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Threshold Dynamics of a Time-periodic Reaction-Diffusion Malaria Model with Distributed Latencies*

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Abstract It is well-known that the transmission of malaria is caused by the bites of mosquitoes. Since the life habit of mosquitoes is influenced by seasonal factors such as temperature, humidity and rainfall, the transmission of malaria presents clear seasonable changes. In this paper, in order to take into account the incubation periods in humans and mosquitoes, we study the threshold dynamics of two periodic reaction-diffusion malaria models with distributed delay in terms of the basic reproduction number. Firstly, the basic reproduction number R_0 is introduced by virtue of the next generation operator method and the Poincaré mapping of a linear system. Secondly, the threshold dynamics is established in terms of R_0. It is proved that if $R_0 < 1$, then the disease-free periodic solution of the model is globally asymptotically stable; and if $R_0 > 1$, then the disease is persistent.

Keywords Incubation period, the basic reproduction number, periodic solution, distributed latency, uniform persistence.

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1. Introduction

Malaria is a serious infectious disease with a long history of development. It is widely believed in biology that malaria originates in Africa from a parasite commonly known as Plasmodium, which was first found in chimpanzees. Malaria is spread widely among the population through female adult mosquitoes and poses a great threat to human health. According to the latest WHO malaria report, there are more than 200 million people suffering from malaria in the world. The data shows that 90% of malaria cases occur in African countries. In addition, India is also the main country of malaria infection. According to statistics, the number of cases in malaria-prone countries increased by nearly 3.6 million in 2018, with 40% of the deaths due to the illness. In China, malaria often occurs in Sichuan, Yunnan and

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Guangxi. Due to the different sources of infection, temperature, humidity and so on, it is very difficult to control and eradicate malaria. Based on the latest news, there are about 2600 cases of malaria in China, and the proportion of deaths has increased. This shows that malaria still has a huge hidden danger to human life and there is a long way to go to achieve the goal of malaria-free.

Mathematical model is a basic and effective tool for studying the mechanism of disease transmission. A reasonable mathematical model reflects the law of disease development and predict its changing trend. It can provide good suggestions and guides for people to prevent,control and eradicate disease. Therefore, many scientists begin to study the dynamics of disease transmission through mathematical model.

The first mathematical model for malaria transmission was introduced by Ross [37] in 1911. Ross proposed a system of ordinary differential equations which studys the malaria transmission between humans and mosquitoes. It proved that the prevalence of diseases would be controlled when the number of mosquitoes was less than the threshold value. Subsequent contributions have been made by Macdonald [32, 33] to the generalization of the classical Ross-Macdonald model, that is,

$$\begin{cases} \frac{dh(t)}{dt} = ab\frac{H-h(t)}{H}v(t) - rh(t), \\ \frac{dv(t)}{dt} = ac\frac{h(t)}{H}(V - v(t)) - dv(t). \end{cases}$$

Here H and V are the total populations of humans and mosquitoes, respectively. $h(t)$ and $v(t)$ are the numbers of infected humans and mosquitoes at time t, a is the rate of biting on humans by a single mosquito, b and c are the transmission probabilities from infected mosquitoes to susceptible humans and from infected humans to susceptible mosquitoes, respectively, $\frac{1}{r}$ is the duration of the disease in humans and d is the mortality rate of mosquitoes.

Macdonald obtained several interesting conclusions through researching the Ross-Macdonald model. Firstly, the result states that malaria can persist in a population only if the number of mosquitoes is greater than a given threshold. Secondly, the prevalence of infection in the human and the mosquito hosts depends directly on the basic reproduction number and the relationship is nonlinear. Thirdly, the model has a stable positive equilibrium when the basic reproduction number is greater than 1. This means that temporary intervention can lead to a temporary reduction of prevalence, when the intervention is relaxed prevalence again increased to the original values. Moreover, Macdonald performed a sensitivity analysis of the basic reproduction number. He found that halving the mosquito population reduces R_0 by a factor of two, meanwhile halving biting rate reduces R_0 by a factor of four. The largest reduction of R_0 is expected for increase in adult mosquito mortality. The work of Macdonald had a very beneficial impact on the collection, analysis, and interpretation of epidemic data on malaria infection and guided the enormous global malaria-eradication campaign of his era.

