Journal of Nonlinear Modeling and Analysis
Volume 1, Number 4, December 2019, 443–459
http://jnma.ca; http://jnma.ijournal.cn
DOI:10.12150/jnma.2019.443

The Formats of Julia Sets for Complex Dynamic Systems

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Abstract In this paper, the formats of Julia sets for a class of nonlinear complex dynamic systems with variable coefficients were studied under certain conditions. For the complex dynamic systems in piecewise cases, we proposed some methods to control the forms of their Julia sets and stable domains analytically. What's more, we illustrated that our methods worked well by computational simulations. Our work provides a better understanding about how to control the Julia sets of certain complex dynamic systems.

Keywords Fractals, variable coefficients, complex dynamic system, stable domain, Julia sets.

MSC(2010) 28A80, 37F50.

1. Introduction

People began to learn the complex dynamic systems during World War I. Inspired by the method of Newton iterative operations and the ultimate sets of Mobius transform group, French mathematicians P. Fatou and G. Julia found some interesting results on Riemman sphere [8,12]. From 1918 to 1920, they applied the new theories of normal train (such as Montel theorem) on dynamic systems to prove a series of valuable results, which completing the fundamental work of complex dynamic systems, defined the famous fractal set-Julia set, and forming the classical Fatou-Julia theory. The Mandelbrot set [13] is highly related to the Julia set, which was defined in 1980 by Benoit B. Mandelbrot. It is the result of iterating the dynamic systems. Though the systems and the iterative operations are simple, the shapes and the fine structures of the results are shocking.

At present, the research of complex dynamic systems is still the focus, involving its qualitative theory [27] and the control of bounded domains for the fractal sets [26, 28]. In addition, it provides novel methods for studying all kinds of complex shapes and structures in the nature. So it is widely applied in astronomy [6, 24], geography [16, 18, 19, 23], physics [3, 10, 20, 22], chemistry [5, 11], biology [1, 9, 15, 17], materials [2, 14, 21, 25], sociology [4] and so on.

In particular, lots of problems involve the stability of systems in engineering and technology. The stability of systems relate to their stable domains, or the shapes and sizes of the stable areas. Julia sets and Mandelbrot sets can describe the the

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shapes of the stable fields for the systems. Here, we give the definition of the formats of complex dynamic systems and introduce Julia set as follows.

Definition 1.1. For the nonlinear time delay complex dynamic system:

$$z_{n+1} = f(a_n, c_n, z_n, z_{n-k}) \ , \tag{1.1}$$

where a_n and c_n are known sequences of complex numbers, k is a quantity of time delay and a nonnegative integer. The formats of complex dynamic systems are the process which makes the stable domain of system (1.1) to achieve some enacted aims by adjusting a_n, c_n and k.

Especially, when analyzing system (1.1) through the definition of Julia sets, the stable domains are the formats of Julia set of system (1.1).

For (1.1), let C is a set of all complex numbers, and we call $\partial(C)$ the boundary of C. The definition of Julia sets is as follows [7].

Definition 1.2. For complex polynomial f, let

$$\mathcal{K}(f) = \{ \ z \in C : f^k(z) \nrightarrow \infty, \ k \to \infty \ \} \ ,$$

$\mathcal{J}(f) \triangleq \partial(\mathcal{K}(f))$. We call $\mathcal{J}(f)$ the Julia set of system (1.1).

In this paper, firstly, we will study the formats of Julia sets for above system in general conditions. Then we analyze the formats of Julia sets of special conditions of the system above, which are generated through modifying the complex sequences a_n and c_n. At last, we study the formats of Julia sets for the system above in the piecewise cases. We will illustrate our results through simulations.

For (1.1), we consider that

$$f(\cdot) = a(z+b)^2 - c \ .$$

For the complex dynamic system:

$$z_{n+1} = a_n(z_n + c_n)^2 - c_{n+1} \ , \tag{1.2}$$

it becomes to the well-known classical system

$$z_{n+1} = z_n^2 + c, \tag{1.3}$$

when $a_n \equiv 1, \ c_n \equiv 0$ and $c_{n+1} \equiv -c$, which was used to study its Julia set originally.

Setting $c = -0.5(1+i)$, the Julia set of system (1.3) has the form showed in Figure 1.

We can see that this form of the Julia set of system (1.3) is irregular, which implies that its stable domain is also irregular. What's more, we should note that system (1.3) is a special form of system (1.2).

For achieving certain requirements of stability for the systems, we need to obtain the regular domains of their Julia sets. As a result, we will focus on studying the regular formats of Julia sets for system (1.2) in this article.

2. The formats of Julia sets for system (1.2)

We will focus on the formats of Julia sets for system (1.2). Here we use the symbol $\mathcal{J}(1.2)$ to denote the Julia sets of system (1.2).

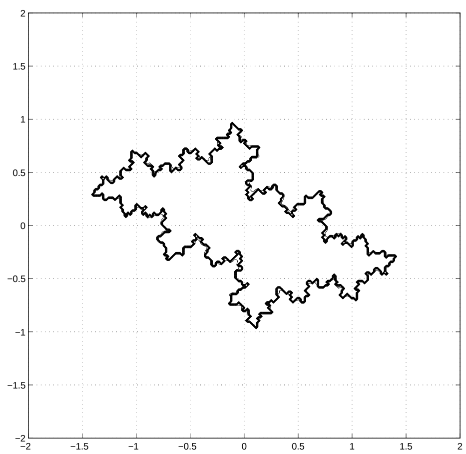

Figure 1. The Julia set of system (1.3) when $c = -0.5(1+i)$.

