Journal of Nonlinear Modeling and Analysis
Volume 1, Number 3, September 2019, 415–424
http://jnma.ca; http://jnma.ijournal.cn
DOI:10.12150/jnma.2019.415

Sufficient Conditions of Blow-up and Bound Estimations of Blow-up Time for a Parabolic Equation in Multi-dimensional Space*

Xiaona Miao[1], Xiaomin Xue[1] and Fushan Li[1,†]

Abstract In this paper, we establish some sufficient conditions on the heat source function and the heat conduction function of the parabolic equation to guarantee that $u(\boldsymbol{x}, t)$ blows up at finite time, and give upper and lower bounds of the blow-up time in multi-dimensional space.

Keywords Sufficient conditions, blow-up, upper and lower bounds, multi-dimensional space.

MSC(2010) 35K55, 35K60.

1. Introduction

In this paper, we deal with the initial-boundary value problem

$$\begin{cases} u_t - \Delta u = f(u), & (\boldsymbol{x}, t) \in \Omega \times (0, t^*), \\ u = 0, & (\boldsymbol{x}, t) \in \Gamma_0 \times (0, t^*), \\ \dfrac{\partial u}{\partial \boldsymbol{n}} = g(u), & (\boldsymbol{x}, t) \in \Gamma_1 \times (0, t^*), \\ u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}) \geq 0, & \boldsymbol{x} \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^N$ $(N \geq 2)$ with smooth boundary $\Gamma := \partial\Omega$, $\Gamma = \Gamma_0 \cup \Gamma_1$, meas $(\Gamma_0 \cap \Gamma_1) = 0$, meas $(\Gamma_0) \geq 0$, meas $(\Gamma_1) > 0$ and $\boldsymbol{n} = (n_1, n_2, \cdots, n_N)$ is the unit outward normal vector on Γ_1, $u_0 \in C^1\left(\overline{\Omega}\right)$, $u_0(\boldsymbol{x}) \geq 0$, $u_0 \not\equiv 0$, and t^* is the blow-up time if blow-up occurs. From the physical standpoint, f is the heat source function and g is the heat conduction function transmitting into interior of Ω from the boundary Γ_1.

The blow-up phenomena of solutions to evolution partial differential equations has received considerable attentions in recent years. For the work in this area, the reader can refer to the book Quittner [9] and papers [1, 4]. Many methods have been used to determine the blow-up of solutions and to indicate an upper bound of the blow-up time. To our knowledge, the first work on lower bound of t^* was given by Weissler [10, 11]. Recently, a number of papers deriving lower bound of t^* in various problems have appeared (see [2, 3, 6–8, 12, 13] and the references therein).

†the corresponding author.
Email address:fushan99@163.com (F. Li), xnmiao0422@163.com (X. Miao), xuexiaomin31415926@163.com (X. Xue)
[1]School of Mathematical Sciences, Qufu Normal University, Qufu, Shangdong 273165, China
*The authors were supported by Natural Science of Shandong Province of China (ZR2019MA067).

The blow-up for nonlinear equations with *Neumann boundary* conditions has received considerable attentions. Payne and Schaefer [6] considered homogeneous equation without heat source term

$$u_t = \Delta u, \quad (\boldsymbol{x}, t) \in \Omega \times (0, t^*).$$

Under suitable nonlinear conditions, they deduced a lower bound of the blow-up time when blow-up occurs only in *three-dimensional* space. Mizoguchi [5] studied the semilinear heat equation with a power function heat source term

$$u_t = \Delta u + u^p, \quad (\boldsymbol{x}, t) \in \Omega \times (0, T),$$

and showed that if u blows up at $t = T$, then $|u(t)|_\infty \leq C(T-t)^{-\frac{1}{p-1}}$ for some $C > 0$. Payne etc [8] considered heat equation with general heat source term

$$u_t = \Delta u - f(u) \quad \boldsymbol{x} \in \Omega, t \in (0, t^*),$$

and established conditions on nonlinearities to guarantee that $u(\boldsymbol{x}, t)$ blows up at some finite time t^*. Moreover, an upper bound for t^* was derived. Under some more restrictive conditions, a lower bound for t^* was derived only in *three-dimensional* space. Li and Li [2] investigated nonhomogeneous divergence form parabolic equation

$$u_t = \sum_{i=1}^{N} (a_{ij}(\boldsymbol{x}) u_{x_i})_{x_j} - f(u), \quad t \in (0, t^*),\ \boldsymbol{x} = (x_1, x_2, \cdots, x_N) \in \Omega,$$

and gave the conditions on nonlinearities to guarantee that $u(\boldsymbol{x}, t)$ exists globally or blows up at some finite time respectively. If blow-up occurs, they obtained upper and lower bounds of the blow-up time, but the lower bound of t^* was valid only in *three-dimensional* space.