However, the Ross-Macdonald model ignores many ecological and epidemiological factors, such as the age structure, acquired immunity in humans, spatial heterogeneity, temperature, climate and latency and so on. But these factors play a great influence on the dynamics of malaria transmission. So a number of researchers begin to focus on this important aspect by including these factors. For example, Forouzannia and Gumel [14] take into account the age structure. An important conclusion is that the disease-free equilibrium is locally asymptotically stable if $R_0 < 1$, and the disease-free equilibrium in case of neglecting the disease-induced mortality rate is globally asymptotically stable if $R_0 < 1$. Dietz et al. [12], Bailey [6], Aron [3] and Aron and May [4] presented many malaria models with acquired immunity in humans. In order to consider spatially heterogeneous environments, Gao and Ruan [16] provided malaria models with spatial effects and investigated the spatial spread of malaria between humans and mosquitoes. In 2010s, Lou and Zhao [29] proposed a climate-based malaria transmission model with structured vector population. They incorporated the seasonality into Ross-Macdonald model. Due to the effects of latencies of parasites in humans and mosquitoes populations, Ruan et al. [38] established the delayed Ross-Macdonald model for malaria transmission. In the paper, the sensibility of the basic reproduction number was analyzed by authors. They concluded that the basic reproduction number is a decreasing function of both time delays. Lou and Zhao [30] derived a reaction-diffusion epidemic model with time-delay and non-locality and investigated the threshold dynamics of the epidemic model by means of the basic reproduction number R_0. Bai et al. [5] considered a time-periodic model with seasonality and incubation period. It shows that the symbol of $R_0 - 1$ determines whether malaria is dead or not. Numerical simulations indicate that prolonging the extrinsic incubation period may be helpful in the disease control. In addition, there have been other papers studying epidemic models, see [2, 13, 17, 22, 31, 42, 47, 48] and the references therein.

In the classical Ross-Macdonald model, mosquitoes bite preferences is not taken into account. But malaria patients are more attractive to mosquitoes in real life. In the year of 1987, Kingsolver [24] firstly proposed a vector-bias malaria model, which is extended from the Ross-Macdonald model, and accounted for the greater attractiveness of infectious humans to mosquitoes. Following Kingsolver's work, Hosack et al.(2008) introduced an extrinsic incubation time in mosquitoes to study the transmission of the disease. Chamchod and Britton [10] redefined the attractiveness and interpreted the spread of malaria in term of a reproduction number. Then, Buonomo and Vargas-De-León [8] expanded the modeling in [10]. The classical threshold for the basic reproductive number, R_0, is obtained. the occurrence of a backward bifurcation at $R_0 = 1$ is shown possible. Besides, great progress has been made in vector-bias modeling in malaria, see, e.g., [23, 25, 30, 43, 45].

As we all known, it is common that the length of the latent period is different for various diseases. For example, the latent period for dengue is about 13 days. The viral hepatitis type A exposures after 45 days. And the latent period of AIDS has nearly several years. Smilarly, for the same disease, the incubation period of the disease is different for different individuals due to the physical differences among individuals. It is longer in some people and shorter in others. Based on this case, the dynamics of a model with discrete time delay is not longer convincing, and more scholars use distributed delay to describe the changes of the density of the infected population.

In 2001s, a SIR model with distributed delay was first introduced by Beretta et al. in [7]. The authors studied the global attractivity of the disease-free state E_0 and the endemic state E_+. Subsequent McCluskey [36] proved the complete global stability for a SIR epidemic model with delay-distributed or discrete by means of Lyapunov function. The result shows that if $R_0 > 1$, the endemic equilibrium is globally asymptotically stable whatever distributed delay or discrete delay. Given the effect of distributed latency, Zhao et al. [46] analyzed a time-periodic and two-group reaction-diffusion epidemic model with distributed delay. The threshold dynamics of this model is investigated by deriving the basic reproduction number R_0. The other papers related to distributed delay are in [9, 21, 26, 41].

In this paper, since different individuals have different immune cycles, it is assumed that malaria patients can obtain permanent immunity after recovery in order to reduce the complexity. The rest of the paper is organized as follows. In the next section, we formulate a periodic vector-bias malaria model with diffusion and distributed delay. And the well-posedness of the solution for the system is studyed. In section 3 and 4, we introduce the basic reproduction number R_0 for the model via the next generation operator method and establish the threshold dynamics of the transmission model in term of R_0. Namely, the disease eventually die out if $R_0 < 1$, while the disease is persistent if $R_0 > 1$.