Theorem 2.1. *For the complex dynamic system (1.2), assume that* $||a_n|| \le a$ *and* $||c_n|| \le c$ *for* $\forall n$*, then* z_n *will be bounded when* $||z_0 + c_0|| \le \frac{1}{a}$ *in the case of* $n \to \infty$*, and*

$$\mathcal{J}(1.2) = \partial\{\ z_0 : ||z_0 + c_0|| \le r,\ r \ge \frac{1}{a}\ \}\ .$$

Proof. For system (1.2), we note that

$$\begin{aligned} z_n &= a_{n-1}(z_{n-1} + c_{n-1})^2 - c_n\ , \\ z_{n-1} &= a_{n-2}(z_{n-2} + c_{n-2})^2 - c_{n-1}\ , \\ &\ \vdots \\ z_1 &= a_0(z_0 + c_0)^2 - c_1\ . \end{aligned}$$

Through iterating in sequence, we obtain

$$\begin{aligned} z_{n+1} &= a_n[a_{n-1}(z_{n-1} + c_{n-1})^2 - c_n + c_n]^2 - c_{n+1} \\ &= a_n a_{n-1}^2 (z_{n-1} + c_{n-1})^4 - c_{n+1} \\ &= a_n a_{n-1}^2 [a_{n-2}(z_{n-2} + c_{n-2})^2 - c_{n-1} + c_{n-1}]^4 - c_{n+1} \\ &= a_n a_{n-1}^2 a_{n-2}^4 (z_{n-2} + c_{n-2})^8 - c_{n+1} \\ &\ \vdots \\ &= a_n a_{n-1}^2 \cdots a_1^{2^{n-1}} a_0^{2^n} (z_0 + c_0)^{2^{n+1}} - c_{n+1}\ . \end{aligned}$$

Then we have

$$\begin{aligned} ||z_{n+1}|| &= ||a_n a_{n-1}^2 \cdots a_1^{2^{n-1}} a_0^{2^n} (z_0 + c_0)^{2^{n+1}} - c_{n+1}|| \\ &\le ||a_n|| \cdot ||a_{n-1}||^2 \cdots ||a_1||^{2^{n-1}} \cdot ||a_0||^{2^n} \cdot ||z_0 + c_0||^{2^{n+1}} + ||c_{n+1}||\ . \end{aligned}$$

For $||a_n|| \le a$ and $||c_n|| \le c$ for $\forall n$, we have

$$||z_{n+1}|| \le a \cdot a^2 \cdot a^4 \cdots a^{2^{n-1}} \cdot a^{2^n} \cdot ||z_0 + c_0||^{2^{n+1}} + c$$

$$
\begin{aligned}
&= a^{(2^0+2^1+\cdots+2^{n-1}+2^n)} \cdot ||z_0+c_0||^{2^{n+1}} + c \\
&= a^{\frac{1-2^{n+1}}{1-2}} \cdot ||z_0+c_0||^{2^{n+1}} + c \\
&= a^{2^{n+1}-1} \cdot ||z_0+c_0||^{2^{n+1}} + c \\
&= ||z_0+c_0|| \cdot (a||z_0+c_0||)^{2^{n+1}-1} + c \ .
\end{aligned}
$$

According to the proof above, we obtain that $||z_{n+1}|| \le \frac{1}{a} + c$ when $n \to \infty$ and $||z_0+c_0|| \le \frac{1}{a}$, therefore z_n is bounded. So $||z_n||$ is bounded for $||z_0+c_0|| \le r$ when $n \to \infty$, and $r \ge \frac{1}{a}$.

From definition 1.2, the Julia set of system (1.2) is the boundary of all the z_0 which satisfies $||z_0+c_0|| \le r$. As a result,

$$
\mathcal{J}(1.2) = \partial\{\ z_0 : ||z_0+c_0|| \le r,\ r \ge \frac{1}{a}\ \} \ .
$$

This implies that it is a circle with the center of $-c_0$ and the radius of some value equal to or greater than $\frac{1}{a}$. □

Remark 2.1. According to the appearance of a in the proof above, we can make sure that z_n is bounded in the case of $n \to \infty$ if we choose $||z_0+c_0|| \le \frac{1}{a}$.

Example 2.1. In system (1.2), let $c_n \equiv 0$ for $\forall n$, $a_0 = 4$ and $a_n \equiv 1$ when $n \ge 1$. Then it becomes to

$$
z_{n+1} = \begin{cases} 4z_n^2\ , & n = 0\ ; \\ z_n^2\ , & n \ge 1\ . \end{cases} \tag{2.1}
$$

Note that $||a_n|| \le ||a_0|| = 4$ and $c_n \equiv 0$ for $\forall n$, from theorem 2.1, we have

$$
\mathcal{J}(2.1) = \partial\{\ z_0 : ||z_0|| \le r,\ r \ge \frac{1}{4}\ \} \ .
$$

This implies that its Julia set is a circle with the center of $-c_0 = 0$ and the radius of some value equal to or greater than $\frac{1}{4}$.

We can proof it from the following analysis. Note that

$$
\begin{aligned}
z_{n+1} &= a_n a_{n-1}^2 \cdots a_1^{2^{n-1}} a_0^{2^n} z_0^{2^{n+1}} \\
&= 4^{2n} \cdot z_0^{2^{n+1}} \\
&= (4z_0^2)^{2n} \ ,
\end{aligned}
$$

then

$$
||z_{n+1}|| = (4||z_0||^2)^{2n} \ .
$$

So in the case of $n \to \infty$, z_{n+1} is bounded if and only if $4||z_0||^2 \le 1$. It implies $||z_n||$ is bounded if and only if $||z_0|| \le 0.5$. According to definition 1.2, its Julia set is the boundary of $||z_0|| \le 0.5$. This means that its Julia set is a circle with the center of origin and the radius of 0.5, which satisfies the theorem 2.1. Its Julia set is demonstrated in Figure 2.

We can see that the result of the simulation matches the theorem 2.1 well from above figure.

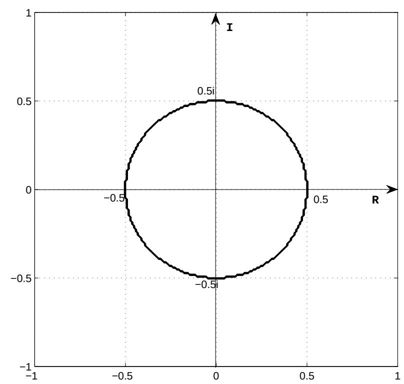

Figure 2. The Julia set of system (2.1).