Motivated by the above work, we intend to study the blow-up phenomena for problem (1.1). It is well known that the data f and g may greatly affect the behavior of $u(\boldsymbol{x}, t)$ with the development of time. The larger the heat source function f and conduction function g are, the greater possibility the blow-up will occur, and the earlier blow-up time will be. The main contributions of this paper are: (i) the conditions of blow-up are derived naturally by means of calculation process, and some examples satisfying the conditions are given; (ii) the lower bound of blow-up time is given under the conditions that ensure occurrence of blow-up phenomena; (iii) the lower bound of blow-up time is obtained in multi-dimensional space which improves the situation discussed in three-dimensional space.

The present work is organized as follows. In Section 2, we derive the conditions on f, g to ensure that the solutions blow up at finite time and obtain an upper bound of the blow-up time. In Section 3, under the conditions on f and g that guarantee the occurrence of blow-up, we get a lower bound of blow-up time t^* in multi-dimensional space.

2. Blow-up and upper bound estimation of t^*

In order to derive the sufficient conditions for blow-up phenomena and the upper bound of blow-up time, we first give the following calculation. From the physical background and characteristics of the equation, we know that if the functions f, g are nonnegative, then the solution to (1.1) is nonnegative and smooth.

Multiplying the equation of (1.1) by u and integrating on Ω, we obtain

$$\int_\Omega u u_t d\boldsymbol{x} = \int_\Omega u \Delta u d\boldsymbol{x} + \int_\Omega u f(u) d\boldsymbol{x},$$

that is

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 d\boldsymbol{x} = \int_{\Gamma_1} u \frac{\partial u}{\partial \boldsymbol{n}} dS - \int_\Omega |\nabla u|^2 d\boldsymbol{x} + \int_\Omega u f(u) d\boldsymbol{x}. \tag{2.1}$$

These calculations inspire us to define

$$\Phi(t) := \frac{1}{2}\int_\Omega u^2 d\boldsymbol{x}.$$

If

$$u g(u) \geq \gamma G(u),\ \ u f(u) \geq \gamma F(u), \tag{2.2}$$

where

$$G(\xi) = \int_0^\xi g(s) ds,\ F(\xi) = \int_0^\xi f(s) ds,\ \gamma \geq 2, \tag{2.3}$$

then, from (2.1), we have

$$\begin{aligned}\Phi'(t) &= \int_{\Gamma_1} u g(u) dS - \int_\Omega |\nabla u|^2 d\boldsymbol{x} + \int_\Omega u f(u) d\boldsymbol{x} \\ &\geq \gamma \left[\int_{\Gamma_1} G(u) dS - \frac{1}{2}\int_\Omega |\nabla u|^2 d\boldsymbol{x} + \int_\Omega F(u) d\boldsymbol{x}\right]. \end{aligned}\tag{2.4}$$

Denote

$$\Theta(t) := \int_{\Gamma_1} G(u) dS - \frac{1}{2}\int_\Omega |\nabla u|^2 d\boldsymbol{x} + \int_\Omega F(u) d\boldsymbol{x},$$

then with (2.4) we see

$$\Phi'(t) \geq \gamma \Theta(t). \tag{2.5}$$

Since

$$\Theta'(t) = \int_{\Gamma_1} g(u) u_t dS - \frac{1}{2}\int_\Omega (|\nabla u|^2)_t d\boldsymbol{x} + \int_\Omega f(u) u_t d\boldsymbol{x}, \tag{2.6}$$

and noting

$$\frac{1}{2}\left(|\nabla u|^2\right)_t = \operatorname{div}(u_t \nabla u) - u_t \Delta u,$$