2. Model formulation

Assume that the habitate $\Omega \in \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$. We divide each population into third epidemiological classes: susceptible, latent and infectious. Denote the densities of three classes at time t and location x by $S_i(t,x)$, $L_i(t,x)$ and $I_i(t,x)$ respectively, where $i = 1, 2$ and $(t,x) \in \mathbb{R}^+ \times \bar{\Omega}$. The symbol with the subscript 1(2) indicates the distribution density of the human(mosquitoes) population in each class. Meanwhile, we assume that the total number of humans stabilizes at $H(x)$.

To incorporate a vector-bias term into the model, we introduce the parameters p and l to describe the probabilities that a mosquito arrives at a human at random and picks the human if he is infectious and susceptible, respectively. Since infectious humans are more attractive to mosquitoes, we suppose that $p \geq l \geq 0$. The biting rate $\beta(t,x)$ of mosquitoes is the number of bites per mosquito per unit time at time t and location x. We suppose that the total number of bites made by mosquitoes is same as the number of bites received by humans. We also suppose that a mosquito will not bite the same person more than once. Then $\beta(t,x)I_2(t,x)$ indicates the number of humans who are bittten by infectious mosquitoes per unit time at time t and location x. Assume the infected people can obtain permanent immunity after recovery. Thus, only those originally susceptible may give rise to increase of infectious humans I_1. Hence, we derive that the probability of a mosquito picking a susceptible human equals $\frac{lS_1(t,x)}{pI_1(t,x)+l[H(x)-I_1(t,x)]}$. Similarly, $\frac{pI_1(t,x)}{pI_1(t,x)+l[H(x)-I_1(t,x)]}$ is the probability of a mosquito picking a infectious human. Then the number of newly infectious humans and newly infectious mosquitoes per unit time at time t and location x is respectively

$$c\beta(t,x)\frac{lS_1(t,x)}{pI_1(t,x)+l[H(x)-I_1(t,x)]}I_2(t,x),$$

and

$$b\beta(t,x)\frac{pI_1(t,x)}{pI_1(t,x)+l[H(x)-I_1(t,x)]}S_2(t,x),$$

where $c(b)$ is the transmission probability per bite from an infectious mosquito (human) to a susceptible human (mosquito).

Let $y_i(t,a,x)$ be the density of a population at time $t \geq 0$, infectious age variable $a \geq 0$ and location $x \in \bar{\Omega}$. Then $y_i(t,a,x)(i=1,2)$ satisfy

$$\frac{\partial y_1(t,a,x)}{\partial t} + \frac{\partial y_1(t,a,x)}{\partial a} = D_1 \triangle y_1(t,a,x) - (d_1(t,x)+\rho) y_1(t,a,x),$$

and

$$\frac{\partial y_2(t,a,x)}{\partial t} + \frac{\partial y_2(t,a,x)}{\partial a} = D_2 \triangle y_2(t,a,x) - d_2(t,x) y_2(t,a,x).$$

Here, D_i represents the diffusion rate of the population. $d_i(t,x)$ is the natural death rate of the population. We assume it is ω-periodic in t for every i. $\rho > 0$ means the recovery rate of the humans. For a convenience, we write $M_1(t,x) = d_1(t,x) + \rho$ and $M_2(t,x) = d_2(t,x)$. Then the above formulas can be rewritten as

$$\frac{\partial y_i(t,a,x)}{\partial t} + \frac{\partial y_i(t,a,x)}{\partial a} = D_i \triangle y_i(t,a,x) - M_i(t,x) y_i(t,a,x), \qquad (2.1)$$
$$t,\ a \geq 0,\ x \in \Omega,\ i = 1,2.$$