Example 2.2. In system (1.2), let $a_n = \frac{1}{2(n+1)} + \frac{\sqrt{3}}{2(n+1)} i$ and $c_n = -\frac{2}{(n+1)^2} + \frac{3}{n+2} i$ for $\forall n$, then it leads to

$$\begin{aligned} z_{n+1} = \ & [\frac{1}{2(n+1)} + \frac{\sqrt{3}}{2(n+1)} i][z_n + (-\frac{2}{(n+1)^2} + \frac{3}{n+2} i)]^2 \\ & -(-\frac{2}{(n+2)^2} + \frac{3}{n+3} i) \ . \end{aligned} \tag{2.2}$$

Because $||a_n|| \le ||a_0|| = 1$ and $||c_n|| \le ||c_0|| = 2.5$ for $\forall n$, according to theorem 2.1, we have

$$\mathcal{J}(2.2) = \partial \{ \ z_0 : ||z_0 - 2 + 1.5i|| \le r, \ r \ge 1 \ \} \ .$$

This implies that its Julia set is a circle with the center of $-c_0 = 2 - 1.5i$ and the radius of 1. Its Julia set is demonstrated in Figure 3.

We can see that the result of the simulation matches theorem 2.1 well from Figure 3.

3. The formats of Julia sets for system (1.2) when a_n is a constant

For system (1.2), let $a_n \equiv a_0 \ne 0$, then it becomes to the following system

$$z_{n+1} = a_0 (z_n + c_n)^2 - c_{n+1} \ . \tag{3.1}$$

We will analyze the formats of Julia sets for system (3.1).

Theorem 3.1. *For the complex dynamic system (3.1), assume that $||c_n|| \le c$ for $\forall n$, then z_n is bounded if and only if $||z_0 + c_0|| \le \frac{1}{||a_0||}$ in the case of $n \to \infty$. Moreover,*

$$\mathcal{J}(3.1) = \partial \{ \ z_0 : ||z_0 + c_0|| \le \frac{1}{||a_0||} \ \} \ .$$

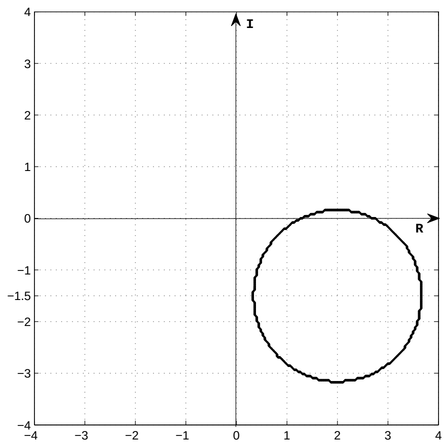

Figure 3. The Julia set of system (2.2).

Proof. According to the proof of theorem 3.1, we have

$$z_{n+1} = (z_0 + c_0)[a_0(z_0 + c_0)]^{2^{n+1}-1} - c_{n+1} \ ,$$
$$||z_{n+1}|| \le ||z_0 + c_0||(||a_0|| \cdot ||z_0 + c_0||)^{2^{n+1}-1} + ||c_{n+1}|| \ .$$

Because $||c_n|| \le c$ for $\forall n$, we have

$$||z_{n+1}|| \le ||z_0 + c_0||(||a_0|| \cdot ||z_0 + c_0||)^{2^{n+1}-1} + c \ .$$

Then $||z_{n+1}|| \le \frac{1}{||a_0||} + c$ if and only if $||a_0|| \cdot ||z_0 + c_0|| \le 1$ in the case of $n \to \infty$. This implies z_n is bounded if and only if $||z_0 + c_0|| \le \frac{1}{||a_0||}$ in the case of $n \to \infty$.

From definition 1.2, we have

$$\mathcal{J}(3.1) = \partial\{ \ z_0 : ||z_0 + c_0|| \le \frac{1}{||a_0||} \ \} \ ,$$

which means the Julia set of system (3.1) is a circle with the center of $-c_0$ and the radius of $\frac{1}{||a_0||}$. □

Example 3.1. In system (3.1), let $a_n \equiv 2i$ and $c_n = -\frac{0.5+i}{n+1}$ for $\forall n$, then it becomes to

$$z_{n+1} = 2i(z_n - \frac{0.5+i}{n+1})^2 + \frac{0.5+i}{n+2} \ . \tag{3.2}$$

Because $||c_n|| \le ||c_0|| = \sqrt{1.25}$ for $\forall n$, from theorem 3.1, we have

$$\mathcal{J}(3.2) = \partial\{ \ z_0 : ||z_0 - 0.5 - i|| \le 0.5 \ \} \ .$$

This means that its Julia set is a circle with the center of $-c_0 = 0.5 + i$ and the radius of $\frac{1}{||a_0||} = 0.5$. Its Julia set is illustrated in figure 4.

We can see that the result of the simulation matches theorem 3.1 well from the Figure 4.

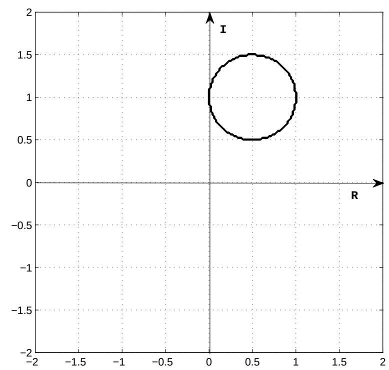

Figure 4. The Julia set of system (3.2).

Remark 3.1. The reason of studying the particular case of system (1.2) is that, from system (3.1) and theorem 3.1, we can construct a specific system that its Julia set is a circle with the center and the radius of any given value on the complex plane.

We will use the following example to illustrate it.

Example 3.2. Finding out a system that its Julia set is a circle with the center of $5 + 6\sqrt{2}i$ and the radius of 10 on the complex plane.

