u(x_1)|^{\alpha-\sum\limits_{i=1}^{3}\frac{p_i}{3}}|v(x_2)|^\beta|w(x_3)|^\gamma. \tag{2.16}$$

Similarly, we can get

$$3 \le C_G\int_\Omega g(x)dx|u(x_1)|^\alpha|v(x_2)|^{\beta-\sum\limits_{i=1}^{3}\frac{q_i}{3}}|w(x_3)|^\gamma, \tag{2.17}$$

$$3 \le C_H\int_\Omega h(x)dx|u(x_1)|^\alpha|v(x_2)|^\beta|w(x_3)|^{\gamma-\sum\limits_{i=1}^{3}\frac{r_i}{3}}, \tag{2.18}$$

where $C_G = C_M^{\sum\limits_{i=1}^{3}\frac{q_i}{3}} r_\Omega^{\sum\limits_{i=1}^{3}\frac{sq_i}{3}-N}$, $C_H = C_M^{\sum\limits_{i=1}^{3}\frac{r_i}{3}} r_\Omega^{\sum\limits_{i=1}^{3}\frac{sr_i}{3}-N}$.

Raising inequality (2.16) to a power $e_1 > 0$, inequality (2.17) to a power $e_2 > 0$, inequality (2.18) to a power $e_3 > 0$ and multiplying the resulting inequalities, we choose e_1, e_2 and e_3 to solve the homogeneous linear system:

$$\begin{cases}(\alpha-\sum\limits_{i=1}^{3}\dfrac{p_i}{3})e_1+\alpha e_2+\alpha e_3=0,\\ \beta e_1+(\beta-\sum\limits_{i=1}^{3}\dfrac{q_i}{3})e_2+\beta e_3=0,\\ \gamma e_1+\gamma e_2+(\gamma-\sum\limits_{i=1}^{3}\dfrac{r_i}{3})e_3=0.\end{cases}$$

Using (1.5), we may take

$$\begin{cases}e_1=\dfrac{3\alpha}{\sum\limits_{i=1}^{3}p_i},\\ e_2=\dfrac{3\beta}{\sum\limits_{i=1}^{3}q_i},\\ e_3=\dfrac{3\gamma}{\sum\limits_{i=1}^{3}r_i}.\end{cases}$$

Therefore, we get

$$3 \leq C_M^{\alpha+\beta+\gamma} r_\Omega^{s(\alpha+\beta+\gamma)-N} \Big(\int_\Omega f(x)dx\Big)^{\frac{3\alpha}{\sum_{i=1}^{3} p_i}} \Big(\int_\Omega g(x)dx\Big)^{\frac{3\beta}{\sum_{i=1}^{3} q_i}} \Big(\int_\Omega h(x)dx\Big)^{\frac{3\gamma}{\sum_{i=1}^{3} r_i}},$$

which yields

$$\Big(\int_\Omega f(x)dx\Big)^{\frac{3\alpha}{\sum_{i=1}^{3} p_i}} \Big(\int_\Omega g(x)dx\Big)^{\frac{3\beta}{\sum_{i=1}^{3} q_i}} \Big(\int_\Omega h(x)dx\Big)^{\frac{3\gamma}{\sum_{i=1}^{3} r_i}} \geq \frac{3}{C_M^{\alpha+\beta+\gamma} r_\Omega^{s(\alpha+\beta+\gamma)-N}}.$$

The proof is completed. □

Our third result is the following Lyapunov inequality for problem (1.2) in the case $sp_i < N$, $sq_i < N$, $(i = 1, 2)$.

Theorem 2.3. *Let $f, g \in L^\theta(\Omega)$, $\frac{N}{sp_i} < \theta < \infty$, $\frac{N}{sq_i} < \theta < \infty$, $(i = 1, 2)$ be a pair of non-negative weights. Suppose that problem (1.2) with $sp_i < N$, $sq_i < N$ has a non-trivial weak solution $(u, v) \in W_0^{s,p}(\Omega) \times W_0^{s,q}(\Omega)$. Then*

$$\Big(\int_\Omega f^\theta(x)dx\Big)^{\frac{2\alpha}{p_1+p_2}} \Big(\int_\Omega g^\theta(x)dx\Big)^{\frac{2\beta}{q_1+q_2}} \geq \frac{2^\theta}{r_\Omega^{s\theta(\alpha+\beta)-N} C_S^{\theta-M_1} C_H^{M_1}}, \tag{2.19}$$

where

$$M_1 = \frac{\alpha N}{sp_1p_2} + \frac{\beta N}{sq_1q_2}$$

with C_H and C_S (universal constant) given by Lemmas 2.1 and 2.2.