by using divergence theorem, we get that

$$\frac{1}{2}\int_\Omega \left(|\nabla u|^2\right)_t d\boldsymbol{x} = \int_\Omega \left[\operatorname{div}(u_t \nabla u) - u_t \Delta u\right] d\boldsymbol{x}$$

$$= \int_{\Gamma_1} u_t \frac{\partial u}{\partial \boldsymbol{n}} dS - \int_\Omega u_t \Delta u d\boldsymbol{x}$$
$$= \int_{\Gamma_1} u_t g(u) dS - \int_\Omega u_t \Delta u d\boldsymbol{x}. \tag{2.7}$$

Substituting (2.7) into (2.6), we have

$$\Theta'(t) = \int_\Omega u_t \left[\Delta u + f(u)\right] d\boldsymbol{x} = \int_\Omega u_t^2 d\boldsymbol{x} \geq 0. \tag{2.8}$$

Assuming

$$\Theta(0) = \int_{\Gamma_1} G(u_0) dS - \frac{1}{2}\int_\Omega |\nabla u_0|^2 d\boldsymbol{x} + \int_\Omega F(u_0) > 0, \tag{2.9}$$

we have $\Theta(t) > 0$ for any $t \in (0, t^*)$. Using the definition of $\Phi(t)$, (2.8) and Hölder's inequality, we have

$$(\Phi'(t))^2 = \left(\int_\Omega u u_t d\boldsymbol{x}\right)^2 \leq \left(\int_\Omega u^2 d\boldsymbol{x}\right)\left(\int_\Omega u_t^2 d\boldsymbol{x}\right) = 2\Phi(t)\Theta'(t). \tag{2.10}$$

Combining (2.10) and (2.5), we obtain

$$\Phi(t)\Theta'(t) \geq \frac{1}{2}(\Phi'(t))^2 \geq \frac{\gamma}{2}\Phi'(t)\Theta(t),$$

that is

$$\Phi(t)\Theta'(t) \geq \frac{\gamma}{2}\Phi'(t)\Theta(t). \tag{2.11}$$

Multiplying the both sides of (2.11) by $\Phi^{-(\frac{\gamma}{2}+1)}$, we deduce

$$\left(\Theta(t)\Phi^{-\frac{\gamma}{2}}(t)\right)' = \Phi^{-(\frac{\gamma}{2}+1)}(t)\left(\Phi(t)\Theta'(t) - \frac{\gamma}{2}\Phi'(t)\Theta(t)\right) \geq 0. \tag{2.12}$$

Integrating (2.12) over $[0, t]$ yields

$$\Theta(t)\Phi^{-\frac{\gamma}{2}}(t) \geq \Theta(0)\Phi^{-\frac{\gamma}{2}}(0) =: M,$$

that is

$$\Theta(t) \geq M\Phi^{\frac{\gamma}{2}}(t). \tag{2.13}$$

By (2.5) and (2.13), we get

$$\Phi'(t) \geq \gamma\Theta(t) \geq \gamma M \Phi^{\frac{\gamma}{2}}(t), \tag{2.14}$$

which implies (if $\gamma > 2$) that

$$\Phi^{-\frac{\gamma}{2}}(t)\Phi'(t) \geq \gamma M \Rightarrow \frac{2}{2-\gamma}\left(\Phi^{-\frac{\gamma}{2}+1}(t)\right)' \geq \gamma M$$
$$\Rightarrow \left(\Phi^{-\frac{\gamma}{2}+1}(t)\right)' \leq \frac{2-\gamma}{2}\gamma M$$

$$
\begin{aligned}
&\Rightarrow \Phi^{-\frac{\gamma}{2}+1}(t) \leq \Phi^{-\frac{\gamma}{2}+1}(0) + \frac{2-\gamma}{2}\gamma M t \\
&\Rightarrow \Phi^{\frac{\gamma}{2}-1}(t) \geq \frac{1}{\Phi^{-\frac{\gamma}{2}+1}(0) - \frac{\gamma-2}{2}\gamma M t} \\
&\Rightarrow \Phi(t) \geq \frac{1}{\left[\Phi^{-\frac{\gamma}{2}+1}(0) - \frac{\gamma-2}{2}\gamma M t\right]^{\frac{2}{\gamma-2}}}.
\end{aligned} \tag{2.15}
$$

Therefore,

$$\lim_{t\to t^*} \Phi(t) = +\infty, \tag{2.16}$$

where $t^* \leq T = \frac{2\Phi(0)}{\gamma(\gamma-2)\Theta(0)}$ (by the definition of M).