From theorem 3.1, the system we want is one of specific cases of system (3.1). It requires c_n is bounded, $c_0 = -5 - 6\sqrt{2}i$ and $\frac{1}{||a_0||} = 10$ in system (6). So a specific case of system (6) which satisfies these three conditions is just the system we need. Let $c_n \equiv c_0 = -5 - 6\sqrt{2}i$ and $a_0 = 0.1$, system (3.1) becomes to the specific case

$$z_{n+1} = 0.1(z_n - 5 - 6\sqrt{2}i)^2 + 5 + 6\sqrt{2}i \ .$$

It is one case of the system we want.

For system (3.1), let $c_n \equiv c_0$, it becomes to a special system

$$z_{n+1} = a_0(z_n + c_0)^2 - c_0 \ . \tag{3.3}$$

From theorem 3.1, we have the following corollary.

Corollary 3.1. *For the complex dynamic system (3.3), z_n is bounded in the case of $n \to \infty$ if and only if $||z_0 + c_0|| \leq \frac{1}{||a_0||}$. What's more,*

$$\mathcal{J}(3.3) = \partial\{ \ z_0 : ||z_0 + c_0|| \leq \frac{1}{||a_0||} \ \} \ .$$

4. The formats of Julia sets of piecewise cases for system (1.2)

In this section, we will study the formats of Julia sets of piecewise cases for system (1.2):

$$z_{n+1} = \begin{cases} a_{1n}(z_n + c_{1n})^2 - c_{1(n+1)} : & ||z_0 + c|| < r_1 \ ; \\ a_{2n}(z_n + c_{2n})^2 - c_{2(n+1)} : & r_1 \leq ||z_0 + c|| < r_2 \ ; \\ \cdots, \cdots & \cdots \\ a_{sn}(z_n + c_{sn})^2 - c_{s(n+1)} : & r_s \leq ||z_0 + c|| \ . \end{cases}$$

Firstly, we give the following theorem.

Theorem 4.1. *For the complex dynamic system (1.2), assume that* $0 < a' \leq ||a_n|| \leq a$ *and* $||c_n|| \leq c$ *for* $\forall n$*, then* z_n *is bounded when* $||z_0 + c_0|| \leq \frac{1}{a}$ *in the case of* $n \to \infty$*. Moreover,*

$$\mathcal{J}(1.2) = \partial\{\ z_0 : ||z_0 + c_0|| \leq r,\ \frac{1}{a} \leq r \leq \frac{1}{a'}\ \}\ .$$

Proof. We have proved that z_n was bounded when $||z_0 + c_0|| \leq \frac{1}{a}$ in the case of $n \to \infty$ and $r \geq \frac{1}{a}$. Here, we deal with $r \leq \frac{1}{a'}$.

According to the proof of theorem 2.1,

$$||z_{n+1}|| = ||a_n a_{n-1}^2 \cdots a_1^{2^{n-1}} a_0^{2^n} (z_0 + c_0)^{2^{n+1}} - c_{n+1}||\ ,$$

we have

$$||z_{n+1}|| \geq ||a_n|| \cdot ||a_{n-1}||^2 \cdots ||a_1||^{2^{n-1}} \cdot ||a_0||^{2^n} \cdot ||z_0 + c_0||^{2^{n+1}} - ||c_{n+1}||\ .$$

Because $||a_n|| \geq a' > 0$ and $||c_n|| \leq c$ for $\forall n$, we can get

$$\begin{aligned} ||z_{n+1}|| &\geq a' \cdot (a')^2 \cdot (a')^4 \cdots (a')^{2^{n-1}} \cdot (a')^{2^n} \cdot ||z_0 + c_0||^{2^{n+1}} - c \\ &= (a')^{(2^0+2^1+\cdots+2^{n-1}+2^n)} \cdot ||z_0 + c_0||^{2^{n+1}} - c \\ &= (a')^{\frac{1-2^{n+1}}{1-2}} \cdot ||z_0 + c_0||^{2^{n+1}} - c \\ &= (a')^{2^{n+1}-1} \cdot ||z_0 + c_0||^{2^{n+1}} - c \\ &= ||z_0 + c_0|| \cdot (a'||z_0 + c_0||)^{2^{n+1}-1} - c\ . \end{aligned}$$

Then we have that $||z_{n+1}|| \geq \frac{1}{a'} - c$ when $a'||z_0 + c_0|| \leq 1$ in the case of $n \to \infty$. This implies that z_n may be bounded only when $||z_0 + c_0|| \leq \frac{1}{a'}$ in the case of $n \to \infty$. As a result, $||z_n||$ is bounded when $n \to \infty$ if $||z_0 + c_0|| \leq r$ and $r \leq \frac{1}{a'}$.

From definition 1.2, the Julia set of system (1.2) is the boundary of all the z_0 which satisfies $||z_0 + c_0|| \leq r$. According to theorem 2.1 and the proof above, we have

$$\mathcal{J}(1.2) = \partial\{\ z_0 : ||z_0 + c_0|| \leq r,\ \frac{1}{a} \leq r \leq \frac{1}{a'}\ \}\ .$$

This means that it is a circle with the center of $-c_0$ and the radius of some value between $\frac{1}{a}$ and $\frac{1}{a'}$. □

Remark 4.1. According to the appearance of a' in the proof above, we can make sure that z_n is bounded in the case of $n \to \infty$ if we choose $||z_0 + c_0|| \leq \frac{1}{a'}$.

4.1. The cases that Julia sets are concentric circles

Let system (1.2) be the following system

$$z_{n+1} = \begin{cases} a_{1n}(z_n + c_{1n})^2 - c_{1(n+1)} : & ||z_0 + c|| < a \ ; \\ a_{2n}(z_n + c_{2n})^2 - c_{2(n+1)} : & ||z_0 + c|| \geq a \ . \end{cases} \tag{4.1}$$

In system (4.1), $a_{1n},\ a_{2n},\ c_{1n}, c_{2n},\ c \in C \neq \varnothing$, and $a > 0,\ n = 0, 1, 2, \cdots$.