Proof. Let

$$p_i' = \lambda_i p_i + (1-\lambda_i)p_i^*, \quad q_i' = \delta_i q_i + (1-\delta_i)q_i^*, \quad i = 1, 2,$$

where

$$\lambda_i = \frac{1}{\theta-1}(\theta - \frac{N}{sp_i}), \qquad \delta_i = \frac{1}{\theta-1}(\theta - \frac{N}{sq_i}),$$

and

$$p_i^* = \frac{Np_i}{N-sp_i}, \qquad q_i^* = \frac{Nq_i}{N-sq_i}.$$

Observe that λ_i, $\delta_i \in (0, 1)$ and $p_i' = p_i\theta'$, $q_i' = q_i\theta'$, where $\frac{1}{\theta} + \frac{1}{\theta'} = 1$. From (1.3), we have $\frac{1}{\theta} + \frac{2\alpha}{p_1'+p_2'} + \frac{2\beta}{q_1'+q_2'} = 1$.

Using Hölder's inequality, we get

$$\begin{aligned}
&\frac{1}{r_\Omega^{\frac{\lambda_1 sp_1+\lambda_2 sp_2}{2}}} \int_\Omega |u(x)|^{\frac{p_1'+p_2'}{2}} dx \leq \int_\Omega \frac{|u(x)|^{\frac{p_1'+p_2'}{2}}}{d(x,\partial\Omega)^{\frac{\lambda_1 sp_1+\lambda_2 sp_2}{2}}} dx \\
&\leq \Big(\int_\Omega \frac{|u(x)|^{\lambda_1 p_1}|u(x)|^{(1-\lambda_1)p_1^*}}{d(x,\partial\Omega)^{\lambda_1 sp_1}} dx\Big)^{\frac{1}{2}} \Big(\int_\Omega \frac{|u(x)|^{\lambda_2 p_2}|u(x)|^{(1-\lambda_2)p_2^*}}{d(x,\partial\Omega)^{\lambda_2 sp_2}} dx\Big)^{\frac{1}{2}} \\
&\leq \Big(\int_\Omega \frac{|u(x)|^{p_1}}{d(x,\partial\Omega)^{sp_1}} dx\Big)^{\frac{\lambda_1}{2}} \Big(\int_\Omega |u(x)|^{p_1^*} dx\Big)^{\frac{(1-\lambda_1)}{2}}
\end{aligned}$$

$$
\begin{aligned}
&\times\Big(\int_\Omega \frac{|u(x)|^{p_2}}{d(x,\partial\Omega)^{sp_2}}dx\Big)^{\frac{\lambda_2}{2}}\Big(\int_\Omega |u(x)|^{p_2^*}dx\Big)^{\frac{(1-\lambda_2)}{2}}\\
&\le C_S^{\frac{\lambda_1}{2}}[u]_{s,p_1}^{\frac{p_1\lambda_1}{2}}C_H^{\frac{(1-\lambda_1)p_1^*}{2p_1}}[u]_{s,p_1}^{\frac{(1-\lambda_1)p_1^*}{2}}C_S^{\frac{\lambda_2}{2}}[u]_{s,p_2}^{\frac{p_2\lambda_2}{2}}C_H^{\frac{(1-\lambda_2)p_2^*}{2p_2}}[u]_{s,p_2}^{\frac{(1-\lambda_2)p_2^*}{2}}\\
&= C_1\{[u]_{s,p_1}^{\frac{p_1}{2}}[u]_{s,p_2}^{\frac{p_2}{2}}\}^{\theta'}\\
&\le \Big(\frac{1}{2}\Big)^{\theta'}C_1\Big(\int_\Omega f(x)|u(x)|^\alpha|v(x)|^\beta dx\Big)^{\theta'}\\
&\le \Big(\frac{1}{2}\Big)^{\theta'}C_1\Big(\int_\Omega f^\theta(x)dx\Big)^{\frac{\theta'}{\theta}}\Big(\int_\Omega |u(x)|^{\frac{p_1'+p_2'}{2}}dx\Big)^{\frac{2\alpha\theta'}{p_1'+p_2'}}\Big(\int_\Omega |v(x)|^{\frac{q_1'+q_2'}{2}}dx\Big)^{\frac{2\beta\theta'}{q_1'+q_2'}},
\end{aligned}
$$

where $C_1=C_S^{\frac{\lambda_1+\lambda_2}{2}}C_H^{\frac{(1-\lambda_1)p_1^*}{2p_1}+\frac{(1-\lambda_2)p_2^*}{2p_2}}$, that is

$$
\begin{aligned}
&\frac{2^{\theta'}}{r_\Omega^{\frac{\lambda_1 sp_1+\lambda_2 sp_2}{2}}}\\
&\le C_1\Big(\int_\Omega f^\theta(x)dx\Big)^{\frac{\theta'}{\theta}}\Big(\int_\Omega |u(x)|^{\frac{p_1'+p_2'}{2}}dx\Big)^{\frac{2\alpha\theta'}{p_1'+p_2'}-1}\Big(\int_\Omega |v(x)|^{\frac{q_1'+q_2'}{2}}dx\Big)^{\frac{2\beta\theta'}{q_1'+q_2'}}, \quad (2.20)
\end{aligned}
$$

where we used (2.4), (2.5) and Lemmas 2.1, 2.2.