Summarizing the above conditions and processes, we can get the following theorem.

Theorem 2.1. *Supposed that $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain, and the nonnegative integrable functions f and g satisfy the conditions for some constant $\gamma > 2$:*

$$\xi f(\xi) \geq \gamma F(\xi), \quad \xi g(\xi) \geq \gamma G(\xi), \forall \xi \geq 0,$$

with

$$F(\xi) := \int_0^\xi f(s)ds, \quad G(\xi) := \int_0^\xi g(s)ds.$$

Moreover, we assume $\Theta(0) > 0$ with

$$\Theta(t) = \int_{\Gamma_1} G(u)dS - \frac{1}{2}\int_\Omega |\nabla u|^2 d\boldsymbol{x} + \int_\Omega F(u)d\boldsymbol{x}.$$

Then the nonnegative classical solution $u(\boldsymbol{x},t)$ of problem (1.1) blows up at time $t^* \leq T$ with

$$T = \frac{2\Phi(0)}{\gamma(\gamma-2)\Theta(0)},$$

where $\Phi(t) = \frac{1}{2}\int_\Omega u^2 d\boldsymbol{x}$.

Remark 2.1. (1) If we choose $f(u) = u^\alpha$, $g(u) = u^\beta$, $(\alpha, \beta > 1)$, $u_0(x) = \text{constant} > 0$, then all the conditions in the theorem are satisfied.

(2) When the equation has no heat source, that is $f \equiv 0$, we can choose $g(u) = u^\beta$, $(\beta > 1)$, $u_0(x) = \text{constant} > 0$, then all the conditions in the theorem are also satisfied. This situation shows that blow-up phenomena only depending on boundary heat conduction may also occur, but the blow-up time will be delayed.

(3) When the boundary is adiabatic, that is $g \equiv 0$, we can choose $f(u) = u^\alpha$, $(\alpha > 1)$, $u_0(x) = \text{constant} > 0$, then all the conditions in the theorem are also satisfied. This situation shows that blow-up phenomena only depending on heat source may also occur, but the blow-up time will be delayed.

(4) When the equation has no heat source and the boundary is adiabatic, that is $f \equiv 0$ and $g \equiv 0$, from (2.1), we know that the energy functional $\Phi(t)$ is decreasing, so blow-up phenomena will not occur.

Remark 2.2. Our theorem can be used to explain the results for some problems in literatures. For example, in [12, 13], the authors dealt with a heat equation with a local nonlinear Neumann boundary conditions:

$$\begin{cases} u_t = \Delta u, & in\ \Omega \times (0, t^*), \\ \dfrac{\partial u}{\partial \boldsymbol{n}} = u^q, & on\ \Gamma_1 \times (0, t^*), \\ \dfrac{\partial u}{\partial \boldsymbol{n}} = 0, & on\ \Gamma_2 \times (0, t^*), \\ u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}), & in\ \Omega, \end{cases}$$

where $q > 1$, $u_0 \in C^1(\overline{\Omega})$, $u_0(\boldsymbol{x}) \geq 0$, and $u_0(\boldsymbol{x}) \not\equiv 0$.

In fact, for $q > 1$, there exists a constant $\gamma > 2$ such that $\frac{\gamma}{1+q} \leq 1$, that is

$$ug(u) = u^{q+1} \geq \gamma \int_0^u s^q ds = \frac{\gamma}{1+q} u^{q+1}.$$

One can choose a suitable $u_0(\boldsymbol{x})$ such that all the conditions in the Theorem 2.1 are satisfied. Consequently, the blow-up phenomena occurs.