On account of the different length of incubation periods in humans and mosquitoes, we suppose that the infected individuals have capable of infecting others when their infection age $a > \tau_1 (\tau_1 \in \mathbb{R}_+)$ for human population. And the infected individuals can infect others effectively when $a > \tau_2 (\tau_2 \in \mathbb{R}_+)$ for mosquito population. But, if $a < \tau_i (i=1,2)$, the infected individuals may or may not have an infection ability. Further, we assume that $f_i(r)dr$ represents the probability of infected of individuals between the infection age r and $r+dr$ and $F_i(a) := \int_0^a f_i(r)dr$ denotes the probability of turning into the individuals with infecting others before the infection age a. Then we have

$$L_i(t,x) = \int_0^{\tau_i} (1 - F_i(a)) y_i(t,a,x) da, \quad i = 1,2,$$

and

$$I_i(t,x) = \int_0^{\tau_i} F_i(a) y_i(t,a,x) da + \int_{\tau_i}^{+\infty} y_i(t,a,x) da, \quad i = 1,2.$$

In fact, $f_i(a) \geq 0$ for $a \in [0,\tau_i)$ and $F_i(a) \equiv 1$ for $a \in [\tau_i, +\infty)$ for $i = 1,2$. Differentiating the above formulas with respect to t and using (2.1) respectively, we get

$$\frac{\partial L_i(t,x)}{\partial t} = D_i \triangle L_i(t,x) - M_i(t,x) L_i(t,x) - \int_0^{\tau_i} f_i(a) y_i(t,a,x) da + y_i(t,0,x) \qquad (2.2)$$

and

$$\frac{\partial I_i(t,x)}{\partial t} = D_i \triangle I_i(t,x) - M_i(t,x) I_i(t,x) + \int_0^{\tau_i} f_i(a) y_i(t,a,x) da - y_i(t,+\infty,x), \qquad (2.3)$$

where $i = 1,2$. Biologically, we assume that $y_i(t,+\infty,x) = 0\,(i=1,2)$. As the new infection individuals come from the contact of the infectious and susceptible individuals, it follows that

$$y_1(t,0,x) = c\beta(t,x) \frac{l S_1(t,x)}{p I_1(t,x) + l[H(x) - I_1(t,x)]} I_2(t,x) = \frac{cl\beta(t,x) S_1(t,x) I_2(t,x)}{(p-l) I_1(t,x) + l H(x)},$$

and

$$y_2(t,0,x) = b\beta(t,x)\frac{pI_1(t,x)}{pI_1(t,x)+l[H(x)-I_1(t,x)]}S_2(t,x) = \frac{bp\beta(t,x)S_2(t,x)I_1(t,x)}{(p-l)I_1(t,x)+lH(x)}.$$

Now we derive function $y_i(t,a,x)$ by the method of characteristics. For any $\xi \geq 0$, consider the solution of (2.1) along the characteristic line $t = a+\xi$ by letting $v_i(\xi,a,x) = y_i(a+\xi,a,x)(i=1,2)$. Then for $a \in [0,\tau_i)$, we have

$$\begin{cases} \dfrac{\partial v_i(\xi,a,x)}{\partial a} = \left[\dfrac{\partial y_i(t,a,x)}{\partial t} + \dfrac{\partial y_i(t,a,x)}{\partial a}\right]_{t=a+\xi} \\ \qquad\qquad\quad = D_i \triangle y_i(a+\xi,a,x) - M_i(t,x)y_i(a+\xi,a,x) \\ \qquad\qquad\quad = D_i \triangle v_i(\xi,a,x) - M_i(t,x)v_i(\xi,a,x), \quad x \in \Omega,\ i=1,2, \\ v_i(\xi,0,x) = y_i(\xi,0,x), \quad x \in \Omega,\ i=1,2, \\ \frac{\partial v_i(\xi,a,x)}{\partial n} = 0, \quad x \in \partial\Omega,\ i=1,2. \end{cases}$$

Here, n is the outward unit normal vector on $\partial\Omega$. By [15], we obtain the solution of the above formula, that is,

$$v_i(\xi,a,x) = \int_\Omega \Gamma_i(\xi+a,\xi,x,y)y_i(\xi,0,y)dy, \tag{2.4}$$

where $\Gamma_i(t,s,x,y)$ with $t \geq s$ and $x,y \in \Omega$ is the fundamental solution of the operator $\partial_t - D_i\triangle + M_i(t,\cdot)$ associated with the Neumann boundary condition for $i=1,2$. Note that $\Gamma_i(t,s,x,y) = \Gamma_i(t+\omega,s+\omega,x,y)$ for all $t > s \geq 0$ and $x,y \in \Omega$. Since $y_i(t,a,x) = v_i(t-a,a,x)(t \geq a \geq 0)$, it follows that