Similarly, we have

$$
\begin{aligned}
&\frac{2^{\theta'}}{r_\Omega^{\frac{\delta_1 sq_1+\delta_2 sq_2}{2}}}\\
&\le C_2\Big(\int_\Omega g^\theta(x)dx\Big)^{\frac{\theta'}{\theta}}\Big(\int_\Omega |u(x)|^{\frac{p_1'+p_2'}{2}}dx\Big)^{\frac{2\alpha\theta'}{p_1'+p_2'}}\Big(\int_\Omega |v(x)|^{\frac{q_1'+q_2'}{2}}dx\Big)^{\frac{2\beta\theta'}{q_1'+q_2'}-1}, \quad (2.21)
\end{aligned}
$$

where $C_2=C_S^{\frac{\delta_1+\delta_2}{2}}C_H^{\frac{(1-\delta_1)q_1^*}{2q_1}+\frac{(1-\delta_2)q_2^*}{2q_2}}$.

Raising inequality (2.20) to a power $e_1>0$, inequality (2.21) to a power $e_2>0$ and multiplying the resulting inequalities, we choose e_1, e_2 to solve the homogeneous linear system:

$$
\begin{cases}
(\dfrac{2\alpha\theta'}{p_1'+p_2'}-1)e_1+\dfrac{2\alpha\theta'}{p_1'+p_2'}e_2=0,\\
\dfrac{2\beta\theta'}{q_1'+q_2'}e_1+(\dfrac{2\beta\theta'}{p_1'+p_2'}-1)e_2=0.
\end{cases}
$$

Using (1.3), we may take

$$
\begin{cases}
e_1=\dfrac{2\alpha}{p_1+p_2},\\
e_2=\dfrac{2\beta}{q_1+q_2}.
\end{cases}
$$

Therefore, we get

$$
\begin{aligned}
&\frac{2^{\theta'}}{r_\Omega^{\frac{\alpha(\lambda_1 sp_1+\lambda_2 sp_2)}{p_1+p_2}+\frac{\beta(\delta_1 sq_1+\delta_2 sq_2)}{q_1+q_2}}}\\
&\le C_1^{\frac{2\alpha}{p_1+p_2}}C_2^{\frac{2\beta}{q_1+q_2}}\Big(\int_\Omega f^\theta(x)dx\Big)^{\frac{2\alpha\theta'}{\theta(p_1+p_2)}}\Big(\int_\Omega g^\theta(x)dx\Big)^{\frac{2\beta\theta'}{\theta(q_1+q_2)}}.
\end{aligned}
$$

Further we get

$$\frac{2^\theta}{r_\Omega^{s\theta(\alpha+\beta)-N}} \leq C_S^{\theta-M_1} C_H^{M_1} \Big(\int_\Omega f^\theta(x)dx\Big)^{\frac{2\alpha}{p_1+p_2}} \Big(\int_\Omega g^\theta(x)dx\Big)^{\frac{2\beta}{q_1+q_2}}.$$

The proof is completed. □

Our fourth result is the following Lyapunov inequality for problem (1.4) in the case $sp_i < N$, $sq_i < N$, $sr_i < N$, $(i = 1, 2, 3)$.

Theorem 2.4. *Let* $f, g, h \in L^\theta(\Omega)$, $\frac{N}{sp_i} < \theta < \infty$, $\frac{N}{sq_i} < \theta < \infty$, $\frac{N}{sr_i} < \theta < \infty$, $(i = 1, 2, 3)$ *be a group of non-negative weights. Suppose that problem (1.4) with* $sp_i < N$, $sq_i < N$, $sr_i < N$ *has a non-trivial weak solution* $(u, v, w) \in W_0^{s,p}(\Omega) \times W_0^{s,q}(\Omega) \times W_0^{s,r}(\Omega)$. *Then*

$$\Big(\int_\Omega f^\theta(x)dx\Big)^{\frac{3\alpha}{\sum_{i=1}^3 p_i}} \Big(\int_\Omega g^\theta(x)dx\Big)^{\frac{3\beta}{\sum_{i=1}^3 q_i}} \Big(\int_\Omega h^\theta(x)dx\Big)^{\frac{3\gamma}{\sum_{i=1}^3 r_i}} \geq \frac{3^\theta}{C_S^{\theta-M_2} C_H^{M_2} r_\Omega^{s\theta(\alpha+\beta+\gamma)-N}},$$