3. Lower bound estimation of t^*

Lemma 3.1. *Let $\Omega \subset R^N$ $(N \geq 2)$ be a bounded star-shaped domain assumed to be convex in $N-1$ orthogonal directions. Then for any nonnegative increasing C^1 function $P(w)$, we have*

$$\int_{\partial\Omega} P(w) dS \leq \frac{N}{\rho_0} \int_\Omega P(w) d\boldsymbol{x} + \frac{d}{\rho_0} \int_\Omega P'(w) |\nabla w| d\boldsymbol{x},$$

where

$$\rho_0 := \min_{\boldsymbol{x} \in \partial\Omega} (\boldsymbol{x} \cdot \boldsymbol{n}), \quad d := \max_{\boldsymbol{x} \in \partial\Omega} |\boldsymbol{x}|.$$

Proof. Since Ω is a bounded star-shaped domain, we have $\rho_0 > 0$. Integrating the identity

$$\operatorname{div}(P(w)\boldsymbol{x}) = NP(w) + P'(w)(\boldsymbol{x} \cdot \nabla w)$$

over Ω, and using the divergence theorem, we get

$$\int_{\partial\Omega} P(w)(\boldsymbol{x} \cdot \boldsymbol{n}) dS = N \int_\Omega P(w) d\boldsymbol{x} + \int_\Omega P'(w)(\boldsymbol{x} \cdot \nabla w) d\boldsymbol{x}.$$

By the definitions of ρ_0 and d, it follows that

$$\begin{aligned} \rho_0 \int_{\partial\Omega} P(w) dS &\leq \int_{\partial\Omega} P(w)(\boldsymbol{x} \cdot \boldsymbol{n}) dS \\ &\leq N \int_\Omega P(w) d\boldsymbol{x} + \int_\Omega P'(w) |\boldsymbol{x}| |\nabla w| d\boldsymbol{x} \\ &\leq N \int_\Omega P(w) d\boldsymbol{x} + d \int_\Omega P'(w) |\nabla w| d\boldsymbol{x}, \end{aligned}$$

which implies the desire conclusion. □

Lemma 3.2. *Assume that $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded star-shaped domain assumed to be convex in $N-1$ orthogonal directions. Let $w(\boldsymbol{x})$ be a nonnegative C^1 function defined in Ω. Then for any constant $\sigma \geq 1$, the following inequality holds*

$$\begin{aligned}&\int_\Omega w^{(1+\frac{1}{2^{N-2}})\sigma} d\boldsymbol{x}\\ \leq &(1+2d)^{N-3}\left[\frac{N}{2\rho_0}\int_\Omega w^\sigma d\boldsymbol{x} + \frac{\sigma}{2}\left(1+\frac{d}{\rho_0}\right)\int_\Omega w^{\sigma-1}|\nabla w| d\boldsymbol{x}\right]^{1+\frac{1}{2^{N-2}}},\end{aligned}$$

where

$$\rho_0 := \min_{\boldsymbol{x}\in\partial\Omega}(\boldsymbol{x}\cdot\boldsymbol{n}) > 0,\ d := \max_{\boldsymbol{x}\in\partial\Omega}|\boldsymbol{x}|.$$

Proof. Using mathematical induction method and iterated integral formula, we can finish the proof. □

In this section, in the multi-dimensional space, we give the lower bound of blow-up time under the conditions that guarantee the occurrence of blow-up phenomena. We assume that functions f and g satisfying

$$f(\xi) \equiv 0,\ \gamma G(\xi) \leq \xi g(\xi) \leq \tau \xi^{2+\frac{1}{2^{N-2}}},\ \xi \geq 0,\ \gamma > 2 \tag{3.1}$$

for $\tau > 0$. For

$$\varphi(t) := \frac{1}{2}\int_\Omega u^2 d\boldsymbol{x}, \tag{3.2}$$

we can show that $\varphi(t)$ satisfies

$$\varphi'(t) \leq \Psi(\varphi) \tag{3.3}$$

for some computable function Ψ. Then it follows that t^* is bounded by

$$t^* \geq \int_{\varphi(0)}^\infty \frac{d\eta}{\Psi(\eta)}. \tag{3.4}$$

Indeed, differentiating (3.2), we have (noting $f \equiv 0$)

$$\begin{aligned}\varphi'(t) = &\int_\Omega u u_t d\boldsymbol{x} = \int_\Omega u\left[\Delta u + f(u)\right] d\boldsymbol{x} = \int_{\Gamma_1} u g(u) dS - \int_\Omega |\nabla u|^2 d\boldsymbol{x}\\ &\leq \tau\int_{\Gamma_1} u^{2+\frac{1}{2^{N-2}}} dS - \int_\Omega |\nabla u|^2 d\boldsymbol{x}.\end{aligned} \tag{3.5}$$