Theorem 4.2. *If piecewise system (4.1) satisfies the following conditions*

1. $0 < a_1' \leq ||a_{1n}|| \leq a_1$ *and* $0 < a_2' \leq ||a_{2n}|| \leq a_2$ *for* $\forall n$*;*
2. $||c_{1n}|| \leq c_1$ *and* $||c_{2n}|| \leq c_2$ *for* $\forall n$*;*
3. $c_{10} = c_{20} = c$*;*
4. $\frac{1}{a_1'} < a < \frac{1}{a_2}$,

we define it as system (4.1.1). Then the Julia set of system (4.1.1) is

$$\mathcal{J}(4.1.1) = \partial\{z_0 : ||z_0 + c|| \leq r_1, \frac{1}{a_1} \leq r_1 \leq \frac{1}{a_1'}\} \bigcup \partial\{z_0 : a \leq ||z_0 + c|| \leq r_2, \frac{1}{a_2} \leq r_2 \leq \frac{1}{a_2'}\} \ .$$

Proof. For system (4.1.1), according to theorem 4.1, we can obtain the conclusion immediately. □

It implies that its Julia set is 3 concentric circles with the center of $-c$ and the radiuses of r_1, a and r_2 respectively, where $\frac{1}{a_1} \leq r_1 \leq \frac{1}{a_1'}$, $\frac{1}{a_2} \leq r_2 \leq \frac{1}{a_2'}$.

Example 4.1. In system (4.1), let $a_{1n} = \sqrt{3} + \frac{i}{n+1},\ c_{1n} = 1 + \frac{i}{n+1},\ a_{2n} = \frac{\sqrt{2}}{4} + \frac{\sqrt{2}i}{4(n+1)},\ c_{2n} = \frac{1}{n+1} + i,\ c = 1 + i$ and $a = 1$, we have

$$z_{n+1} = \begin{cases} (\sqrt{3} + \frac{i}{n+1})[z_n + (1 + \frac{i}{n+1})]^2 - (1 + \frac{i}{n+2}) : & ||z_0 + (1+i)|| < 1 \ ; \\ (\frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{4(n+1)}i)[z_n + (\frac{1}{n+1} + i)]^2 - (\frac{1}{n+2} + i) : & ||z_0 + (1+i)|| \geq 1 \ . \end{cases} \tag{4.2}$$

Because

1. $0 < \sqrt{3} < ||a_{1n}|| \leq ||a_{10}|| = 2$ and $0 < \frac{\sqrt{2}}{4} < ||a_{2n}|| \leq ||a_{20}|| = \frac{1}{2}$ for $\forall n$;
2. $||c_{1n}|| \leq ||c_{10}|| = \sqrt{2}$ and $||c_{2n}|| \leq ||c_{20}|| = \sqrt{2}$ for $\forall n$;
3. $c_{10} = c_{20} = c = 1 + i$;
4. $a = 1,\ \frac{1}{a_1'} = \frac{\sqrt{3}}{3},\ \frac{1}{a_2} = 2$, so $\frac{1}{a_1'} < a < \frac{1}{a_2}$,

by theorem 4.2, we have

$$\mathcal{J}(4.2) = \partial\{\ z_0 : ||z_0 + (1+i)|| \leq r_1,\ 0.5 \leq r_1 < \frac{\sqrt{3}}{3}\ \}$$
$$\bigcup \partial\{\ z_0 : 1 \leq ||z_0 + (1+i)|| \leq r_2,\ 2 \leq r_2 < 2\sqrt{2}\ \} \ .$$

This implies that the Julia set of system (4.2) is 3 concentric circles with the center of $-c = -1 - i$ and the radiuses of r_1, 1 and r_2 respectively, where $0.5 \leq r_1 < \frac{\sqrt{3}}{3}$, $2 \leq r_2 < 2\sqrt{2}$. Its Julia set is demonstrated in Figure 5.

We can see that the simulation matches theorem 4.2 well from the Figure 5.

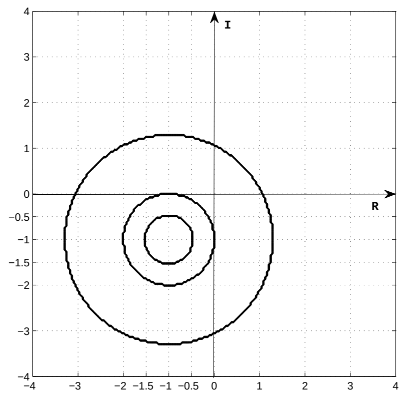

Figure 5. The Julia set of system (4.2).

For system (4.1), let $a_{1n} \equiv a_1 \neq 0,\ a_{2n} \equiv a_2 \neq 0$, and $c_{1n} = c_{2n} \equiv c$, it becomes to a special system

$$z_{n+1} = \begin{cases} a_1(z_n + c)^2 - c : & ||z_0 + c|| < a \ ; \\ a_2(z_n + c)^2 - c : & ||z_0 + c|| \geq a \ . \end{cases} \tag{4.3}$$

From theorem 4.2, we can obtain the following corollary directly.

Corollary 4.1. *For the complex dynamic system (11), when* $\frac{1}{||a_1||} < a < \frac{1}{||a_2||}$*, we have*

$$\mathcal{J}(4.3) = \partial\{\ z_0 : ||z_0 + c|| \leq \frac{1}{||a_1||}\ \} \bigcup \partial\{\ z_0 : a \leq ||z_0 + c|| \leq \frac{1}{||a_2||}\ \} \ .$$

This implies that its Julia set is 3 concentric circles with the center of $-c$ *and the radiuses of* $\frac{1}{||a_1||}$*,* a *and* $\frac{1}{||a_2||}$ *respectively.*