where

$$M_2 = \frac{\sum_{i=1}^3 \frac{\alpha N}{sp_i}}{p_1+p_2+p_3} + \frac{\sum_{i=1}^3 \frac{\beta N}{sq_i}}{q_1+q_2+q_3} + \frac{\sum_{i=1}^3 \frac{\gamma N}{sr_i}}{r_1+r_2+r_3}$$

with C_H *and* C_S *(universal constant) given by Lemmas 2.1 and 2.2.*

Proof. Let

$$p_i' = \lambda_i p_i + (1-\lambda_i)p_i^*, \quad q_i' = \delta_i q_i + (1-\delta_i)q_i^*, \quad r_i' = \eta_i r_i + (1-\eta_i)r_i^*, \quad i = 1, 2, 3,$$

where

$$\lambda_i = \frac{1}{\theta-1}(\theta - \frac{N}{sp_i}), \quad \delta_i = \frac{1}{\theta-1}(\theta - \frac{N}{sq_i}), \quad \eta_i = \frac{1}{\theta-1}(\theta - \frac{N}{sr_i}),$$

and

$$p_i^* = \frac{Np_i}{N-sp_i}, \quad q_i^* = \frac{Nq_i}{N-sq_i}, \quad r_i^* = \frac{Nr_i}{N-sr_i}.$$

Observe that $\lambda_i, \delta_i, \eta_i \in (0, 1)$ and $p_i' = p_i\theta'$, $q_i' = q_i\theta'$, $r_i' = r_i\theta'$, where $\frac{1}{\theta} + \frac{1}{\theta'} = 1$. From (1.5), we have $\frac{1}{\theta} + \frac{3\alpha}{p_1'+p_2'+p_3'} + \frac{3\beta}{q_1'+q_2'+q_3'} + \frac{3\gamma}{r_1'+r_2'+r_3'} = 1$.

Using Hölder's inequality, we get

$$\frac{1}{r_\Omega^{\sum_{i=1}^3 \frac{\lambda_i sp_i}{3}}} \int_\Omega |u(x)|^{\sum_{i=1}^3 \frac{p_i'}{3}} dx \leq \int_\Omega \frac{|u(x)|^{\sum_{i=1}^3 \frac{p_i'}{3}}}{d(x, \partial\Omega)^{\sum_{i=1}^3 \frac{\lambda_i sp_i}{3}}} dx$$
$$\leq \prod_{i=1}^3 \Big(\int_\Omega \frac{|u(x)|^{\lambda_i p_i} |u(x)|^{(1-\lambda_i)p_i^*}}{d(x, \partial\Omega)^{\lambda_i sp_i}} dx\Big)^{\frac{1}{3}}$$

$$
\begin{aligned}
&\le \prod_{i=1}^{3}\Big(\int_\Omega \frac{|u(x)|^{p_i}}{d(x,\,\partial\Omega)^{sp_i}}dx\Big)^{\frac{\lambda_i}{3}}\Big(\int_\Omega |u(x)|^{p_i^*}dx\Big)^{\frac{1-\lambda_i}{3}}\\
&\le \prod_{i=1}^{3} C_S^{\frac{\lambda_i}{3}}[u]_{s,\,p_i}^{\frac{p_i\lambda_i}{3}} C_H^{\frac{(1-\lambda_i)p_i^*}{3p_i}}[u]_{s,\,p_i}^{\frac{(1-\lambda_i)p_i^*}{3}}\\
&= C_A\{[u]_{s,\,p_1}^{\frac{p_1}{3}}[u]_{s,\,p_2}^{\frac{p_2}{3}}[u]_{s,\,p_3}^{\frac{p_3}{3}}\}^{\theta'}\\
&\le \Big(\frac{1}{3}\Big)^{\theta'} C_A\Big(\int_\Omega f(x)|u(x)|^\alpha|v(x)|^\beta|w(x)|^\gamma dx\Big)^{\theta'}\\
&\le \Big(\frac{1}{3}\Big)^{\theta'} C_A\Big(\int_\Omega f^\theta(x)dx\Big)^{\frac{\theta'}{\theta}}\Big(\int_\Omega |u(x)|^{\sum_{i=1}^{3}\frac{p_i'}{3}}dx\Big)^{\frac{3\alpha\theta'}{\sum_{i=1}^{3}p_i'}}\\
&\quad\times\Big(\int_\Omega |v(x)|^{\sum_{i=1}^{3}\frac{q_i'}{3}}dx\Big)^{\frac{3\beta\theta'}{\sum_{i=1}^{3}q_i'}}\Big(\int_\Omega |w(x)|^{\sum_{i=1}^{3}\frac{r_i'}{3}}dx\Big)^{\frac{3\gamma\theta'}{\sum_{i=1}^{3}r_i'}},
\end{aligned}
$$