Using Lemma 3.1, we have

$$\begin{aligned}\int_{\Gamma_1} u^{2+\frac{1}{2^{N-2}}} dS \leq &\int_{\partial\Omega} u^{2+\frac{1}{2^{N-2}}} dS\\ &\leq \frac{N}{\rho_0}\int_\Omega u^{2+\frac{1}{2^{N-2}}} d\boldsymbol{x} + \frac{(2^{N-1}+1)d}{2^{N-2}\rho_0}\int_\Omega u^{1+\frac{1}{2^{N-2}}}|\nabla u| d\boldsymbol{x}.\end{aligned} \tag{3.6}$$

Using Hölder's and Young's inequalities, we get

$$\int_\Omega u^{2+\frac{1}{2^{N-2}}} d\boldsymbol{x} \leq \frac{1}{2}\int_\Omega u^{2+\frac{1}{2^{N-3}}} d\boldsymbol{x} + \frac{1}{2}\int_\Omega u^2 d\boldsymbol{x}$$

$$= \frac{1}{2}\int_\Omega u^{2+\frac{1}{2^{N-3}}}\,d\boldsymbol{x} + \varphi(t), \tag{3.7}$$

and

$$\int_\Omega u^{1+\frac{1}{2^{N-2}}}|\nabla u|d\boldsymbol{x} \le \frac{1}{2\mu}\int_\Omega u^{2+\frac{1}{2^{N-3}}}\,d\boldsymbol{x} + \frac{\mu}{2}\int_\Omega |\nabla u|^2 d\boldsymbol{x}, \tag{3.8}$$

for all $\mu > 0$.

Inserting (3.6)-(3.8) into (3.5), we obtain

$$\begin{aligned}\varphi'(t) \le& \frac{N\tau}{\rho_0}\varphi(t) + \frac{\tau}{2\rho_0}\left(N + \frac{(2^{N-1}+1)d}{2^{N-2}\mu}\right)\int_\Omega u^{2+\frac{1}{2^{N-3}}}\,d\boldsymbol{x}\\ &+\left(\frac{(2^{N-1}+1)\mu\tau d}{2^{N-1}\rho_0} - 1\right)\int_\Omega |\nabla u|^2 d\boldsymbol{x}.\end{aligned} \tag{3.9}$$

Applying Lemma 3.2 and Hölder's inequality, we have

$$\begin{aligned}&\int_\Omega u^{2+\frac{1}{2^{N-3}}}\,d\boldsymbol{x} = \int_\Omega u^{(1+\frac{1}{2^{N-2}})2}d\boldsymbol{x}\\ &\le (1+2d)^{N-3}\left(\frac{N}{2\rho_0}\int_\Omega u^2 d\boldsymbol{x} + \left(1+\frac{d}{\rho_0}\right)\int_\Omega u|\nabla u|d\boldsymbol{x}\right)^{1+\frac{1}{2^{N-2}}}\\ &\le (1+2d)^{N-3}\left(\frac{N}{2\rho_0}\int_\Omega u^2 d\boldsymbol{x} + \left(1+\frac{d}{\rho_0}\right)\left(\int_\Omega u^2 d\boldsymbol{x}\int_\Omega |\nabla u|^2 d\boldsymbol{x}\right)^{\frac{1}{2}}\right)^{1+\frac{1}{2^{N-2}}}\\ &\le C(1+2d)^{N-3}\left[\left(\frac{N}{\rho_0}\right)^{1+\frac{1}{2^{N-2}}}\varphi^{1+\frac{1}{2^{N-2}}}(t)\right.\\ &\quad\left.+\left(1+\frac{d}{\rho_0}\right)^{1+\frac{1}{2^{N-2}}}2^{\frac{1}{2}+\frac{1}{2^{N-1}}}\varphi^{\frac{1}{2}+\frac{1}{2^{N-1}}}(t)\left(\int_\Omega |\nabla u|^2 d\boldsymbol{x}\right)^{\frac{1}{2}+\frac{1}{2^{N-1}}}\right]\end{aligned} \tag{3.10}$$