4.2. The cases that Julia sets are non-concentric circles

Theorem 4.3. *If the piecewise system (4.1) satisfies the following conditions*

1. $0 < a_1' \leq ||a_{1n}|| \leq a_1$ *and* $0 < a_2' \leq ||a_{2n}|| \leq a_2$ *for* $\forall n$*;*
2. $||c_{1n}|| \leq c_1$ *and* $||c_{2n}|| \leq c_2$ *for* $\forall n$*;*
3. $||c_{10} - c|| \leq a - \frac{1}{a_1'}$ *and* $||c_{20} - c|| \geq a + \frac{1}{a_2'}$*,*

we define it as system (4.1.2). Then the Julia set of system (4.1.2) is

$$\mathcal{J}(4.1.2) = \partial\{z_0 : ||z_0 + c_{10}|| \leq r_1, \frac{1}{a_1} \leq r_1 \leq \frac{1}{a_1'}\} \bigcup \partial\{z_0 : ||z_0 + c_{20}|| \leq r_2, \frac{1}{a_2} \leq r_2 \leq \frac{1}{a_2'}\} \ .$$

Proof. For system (4.1.2), according to theorem 4.1, we can obtain the conclusion directly. □

In addition, from the third condition of theorem 4.3, we have

$$\begin{aligned}||c_{10}-c_{20}|| &= ||c_{10}-c+c-c_{20}||\\ &= ||(c-c_{20})-(c-c_{10})||\\ &\geq ||c-c_{20}||-||c-c_{10}||\\ &\geq (a+\frac{1}{a_2'})-(a-\frac{1}{a_1'})\\ &= \frac{1}{a_1'}+\frac{1}{a_2'}\\ &\geq r_1+r_2\ .\end{aligned}$$

As a result, its Julia set is two circles that one with the center of $-c_{10}$ and the radius of r_1 while another with the center of $-c_{20}$ and the radius of r_2, where $\frac{1}{a_1} \leq r_1 \leq \frac{1}{a_1'}$, $\frac{1}{a_2} \leq r_2 \leq \frac{1}{a_2'}$. What's more, These two circles are away from each other when $||c_{10}-c_{20}|| > r_1+r_2$, and they are externally tangent when $||c_{10}-c_{20}|| = r_1+r_2$.

Example 4.2. In system (4.1), let $a_{1n} = \frac{\sqrt{3}}{2}+\frac{i}{2(n+1)}$, $c_{1n} = \frac{3}{2}-\frac{3i}{2(n+1)}$, $a_{2n} = 1+\frac{\sqrt{3}i}{n+1}$, $c_{2n} = -\frac{3}{5(n+1)}+\frac{4i}{5}$, $c = 1.5-1.5i$ and $a = 1.5$, then it leads to

$$z_{n+1}=\begin{cases}(\frac{\sqrt{3}}{2}+\frac{i}{2(n+1)})[z_n+(\frac{3}{2}-\frac{3i}{2(n+1)})]^2-(\frac{3}{2}-\frac{3i}{2(n+2)}):||z_0+(1.5-1.5i)||<1.5;\\ (1+\frac{\sqrt{3}i}{n+1})[z_n+(-\frac{3}{5(n+1)}+\frac{4i}{5})]^2-(-\frac{3}{5(n+2)}+\frac{4i}{5}):||z_0+(1.5-1.5i)||\geq 1.5\ .\end{cases} \tag{4.4}$$

Following the conditions

1. $0 < \frac{\sqrt{3}}{2} < ||a_{1n}|| \leq ||a_{10}|| = 1$ and $0 < 1 < ||a_{2n}|| \leq ||a_{20}|| = 2$ for $\forall n$;
2. $||c_{1n}|| \leq ||c_{10}|| = \frac{3\sqrt{2}}{2}$ and $||c_{2n}|| \leq ||c_{20}|| = 1$ for $\forall n$;
3. $a = 1.5,\ \frac{1}{a_1'} = \frac{2\sqrt{3}}{3},\ \frac{1}{a_2'} = 1,\ c_{10} = 1.5-1.5i,\ c_{20} = -0.6+0.8i,\ c = 1.5-1.5i$, so $||c_{10}-c|| \leq a-\frac{1}{a_1'}$ and $||c_{20}-c|| \geq a+\frac{1}{a_2'}$,

applying theorem 4.3, we have

$$\begin{aligned}\mathcal{J}(4.4) = \partial\{\ z_0 : ||z_0+1.5-1.5i|| \leq r_1,\ 1 \leq r_1 < \frac{2\sqrt{3}}{3}\ \}\\ \bigcup \partial\{\ z_0 : ||z_0-0.6+0.8i|| \leq r_2,\ 0.5 \leq r_2 < 1\ \}\ .\end{aligned}$$

Note that

$$||c_{10}-c_{20}|| > \frac{1}{a_1'}+\frac{1}{a_2'} \geq r_1+r_2\ .$$

The Julia set of system (4.4) is two circles away from each other, one with the center of $-c_{10} = -1.5+1.5i$ and the radius of r_1 while another with the center of $-c_{20} = 0.6-0.8i$ and the radius of r_2, where $1 \leq r_1 < \frac{2\sqrt{3}}{3}$, $0.5 \leq r_2 < 1$. Its Julia set is demonstrated in Figure 6.

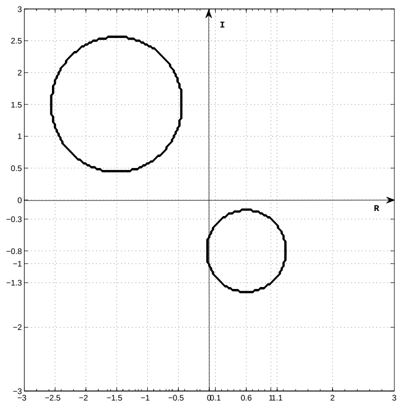

Figure 6. The Julia set of system (4.4).