where $C_A = C_S^{\sum_{i=1}^{3}\frac{\lambda_i}{3}} C_H^{\sum_{i=1}^{3}\frac{(1-\lambda_i)p_i^*}{3p_i}}$, that is

$$
\begin{aligned}
\frac{3^{\theta'}}{r_\Omega^{\sum_{i=1}^{3}\frac{\lambda_i sp_i}{3}}} &\le C_A\Big(\int_\Omega f^\theta(x)dx\Big)^{\frac{\theta'}{\theta}}\Big(\int_\Omega |u(x)|^{\sum_{i=1}^{3}\frac{p_i'}{3}}dx\Big)^{\frac{3\alpha\theta'}{\sum_{i=1}^{3}p_i'}-1}\\
&\quad\times\Big(\int_\Omega |v(x)|^{\sum_{i=1}^{3}\frac{q_i'}{3}}dx\Big)^{\frac{3\beta\theta'}{\sum_{i=1}^{3}q_i'}}\Big(\int_\Omega |w(x)|^{\sum_{i=1}^{3}\frac{r_i'}{3}}dx\Big)^{\frac{3\gamma\theta'}{\sum_{i=1}^{3}r_i'}}, \qquad (2.22)
\end{aligned}
$$

where we used (2.12) and Lemmas 2.1, 2.2.

Similarly, we have

$$
\begin{aligned}
\frac{3^{\theta'}}{r_\Omega^{\sum_{i=1}^{3}\frac{\delta_i sq_i}{3}}} &\le C_B\Big(\int_\Omega g^\theta(x)dx\Big)^{\frac{\theta'}{\theta}}\Big(\int_\Omega |u(x)|^{\sum_{i=1}^{3}\frac{p_i'}{3}}dx\Big)^{\frac{3\alpha\theta'}{\sum_{i=1}^{3}p_i'}}\\
&\quad\times\Big(\int_\Omega |v(x)|^{\sum_{i=1}^{3}\frac{q_i'}{3}}dx\Big)^{\frac{3\beta\theta'}{\sum_{i=1}^{3}q_i'}-1}\Big(\int_\Omega |w(x)|^{\sum_{i=1}^{3}\frac{r_i'}{3}}dx\Big)^{\frac{3\gamma\theta'}{\sum_{i=1}^{3}r_i'}}, \qquad (2.23)
\end{aligned}
$$

where $C_B = C_S^{\sum_{i=1}^{3}\frac{\delta_i}{3}} C_H^{\sum_{i=1}^{3}\frac{(1-\delta_i)q_i^*}{3q_i}}$, and

$$
\begin{aligned}
\frac{3^{\theta'}}{r_\Omega^{\sum_{i=1}^{3}\frac{\eta_i sr_i}{3}}} &\le C_C\Big(\int_\Omega h^\theta(x)dx\Big)^{\frac{\theta'}{\theta}}\Big(\int_\Omega |u(x)|^{\sum_{i=1}^{3}\frac{p_i'}{3}}dx\Big)^{\frac{3\alpha\theta'}{\sum_{i=1}^{3}p_i'}}\\
&\quad\times\Big(\int_\Omega |v(x)|^{\sum_{i=1}^{3}\frac{q_i'}{3}}dx\Big)^{\frac{3\beta\theta'}{\sum_{i=1}^{3}q_i'}}\Big(\int_\Omega |w(x)|^{\sum_{i=1}^{3}\frac{r_i'}{3}}dx\Big)^{\frac{3\gamma\theta'}{\sum_{i=1}^{3}r_i'}-1}, \qquad (2.24)
\end{aligned}
$$

where $C_C = C_S^{\sum_{i=1}^{3}\frac{\eta_i}{3}} C_H^{\sum_{i=1}^{3}\frac{(1-\eta_i)r_i^*}{3r_i}}$.

Raising inequality (2.22) to a power $e_1 > 0$, inequality (2.23) to a power $e_2 > 0$, inequality (2.24) to a power $e_3 > 0$ and multiplying the resulting inequalities, we choose e_1, e_2 and e_3 to solve the homogeneous linear system:

$$\begin{cases} (\frac{3\alpha\theta'}{\sum\limits_{i=1}^{3} p_i'} - 1)e_1 + \frac{3\alpha\theta'}{\sum\limits_{i=1}^{3} p_i'}e_2 + \frac{3\alpha\theta'}{\sum\limits_{i=1}^{3} p_i'}e_3 = 0, \\ \frac{3\beta\theta'}{\sum\limits_{i=1}^{3} q_i'}e_1 + (\frac{3\beta\theta'}{\sum\limits_{i=1}^{3} q_i'} - 1)e_2 + \frac{3\beta\theta'}{\sum\limits_{i=1}^{3} q_i'}e_3 = 0, \\ \frac{3\gamma\theta'}{\sum\limits_{i=1}^{3} r_i'}e_1 + \frac{3\gamma\theta'}{\sum\limits_{i=1}^{3} r_i'}e_2 + (\frac{3\gamma\theta'}{\sum\limits_{i=1}^{3} r_i'} - 1)e_3 = 0. \end{cases}$$