$$y_1(t,a,x) = \int_\Omega \Gamma_1(t,t-a,x,y)\frac{cl\beta(t-a,y)S_1(t-a,y)I_2(t-a,y)}{(p-l)I_1(t-a,y)+lH(y)}dy,$$

and

$$y_2(t,a,x) = \int_\Omega \Gamma_2(t,t-a,x,y)\frac{bp\beta(t-a,y)S_2(t-a,y)I_1(t-a,y)}{(p-l)I_1(t-a,y)+lH(y)}dy.$$

Substituting $y_1(t,a,x)$ and $y_2(t,a,x)$ into (2.2) and (2.3) respectively, and dropping

$L_i(t,x)$, we get

$$\begin{cases}\dfrac{\partial u_{S_1}(t,x)}{\partial t} = D_1\triangle u_{S_1}(t,x) - d_1(t,x)u_{S_1}(t,x) + \mu_1(t,x) \\ \qquad\qquad - \dfrac{cl\beta(t,x)u_{S_1}(t,x)u_2(t,x)}{(p-l)u_1(t,x)+lH(x)}, \qquad t>0, x\in\Omega, \\ \dfrac{\partial u_{S_2}(t,x)}{\partial t} = D_2\triangle u_{S_2}(t,x) - d_2(t,x)u_{S_2}(t,x) + \mu_2(t,x) \\ \qquad\qquad - \dfrac{bp\beta(t,x)u_{S_2}(t,x)u_1(t,x)}{(p-l)u_1(t,x)+lH(x)}, \qquad t>0, x\in\Omega, \\ \dfrac{\partial u_1(t,x)}{\partial t} = D_1\triangle u_1(t,x) - (d_1(t,x)+\rho)u_1(t,x) + \displaystyle\int_0^{\tau_1} f_1(a)\int_\Omega \Gamma_1(t,t-a,x,y) \\ \qquad\qquad \times \dfrac{cl\beta(t-a,y)u_{S_1}(t-a,y)u_2(t-a,y)}{(p-l)u_1(t-a,y)+lH(y)}dyda, \qquad t>0, x\in\Omega, \\ \dfrac{\partial u_2(t,x)}{\partial t} = D_2\triangle u_2(t,x) - d_2(t,x)u_2(t,x) + \displaystyle\int_0^{\tau_2} f_2(a)\int_\Omega \Gamma_2(t,t-a,x,y) \\ \qquad\qquad \times \dfrac{bp\beta(t-a,y)u_{S_2}(t-a,y)u_1(t-a,y)}{(p-l)u_1(t-a,y)+lH(y)}dyda, \qquad t>0, x\in\Omega, \\ \frac{\partial u_{S_i}(t,x)}{\partial n} = \frac{\partial u_i(t,x)}{\partial n} = 0, \qquad t>0, x\in\partial\Omega, i=1,2,\end{cases} \tag{2.5}$$

where $(u_{S_1}, u_{S_2}, u_1, u_2) = (S_1, S_2, I_1, I_2)$. $\mu_i(t,x)(i=1,2)$ is the recruitment rate. Assume $d_i(t,x)$, $\mu_i(t,x)$ and $\beta(t,x)$ are Hölder continuous and nonnegative nontrivial on $\mathbb{R}\times\bar{\Omega}$, and ω-periodic in t. Further, assume all of constant parameters are positive.

Set $\tau = \max\{\tau_1,\tau_2\}$. Define some Banach spaces and corresponding norms as follows

$$X := C(\bar{\Omega},\mathbb{R}^4), \qquad \|\varphi\|_X = \max_{x\in\bar{\Omega}} |\varphi(x)|, \forall\varphi\in X,$$
$$C := C([-\tau,0],X), \qquad \|\varphi\|_C = \max_{-\tau\le\theta\le 0}\|\varphi(\theta)\|_X, \forall\varphi\in C.$$