for some constant C. Using Young's inequality with ε, we have

$$\begin{aligned}&\varphi^{\frac{1}{2}+\frac{1}{2^{N-1}}}(t)\left(\int_\Omega |\nabla u|^2 d\boldsymbol{x}\right)^{\frac{1}{2}+\frac{1}{2^{N-1}}}\\ =&\left(\varepsilon^{\frac{2^{N-2}+1}{2^{N-1}}}\left(\int_\Omega |\nabla u|^2 d\boldsymbol{x}\right)^{\frac{1}{2}+\frac{1}{2^{N-1}}}\right)\left(\varepsilon^{-\frac{2^{N-2}+1}{2^{N-1}}}\varphi^{\frac{1}{2}+\frac{1}{2^{N-1}}}(t)\right)\\ \le&\frac{2^{N-2}+1}{2^{N-1}}\varepsilon\int_\Omega |\nabla u|^2 d\boldsymbol{x} + \frac{2^{N-2}-1}{2^{N-1}}\varepsilon^{-\frac{2^{N-2}+1}{2^{N-2}-1}}\varphi^{\frac{2^{N-2}+1}{2^{N-2}-1}}(t).\end{aligned} \tag{3.11}$$

Combining (3.9)-(3.11), we obtain

$$\varphi'(t) \le c_1\varphi + c_2\varphi^{\frac{2^{N-2}+1}{2^{N-2}}} + c_3\varphi^{\frac{2^{N-2}+1}{2^{N-2}-1}} + c_4\int_\Omega |\nabla u|^2 d\boldsymbol{x}, \tag{3.12}$$

where

$$c_1 = \frac{N\tau}{\rho_0},$$

$$c_2 = \frac{C\tau}{2\rho_0}\left(N + \frac{(2^{N-1}+1)d}{2^{N-2}\mu}\right)(1+2d)^{N-3}\left(\frac{N}{\rho_0}\right)^{1+\frac{1}{2^{N-2}}},$$

$$c_3 = \frac{C\tau}{2\rho_0}\left(N + \frac{(2^{N-1}+1)d}{2^{N-2}\mu}\right)(1+2d)^{N-3}\left(1+\frac{d}{\rho_0}\right)^{1+\frac{1}{2^{N-2}}} 2^{\frac{1}{2}+\frac{1}{2^{N-1}}}\frac{2^{N-2}-1}{2^{N-1}} \cdot \varepsilon^{-\frac{2^{N-2}+1}{2^{N-2}-1}},$$

$$c_4 = \frac{C\tau}{2\rho_0}\left(N + \frac{(2^{N-1}+1)d}{2^{N-2}\mu}\right)(1+2d)^{N-3}\left(1+\frac{d}{\rho_0}\right)^{1+\frac{1}{2^{N-2}}} 2^{\frac{1}{2}+\frac{1}{2^{N-1}}}\frac{2^{N-2}+1}{2^{N-1}}\varepsilon + \frac{(2^{N-1}+1)\mu\tau d}{2^{N-1}\rho_0} - 1.$$

Choosing ε small enough, one can get a positive μ such that $c_4 = 0$. Therefore,

$$\varphi'(t) \le c_1\varphi + c_2\varphi^{\frac{2^{N-2}+1}{2^{N-2}}} + c_3\varphi^{\frac{2^{N-2}+1}{2^{N-2}-1}} := \Psi(\varphi). \tag{3.13}$$

From (3.13), we get

$$\left(\int_{\varphi(0)}^{\varphi(t)} \frac{d\eta}{\Psi(\eta)}\right)' = \frac{\varphi'(t)}{\Psi(\varphi)} \le 1. \tag{3.14}$$

Integrating (3.14) over $[0,t]$, we obtain

$$t \ge \int_{\varphi(0)}^{\varphi(t)} \frac{d\eta}{\Psi(\eta)}, \tag{3.15}$$

which implies

$$t^* \ge \int_{\varphi(0)}^{\infty} \frac{d\eta}{\Psi(\eta)} = \int_{\varphi(0)}^{\infty} \frac{d\eta}{c_1\eta + c_2\eta^{\frac{2^{N-2}+1}{2^{N-2}}} + c_3\eta^{\frac{2^{N-2}+1}{2^{N-2}-1}}}$$

with $\lim_{t\to t^*}\varphi(t) = \infty$ (by Theorem 2.1).