Using (1.5), we may take

$$\begin{cases} e_1 = \frac{3\alpha}{\sum\limits_{i=1}^{3} p_i}, \\ e_2 = \frac{3\beta}{\sum\limits_{i=1}^{3} q_i}, \\ e_3 = \frac{3\gamma}{\sum\limits_{i=1}^{3} r_i}. \end{cases}$$

Therefore, we get

$$\frac{3^{\theta'}}{r_\Omega^{\frac{\sum_{i=1}^{3}\alpha\lambda_i s p_i}{p_1+p_2+p_3} + \frac{\sum_{i=1}^{3}\beta\delta_i s q_i}{q_1+q_2+q_3} + \frac{\sum_{i=1}^{3}\gamma\eta_i s r_i}{r_1+r_2+r_3}}} \leq C_A^{\frac{3\alpha}{p_1+p_2+p_3}} C_B^{\frac{3\beta}{q_1+q_2+q_3}} C_C^{\frac{3\gamma}{r_1+r_2+r_3}}$$
$$\times \Big(\int_\Omega f^\theta(x)dx\Big)^{\frac{3\alpha\theta'}{\sum_{i=1}^{3} p_i\theta}} \Big(\int_\Omega g^\theta(x)dx\Big)^{\frac{3\beta\theta'}{\sum_{i=1}^{3} q_i\theta}} \Big(\int_\Omega h^\theta(x)dx\Big)^{\frac{3\gamma\theta'}{\sum_{i=1}^{3} r_i\theta}}.$$

Further we get

$$\frac{3^\theta}{r_\Omega^{s\theta(\alpha+\beta+\gamma)-N}} \leq C_S^{\theta-M_2} C_H^{M_2} \Big(\int_\Omega f^\theta(x)dx\Big)^{\frac{3\alpha}{\sum_{i=1}^{3} p_i}}$$
$$\times \Big(\int_\Omega g^\theta(x)dx\Big)^{\frac{3\beta}{\sum_{i=1}^{3} q_i}} \Big(\int_\Omega h^\theta(x)dx\Big)^{\frac{3\gamma}{\sum_{i=1}^{3} r_i}}.$$

The proof is completed. □

As a consequence of Theorem 2.1 and Theorem 2.3, we deduce the following case of a single equation:

$$\begin{cases} (-\Delta_{p_1})^s u(x) + (-\Delta_{p_2})^s u(x) = f(x)|u(x)|^{\frac{p_1+p_2}{2}-2}u(x), & \text{in } \Omega, \\ u = 0, & \text{on } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{2.25}$$

where $p_i \geq 2$, $(i = 1, 2)$, $f \geq 0$.

Corollary 2.1. *Let us assume that problem (2.25) exists a nontrivial weak solution. Then*
(i) If $sp_i > N$, $f \in L^1(\Omega)$, we have

$$\int_\Omega f(x)dx \geq \frac{2}{C_M^{\frac{p_1+p_2}{2}} r_\Omega^{\frac{s(p_1+p_2)}{2}-N}}.$$

(ii) If $sp_i < N$, $f \in L^\theta(\Omega)$, we have

$$\int_\Omega f^\theta(x)dx \geq \frac{2^\theta}{r_\Omega^{\frac{s\theta(p_1+p_2)}{2}-N} C_S^{\theta-\frac{N}{2sp_2}-\frac{N}{2sp_1}} C_H^{\frac{N}{2sp_2}+\frac{N}{2sp_1}}}.$$

Remark 2.1. It is interesting to note that when $p_1 = p_2$, corollary 2.1 reduces to theorem 3.1 and 3.3 of [4].

3. Generalized eigenvalues

Inspired by the literature [8,10], we present some applications to generalized eigenvalues related to problem (1.2) in this section.

We consider the generalized eigenvalue problem

$$\begin{cases} (-\Delta_{p_1})^s u(x) + (-\Delta_{p_2})^s u(x) = \lambda w(x)|u(x)|^{\alpha-2}|v(x)|^\beta u(x), & \\ (-\Delta_{q_1})^s v(x) + (-\Delta_{q_2})^s v(x) = \mu w(x)|u(x)|^\alpha |v(x)|^{\beta-2} v(x), & \text{in } \Omega, \\ u = v = 0, & \text{on } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{3.1}$$

If problem (3.1) admits a nontrivial weak solution $(u, v) \in W_0^{s,p}(\Omega) \times W_0^{s,q}(\Omega)$, we say that (λ, μ) is a generalized eigenvalue of (3.1). The set of generalized eigenvalues is called generalized spectrum, which is denoted by σ.