4.3. The cases that Julia sets are not circles

Theorem 4.4. *If piecewise system (4.1) satisfies the following conditions*

1. $0 < a_1' \leq ||a_{1n}|| \leq a_1$ *and* $0 < a_2' \leq ||a_{2n}|| \leq a_2$ *for* $\forall n$*;*
2. $||c_{1n}|| \leq c_1$ *and* $||c_{2n}|| \leq c_2$ *for* $\forall n$*;*
3. $||c_{10} - c|| \geq a + \frac{1}{a_1'}$ *and* $a + \frac{1}{a_2} > ||c_{20} - c||$*;*
4. $||c_{20} - c|| > a - \frac{1}{a_2} \geq 0$ *or* $||c_{20} - c|| > \frac{1}{a_2'} - a \geq 0$*,*

we define it as system (4.1.3). Then the Julia set of system (4.1.3) is

$$\mathcal{J}(4.1.3) = \partial(\ \{ \ z_0 : ||z_0 + c_{20}|| \leq r_2, \ \frac{1}{a_2} \leq r_2 \leq \frac{1}{a_2'} \ \}$$
$$-\{ \ z_0 : ||z_0 + c_{20}|| \leq r_2, \ \frac{1}{a_2} \leq r_2 \leq \frac{1}{a_2'} \ \} \bigcap \{ \ z_0 : ||z_0 + c|| < a \ \} \ ) \ .$$

Proof. For system (4.1.3), according to theorem 4.1, we can obtain the conclusion directly. □

It implies that its Julia set is formed by two intersected circles, one with the center of $-c_{20}$ and the radius of r_2 while another with the center of $-c$ and the radius of a, where $\frac{1}{a_2} \leq r_2 \leq \frac{1}{a_2'}$.

Example 4.3. In system (4.1), let $a_{1n} = 10 + \frac{10i}{n+1}$, $c_{1n} = \frac{16}{n+1}$, $a_{2n} = \frac{1}{2} + \frac{\sqrt{3}i}{2(n+1)}$, $c_{2n} = -2 - \frac{2i}{n+1}$, $c = -2 - 3i$ and $a = 1$, then it leads to

$$z_{n+1} = \begin{cases} (10 + \frac{10i}{n+1})(z_n + \frac{16}{n+1})^2 - \frac{16}{n+2} : & ||z_0 - 2 - 3i|| < 1 \ ; \\ (\frac{1}{2} + \frac{\sqrt{3}i}{2(n+1)})[z_n + (-2 - \frac{2i}{n+1})]^2 - (-2 - \frac{2i}{n+2}) : & ||z_0 - 2 - 3i|| \geq 1 \ . \end{cases} \tag{4.5}$$

For system (4.5) satisfies all the conditions of theorem 4.4, we have

$$\mathcal{J}(4.5) = \partial(\ \{ \ z_0 : ||z_0 - 2 - 2i|| \leq r_2, \ 1 \leq r_2 < 2 \ \}$$

$$-\{\ z_0 : ||z_0 - 2 - 2i|| \le r_2,\ 1 \le r_2 < 2\ \}\bigcap\{\ z_0 : ||z_0 - 2 - 3i|| < 1\ \}\ )\ .$$

It implies the Julia set of system (4.5), the boundary with the shape of crescent moon, is formed by the circle with the center of $-c_{20} = 2 + 2i$ and the radius of r_2 removing the intersecting part of it and the circle with the center of $-c = 2 + 3i$ and the radius of 1, where $1 \le r_2 < 2$. Its Julia set is demonstrated in Figure 7 by the boundary of the shadow region.

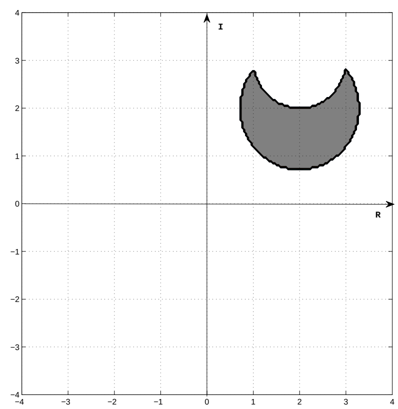

Figure 7. The Julia set of system (4.5).

Example 4.4. In system (4.1), let $a_{1n} = \frac{3}{n+1} + 4i,\ c_{1n} = \frac{9i}{n+3},\ a_{2n} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}i}{2(n+1)},\ c_{2n} = \frac{1}{n+1} - i,\ c = 0.5 - 0.5i$ and $a = 1.5$, it becomes to

$$z_{n+1} = \begin{cases} (\frac{3}{n+1} + 4i)(z_n + \frac{9i}{n+3})^2 - \frac{9i}{n+4} : & ||z_0 + 0.5 - 0.5i|| < 1.5\ ; \\ (\frac{\sqrt{2}}{2} + \frac{\sqrt{2}i}{2(n+1)})[z_n + (\frac{1}{n+1} - i)]^2 - (\frac{1}{n+2} - i) : & ||z_0 + 0.5 - 0.5i|| \ge 1.5\ . \end{cases} \tag{4.6}$$

For system (4.6) satisfies all the conditions of theorem 4.4, we have

$$\begin{aligned} \mathcal{J}(4.6) = \partial(\ &\{\ z_0 : ||z_0 + 1 - i|| \le r_2,\ 1 \le r_2 < \sqrt{2}\ \} \\ &-\{\ z_0 : ||z_0 + 1 - i|| \le r_2,\ 1 \le r_2 < \sqrt{2}\ \}\bigcap\{\ z_0 : ||z_0 + 0.5 - 0.5i|| < 1.5\ \}\ )\ , \end{aligned}$$

It implies the Julia set of system (4.6), the boundary with the shape of crescent moon, is formed by the circle with the center of $-c_{20} = -1 + i$ and the radius of r_2 removing the intersecting part of it and the circle with the center of $-c = -0.5 + 0.5i$ and the radius of 1.5, where $1 \le r_2 < \sqrt{2}$. Its Julia set is demonstrated in Figure 8 by the boundary of the shadow region.