From the above analysis and Theorem 2.1, we can summarize the following theorem on lower bound estimation of blow-up time t^*:

Theorem 3.1. *Let $\Omega \subset R^N$ $(N \ge 3)$ be a bounded star-shaped domain assumed to be convex in $N-1$ orthogonal directions, and the nonnegative f and g satisfy the conditions*

$$f(\xi) \equiv 0,\ \gamma G(\xi) \le \xi g(\xi) \le \tau\xi^{2+\frac{1}{2^{N-2}}},\ \xi \ge 0,\ \gamma > 2.$$

Then the nonnegative solution $u(\boldsymbol{x},t)$ of problem (1.1) blows up at finite time, and the blow-up time t^* is bounded from below by

$$t^* \ge \int_{\varphi(0)}^{\infty} \frac{d\eta}{\Psi(\eta)} = \int_{\varphi(0)}^{\infty} \frac{d\eta}{c_1\eta + c_2\eta^{\frac{2^{N-2}+1}{2^{N-2}}} + c_3\eta^{\frac{2^{N-2}+1}{2^{N-2}-1}}}.$$

Acknowledgements

The authors would like to thank the referees for their valuable comments and suggestion.

References

[1] C. Bandle and H. Brunner, *Blow-up in diffusion equations*, A survey, Journal of Computational and Applied Mathematics, 1998, 97, 3–22.

[2] F. Li and J. Li, *Global existence and blow-up phenomena for nonlinear divergence form parabolic equations with inhomogeneous Neumann boundary conditions*, Journal of Mathematical Analysis and Applications, 2012, 385, 1005–1014.

[3] F. Li and J. Li, *Global existence and blow-up phenomena for p-Laplacian heat equation with inhomogeneous Neumann boundary conditions*, Boundary Value Problems, 2014, 2014: 219.

[4] B. Liu, R. Wu, and L. Chen, *Weak solutions of a reaction difusion system with super diffusion and its optimal control*, Journal of Nonlinear Modeling and Analysis, 2019, 1(2), 271–282.

[5] N. Mizoguchi, *Blow-up rate of solutions for a semilinear heat equation with Neumann boundary condition*, Journal of Differential Equations, 2003, 193, 212–238.

[6] L.E. Payne and P.W. Schaefer, *Bounds for the blow-up time for heat equation under nonlinear boundary condition*, Proceedings of the Royal Society of Edinburgh Section A Mathematics, 2009, 139, 1289–1296.

[7] L.E. Payne, G.A. Philippin and S. Vernier Piro, *Blow-up phenomena for a semilinear heat equation with nonlinear boundary condition*, I, Zeitschrift für Angewandte Mathematik und Physik, 2010, 61, 999–1007.

[8] L.E. Payne, G.A. Philippin and S. Vernier Piro, *Blow-up phenomena for a semilinear heat equation with nonlinear boundary condition*, II, Nonlinear Analysis: Methods and Applications, 2010, 73, 971–978.

[9] R. Quittner and P. Souplet, *Superlinear Parabolic Problems*, in: Blow-up, Global Existence and Steady States, in: Birkhäuser Advanced Texts, Birkhäuser, Basel, 2007.

[10] F.B. Weissler, *Local existence and nonexistence for semilinear parabolic equations in L^p*, Indiana University Mathematics Journal, 1980, 29, 79–102.

[11] F.B. Weissler, *Existence and nonexistence of global solutions for a heat equation*, Israel Journal of Mathematics, 1981, 38 (1-2), 29–40.

[12] X. Yang and Z. Zhou, *Blow-up problems for the heat equation with a local nonlinear Neumann boundary condition*, Journal of Differential Equations, 2016, 261, 2738–2783.

[13] X. Yang and Z. Zhou, *Improvement on lower bounds for the blow-up time under local nonlinear Neumann boundary conditions*, Journal of Differential Equations, 2018, 265, 830–862.