We assume that

$$\alpha \geq 2, \quad \beta \geq 2, \quad p_i \geq 2, \quad q_i \geq 2, \quad i = 1, 2, \quad w \geq 0,$$

and (1.3) is satisfied.

The following result provides lower bounds of the generalized eigenvalues of (3.1).

Theorem 3.1. *Let (λ, μ) be a generalized eigenvalue of (3.1). Then*

$$\mu \geq h(\lambda), \tag{3.2}$$

(i) If $sp_i > N$, $sq_i > N$, $w \in L^1(\Omega)$, the function $h : (0, \infty) \to (0, \infty)$ is defined by

$$h(t) = \Big(\frac{M}{t^{\frac{2\alpha}{p_1+p_2}} \int_\Omega w(x)dx}\Big)^{\frac{q_1+q_2}{2\beta}}, \tag{3.3}$$

with

$$M = \frac{2}{C_M^{\alpha+\beta} r_\Omega^{s(\alpha+\beta)-N}}.$$

(ii) If $sp_i < N$, $sq_i < N$, $w \in L^{\theta}(\Omega)$, the function $h : (0, \infty) \to (0, \infty)$ is defined by

$$h(t) = \Big(\frac{R}{t^{\frac{2\alpha\theta}{p_1+p_2}} \int_{\Omega} w^{\theta}(x)dx} \Big)^{\frac{q_1+q_2}{2\beta\theta}}, \tag{3.4}$$

where

$$R = \frac{2^{\theta}}{r_{\Omega}^{s\theta(\alpha+\beta)-N} C_S^{\theta-M_1} C_H^{M_1}}.$$

with M_1 given in Theorem 2.3.

Proof. Let (λ, μ) be a generalized eigenpair, and let u, v be the corresponding nontrivial weak solutions. For $sp_i > N$, $sq_i > N$, since $f(x) = \lambda w(x)$, $g(x) = \mu w(x)$, using condition (1.3) and Theorem 2.1, we obtain

$$\lambda^{\frac{2\alpha}{p_1+p_2}} \mu^{\frac{2\beta}{q_1+q_2}} \int_{\Omega} w(x)dx \geq M.$$

Hence, we have

$$\mu^{\frac{2\beta}{q_1+q_2}} \geq \frac{M}{\lambda^{\frac{2\alpha}{p_1+p_2}} \int_{\Omega} w(x)dx},$$

which yields

$$\mu \geq \Big(\frac{M}{\lambda^{\frac{2\alpha}{p_1+p_2}} \int_{\Omega} w(x)dx} \Big)^{\frac{q_1+q_2}{2\beta}}.$$

For $sp_i < N$, $sq_i < N$, by replacing the functions $f(x) = \lambda w(x)$, $g(x) = \mu w(x)$ in inequality (2.19), we can get the conclusion from the proof of (i). The proof is completed. □

As consequences of Theorem 3.1, we deduce the following Protter's type results for the generalized spectrum.

Corollary 3.1. *There exists a constant $c_{\Omega} > 0$ that depends on domain Ω such that no point of the generalized spectrum σ is contained in the ball $B(0, c_{\Omega})$, where*

$$B(0, c_{\Omega}) = \{x \in \mathbb{R}^{2N} : \|x\|_{\infty} < c_{\Omega}\},$$

and $\| \cdot \|_{\infty}$ is the Chebyshev norm in $\mathbb{R}^{2N}$.

Proof. Let $(\lambda, \mu) \in \sigma$. For $sp_i > N$, $sq_i > N$, $(i = 1, 2)$, from (3.2) and (3.3), we obtain easily that

$$\lambda^{\frac{2\alpha}{p_1+p_2}} \mu^{\frac{2\beta}{q_1+q_2}} \geq \frac{M}{\int_{\Omega} w(x)dx}. \tag{3.5}$$

On the other hand, using condition (1.3), we have

$$\lambda^{\frac{2\alpha}{p_1+p_2}} \mu^{\frac{2\beta}{q_1+q_2}} \leq \|(\lambda, \mu)\|_{\infty}^{\frac{2\alpha}{p_1+p_2}+\frac{2\beta}{q_1+q_2}} = \|(\lambda, \mu)\|_{\infty}.$$

Therefore, we obtain

$$\|(\lambda,\, \mu)\|_\infty \geq c_{\,\Omega},$$

where

$$c_{\,\Omega} = \frac{M}{\int_\Omega w(x)dx}.$$

Analogously, for $sp_i < N$, $sq_i < N$, $(i = 1,\, 2)$, from (3.2) and (3.4), we obtain easily that

$$\lambda^{\frac{2\alpha\theta}{p_1+p_2}}\mu^{\frac{2\beta\theta}{q_1+q_2}} \geq \frac{R}{\int_\Omega w^\theta(x)dx}. \tag{3.6}$$