Let $X^+ := C(\bar{\Omega},\mathbb{R}^4_+)$ and $C^+ := C([-\tau,0],X^+)$, then (X,X^+) and (C,C^+) are strongly ordered spaces. Given a function $u:[-\tau,\sigma]\to X$ for $\sigma>0$, we define $u_t\in C$ by

$$u_t(\theta) = u(t+\theta), \quad \theta\in[-\tau,0].$$

Setting $Y := C(\bar{\Omega},\mathbb{R})$ and $Y^+ := C(\bar{\Omega},\mathbb{R}_+)$, we consider the following system:

$$\begin{cases}\frac{\partial w_i(t,x)}{\partial t} = D_i\triangle w_i(t,x) - d_i(t,x)w_i(t,x), \quad t>0, x\in\Omega, i=1,2, \\ \frac{\partial w_i(t,x)}{\partial n} = 0, \quad t>0, x\in\partial\Omega, i=1,2,\end{cases}$$

where $D_i>0\,(i=1,2)$ and $d_i(t,x)\,(i=1,2)$ is Hölder continuous and nonnegative nontrivial on $\mathbb{R}\times\bar{\Omega}$, and ω-periodic in t. It follows from [18, Chapter 2] that the system has an evolution operator $V_i(t,s): Y^+\to Y^+$ satisfying $V_i(t+\omega,s+\omega) = V_i(t,s)$ for $i=1,2$. Similarly, the equation

$$\begin{cases}\frac{\partial \bar{w}(t,x)}{\partial t} = D_1\triangle \bar{w}(t,x) - (d_1(t,x)+\rho)\bar{w}(t,x), \quad t>0, x\in\Omega, \\ \frac{\partial \bar{w}(t,x)}{\partial n} = 0, \quad t>0, x\in\partial\Omega,\end{cases}$$

has also an evolution operator $V_3(t,s): Y^+\to Y^+$ and $V_3(t,s)$ is $\omega-periodic$. Moreover, $V_i(t,s)(i=1,2,3)$ is compact and strongly positive for $t\ge s$. By [11, Theorem 6.6] with $\alpha = 0$, we admit that there exist constants $M \geq 1$ and $c \in \mathbb{R}$ such that

$$\| V_i(t,s) \| \leq Me^{-c(t-s)}, \quad \forall t \geq s,\ t,s \in \mathbb{R}, i = 1,2,3.$$

Define $F = (F_{S_1}, F_{S_2}, F_1, F_2)^T : [0,+\infty) \times C^+ \to X$ by

$$\begin{aligned}
F_{S_1}(t,\phi) &:= \mu_1(t,\cdot) - \frac{cl\beta(t,\cdot)\phi_{S_1}(0,\cdot)\phi_2(0,\cdot)}{(p-l)\phi_1(0,\cdot)+lH(\cdot)},\\
F_{S_2}(t,\phi) &:= \mu_2(t,\cdot) - \frac{bp\beta(t,\cdot)\phi_{S_2}(0,\cdot)\phi_1(0,\cdot)}{(p-l)\phi_1(0,\cdot)+lH(\cdot)},\\
F_1(t,\phi) &:= \int_0^{\tau_1}\int_\Omega \Gamma_1(t,t-a,x,y)\frac{cl\beta(t-a,y)\phi_{S_1}(t-a,y)\phi_2(t-a,y)}{(p-l)\phi_1(t-a,y)+lH(y)}dyda,\\
F_2(t,\phi) &:= \int_0^{\tau_2}\int_\Omega \Gamma_2(t,t-a,x,y)\frac{bp\beta(t-a,y)\phi_{S_2}(t-a,y)\phi_1(t-a,y)}{(p-l)\phi_1(t-a,y)+lH(y)}dyda.
\end{aligned}$$

Let $A(t) = \mathrm{diag}(\mathrm{A_1(t), A_2(t), A_3(t), A_2(t)})$. $A_i(t)\,(i = 1,2)$ is defined by

$$\begin{cases} D(A_i) = \{\varphi \in C^2(\bar{\Omega}) : \frac{\partial \varphi(x)}{\partial n} = 0, x \in \partial\Omega\},\\ A_i\varphi = D_i\triangle\varphi - d_i(t,x)\varphi,\ \ \forall\varphi \in D(A_i), i = 1,2, \end{cases}$$

and $A_3(t)$ is defined by

$$\begin{cases} D(A_3(t)) = \{\varphi \in C^2(\bar{\Omega}) : \frac{\partial \varphi(x)}{\partial n} = 0, x \in \partial\Omega\},\\ A_3(t)\varphi = D_1\triangle\varphi - (d_1(t,x)+\rho)\varphi,\ \ \forall\varphi \in D(A_3(t)). \end{cases}$$