Example 4.5. In system (4.1), let $a_{1n} = 8 + \frac{6i}{n+1},\ c_{1n} = \frac{12}{n+2},\ a_{2n} = \frac{1}{3(n+1)} + \frac{\sqrt{3}i}{3},\ c_{2n} = 2 + \frac{9i}{5(n+1)},\ c = 1 + 2i$ and $a = 1$, then it leads to

$$z_{n+1} = \begin{cases} (8 + \frac{6i}{n+1})(z_n + \frac{12}{n+2})^2 - \frac{12}{n+3} : & ||z_0 + 1 + 2i|| < 1\ ; \\ (\frac{1}{3(n+1)} + \frac{\sqrt{3}i}{3})[z_n + (2 + \frac{9i}{5(n+1)})]^2 - (2 + \frac{9i}{5(n+2)}) : & ||z_0 + 1 + 2i|| \ge 1\ . \end{cases} \tag{4.7}$$

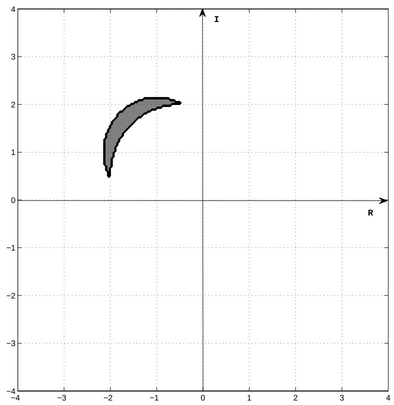

Figure 8. The Julia set of system (4.6).

For system (4.7) satisfies all the conditions of theorem 4.4, we have

$$\begin{aligned}\mathcal{J}(4.7) = \partial(&\ \{\ z_0 : ||z_0 + 2 + 1.8i|| \le r_2,\ 1.5 \le r_2 < \sqrt{3}\ \} \\ &-\{\ z_0 : ||z_0 + 2 + 1.8i|| \le r_2,\ 1.5 \le r_2 < \sqrt{3}\ \} \bigcap \{\ z_0 : ||z_0 + 1 + 2i|| < 1\ \}\ )\ ,\end{aligned}$$

It implies the Julia set of system (15), the boundary with the shape of crescent moon, is formed by the circle with the center of $-c_{20} = -2 - 1.8i$ and the radius of r_2 removing the intersecting part of it and the circle with the center of $-c = -1 - 2i$ and the radius of 1, where $1.5 \le r_2 < \sqrt{3}$. Its Julia set is demonstrated in Figure 9 by the boundary of the shadow region.

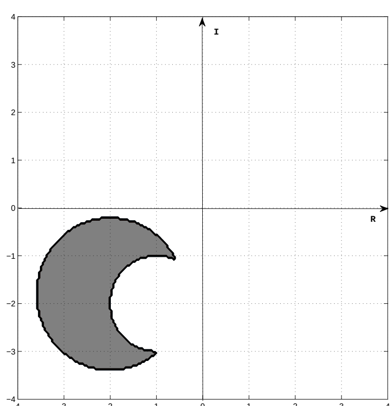

Figure 9. The Julia set of system (4.7).

Example 4.6. In system (4.1), let $a_{1n} = \frac{2}{n+1} + \sqrt{5}i$, $c_{1n} = \frac{7i}{n+1}$, $a_{2n} = 1 +$

$\frac{\sqrt{3}i}{3(n+1)}$, $c_{2n} = -\frac{2}{n+1} + 2i$, $c = -1.5 + 1.5i$ and $a = 1$, then it becomes

$$z_{n+1} = \begin{cases} (\frac{2}{n+1} + \sqrt{5}i)(z_n + \frac{7i}{n+1})^2 - \frac{7i}{n+2} : & ||z_0 - 1.5 + 1.5i|| < 1 \ ; \\ (1 + \frac{\sqrt{3}i}{3(n+1)})[z_n + (-\frac{2}{n+1} + 2i)]^2 - (-\frac{2}{n+2} + 2i) : & ||z_0 - 1.5 + 1.5i|| \geq 1 \ . \end{cases} \tag{4.8}$$

Because system (4.8) satisfies all the conditions of theorem 4.4, we have

$$\mathcal{J}(4.8) = \partial(\ \{ \ z_0 : ||z_0 - 2 + 2i|| \leq r_2, \ \frac{\sqrt{3}}{2} \leq r_2 < 1 \ \} \\ -\{ \ z_0 : ||z_0 - 2 + 2i|| \leq r_2, \ \frac{\sqrt{3}}{2} \leq r_2 < 1 \ \} \bigcap \{ \ z_0 : ||z_0 - 1.5 + 1.5i|| < 1 \ \} \ ) \ ,$$

It implies the Julia set of system (4.8), the boundary with the shape of crescent moon, is formed by the circle with the center of $-c_{20} = 2 - 2i$ and the radius of r_2 removing the intersecting part of it and the circle with the center of $-c = 1.5 - 1.5i$ and the radius of 1, where $\frac{\sqrt{3}}{2} \leq r_2 < 1$. Its Julia set is demonstrated in Figure 10 by the boundary of the shadow region.

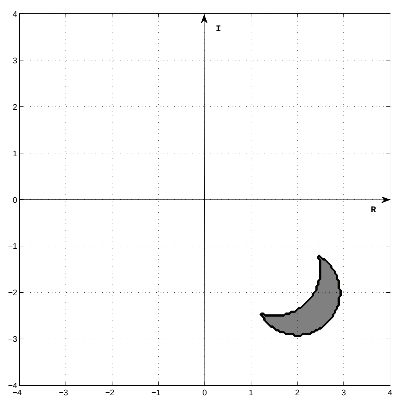

Figure 10. The Julia set of system (4.8).

We can see that all the simulated results match theorem 6 well from these figures above. What's more, here the Julia sets of these four systems are located in four different quadrants respectively. This means that we can control the locations and sizes of their Julia sets.

5. Conclusion

For the general complex dynamic systems with constant coefficients, the related research is well developed and they are applied in multiple fields. However, for the complex dynamic systems with variable coefficients, its relevant study just begins. In this paper, aiming at a class of complex dynamic systems with variable coefficients, we got some theoretical results of the stable domains and formats of

Julia sets for their general and special cases. In addition, we simulated the related Julia sets to illustrate our theoretical study. What's more, we studied some piecewise cases to get the conditions such that their stable domains and Julia sets can achieve certain required forms. We also used simulations to verify our results. Our study provides some ideas about how to control the stable domains and the shapes, sizes and locations of Julia sets for these systems. We will generalize our results to more general complex dynamic systems with variable coefficients in further study.

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