Further we get

$$\|(\lambda,\, \mu)\|_\infty \geq c_{\,\Omega},$$

where

$$c_{\,\Omega} = \Big(\frac{R}{\int_\Omega w^\theta(x)dx}\Big)^{\frac{1}{\theta}}.$$

The proof is completed. □

Corollary 3.2. *Let $(\lambda,\, \mu)$ be fixed and $s(\alpha+\beta) > N$. There exists an domain J of sufficiently small measure such that, if $\Omega \subset J$, then there are no nontrivial weak solutions of (3.1).*

Proof. Suppose that (3.1) admits a nontrivial weak solution. For $sp_i > N$, $sq_i > N$, $(i = 1,\, 2)$, since $M \to \infty$ as $|\Omega| \to 0^+$, where M is defined in Theorem 3.1, there exists $\delta_1 > 0$ such that

$$r_\Omega < \delta_1 \quad \Rightarrow \quad \frac{M}{\int_\Omega w(x)dx} > \lambda^{\frac{2\alpha}{p_1+p_2}}\mu^{\frac{2\beta}{q_1+q_2}}.$$

Let $J = B(x_0,\, \frac{\delta_1}{2})$, $x_0 \in R^N$. Hence, if $\Omega \subset J$, we have

$$\frac{M}{\int_\Omega w(x)dx} > \lambda^{\frac{2\alpha}{p_1+p_2}}\mu^{\frac{2\beta}{q_1+q_2}},$$

which is a contradiction with (3.5).

Analogously, for $sp_i < N$, $sq_i < N$, $(i = 1,\, 2)$, since $R \to \infty$ as $|\Omega| \to 0^+$, where R is defined in Theorem 3.1, there exists $\delta_2 > 0$ such that

$$r_\Omega < \delta_2 \quad \Rightarrow \quad \frac{R}{\int_\Omega w^\theta(x)dx} > \lambda^{\frac{2\alpha\theta}{p_1+p_2}}\mu^{\frac{2\beta\theta}{q_1+q_2}}.$$

Let $J = B(x_0,\, \frac{\delta_2}{2})$, $x_0 \in R^N$. Hence, if $\Omega \subset J$, we have

$$\frac{R}{\int_\Omega w^\theta(x)dx} > \lambda^{\frac{2\alpha\theta}{p_1+p_2}}\mu^{\frac{2\beta\theta}{q_1+q_2}},$$

which is a contradiction with (3.6).

To sum up, if $\Omega \subset J$, there are no nontrivial weak solutions of (3.1). The proof is completed. □

References

[1] L. Ihnatsyeva, J. Lehrbäck, H. Tuominen and A.V. Väahäkangas, *Fractional Hardy inequalities and visibility of the boundary*, Studia Mathematica, 2014, 224(1), 47–80.

[2] M. Jleli, M. Kirane and B. Samet, *Lyapunov-type inequalities for a fractional p-Laplacian system*, Fractional Calculus and Applied Analysis, 2017, 20(6), 1485–1506.

[3] M. Jleli, M. Kirane and B. Samet, *Lyapunov-type inequalities for fractional quasilinear problems via variational methods*, Journal of Nonlinear Sciences and Applications, 2017, 10, 2471–2486.

[4] M. Jleli, M. Kirane and B. Samet, *Lyapunov-type inequalities for fractional partial differential equations*, Applied Mathematies Letters, 2017, 66, 30–39.

[5] M. Jleli and B. Samet, *On Lyapunov-type inequalities for* $(p,\, q)$*-Laplacian systems*, Journal of Inequalities and Applications, 2017. DOI: 10.1186/s13660-017-1377-0

[6] A.M. Lyapunov, *Problème général de la stabilité du mouvement*, Ann. Fac. Sci. Univ. Toulouse Sci. Math. Sci. Phys., 1907, 2, 203–407.

[7] E. Lindgren and P. Lindqvist, *Fractional eigenvalues*, Calculus of Variations and Partial Differential Equations, 2014, 49(1–2), 795–826.

[8] P.L.De Nápoli and J.P. Pinasco, *Estimates for eigenvalues of quasilinear elliptic systems*, Journal of Differential Equations, 2006, 227, 102–115.

[9] E.Di Nezza, G. Palatucci and E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bulletin des Sciences Mathématiques, 2012, 136, 521–573.

[10] M. Protter, *The generalized spectrum of second order elliptic systems*, Rocky Mountain Journal of Mathematics, 1979, 9(3), 503–518.