Then (2.5) can be rewritten as

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = A(t)u(t,x) + F(t,u_t), & t > 0, x \in \Omega,\\ u(\theta,x) = \phi(\theta,x), & \theta \in [-\tau,0], x \in \Omega. \end{cases} \tag{2.6}$$

Let $V(t,s) = \mathrm{diag}(\mathrm{V_1(t,s), V_2(t,s), V_3(t,s), V_2(t,s)})$ and $V(t,s)$ be an evolution operator with compactness and strongly positive. Then (2.6) can be rewritten as the following integral equation

$$u(t,\phi) = V(t,0)\phi(0) + \int_0^t V(t,0)F(s,u_s)ds, \quad t \geq 0, \phi \in C^+. \tag{2.7}$$

A solution of (2.7) is called a mild solution of (2.6).

Lemma 2.1 ([46], Lemma 3.1)**.** *For every initial value function $\varphi \in C^+$, system (2.5) has a unique mild solution $z(t,\cdot,\varphi) \in X^+$ on $[0,+\infty)$. Furthermore, system (2.5) generates a ω-periodic semiflow $Q_t(\cdot) := z_t(\cdot) : C^+ \to C^+$, namely,*

$$Q_t(\varphi)(\theta,x) = z_t(\varphi)(\theta,x) = z(t+\theta,x,\varphi), \qquad \forall t \geq 0,$$

and $Q := Q_\omega : C^+ \to C^+$ has a global attractor in C^+.

Define

$$\begin{aligned} C_1 &:= C([-\tau_1,0],Y), & C_1^+ &:= C([-\tau_1,0],Y^+),\\ C_2 &:= C([-\tau_2,0],Y), & C_2^+ &:= C([-\tau_2,0],Y^+), \end{aligned}$$

$$\tilde{C} := C_1 \times C_2 \times C_2 \times C_1, \qquad \tilde{C}^+ := C_1^+ \times C_2^+ \times C_2^+ \times C_1^+.$$

For any given $\varphi = (\varphi_{S_1}, \varphi_{S_2}, \varphi_1, \varphi_2) \in \tilde{C}^+$, we define $\hat{\varphi} = (\hat{\varphi}_{S_1}, \hat{\varphi}_{S_2}, \hat{\varphi}_1, \hat{\varphi}_2)$ by

$$\hat{\varphi}_{S_1}(\theta) = \begin{cases} \varphi_{S_1}(-\tau_1), & \theta \in [-\tau, -\tau_1], \\ \varphi_{S_1}(\theta), & \theta \in [-\tau_1, 0], \end{cases}$$

$$\hat{\varphi}_{S_2}(\theta) = \begin{cases} \varphi_{S_2}(-\tau_2), & \theta \in [-\tau, -\tau_2], \\ \varphi_{S_2}(\theta), & \theta \in [-\tau_2, 0], \end{cases}$$

$$\hat{\varphi}_1(\theta) = \begin{cases} \varphi_1(-\tau_2), & \theta \in [-\tau, -\tau_2], \\ \varphi_1(\theta), & \theta \in [-\tau_2, 0], \end{cases}$$

$$\hat{\varphi}_2(\theta) = \begin{cases} \varphi_2(-\tau_1), & \theta \in [-\tau, -\tau_1], \\ \varphi_2(\theta), & \theta \in [-\tau_1, 0]. \end{cases}$$

It is clearly that $\hat{\varphi} \in C^+$. By the uniqueness of solution, we get $u(t, x, \varphi) = z(t, x, \hat{\varphi})$ for all $t \in [0, +\infty)$. From Lemma 2.1 we admit system (2.5) has a unique solution $u(t, x, \varphi)$ on $[0, +\infty)$.

In the next result, the existence of a solution of (2.5) in a smaller space $\tilde{C}^+$ is stated. We prove that the Poincaré map related to (2.5) admits a global attractor in $\tilde{C}^+$.

Lemma 2.2. *For any $\varphi \in \tilde{C}^+$, system (2.5) has a unique solution, denoted by $u(t, \cdot, \varphi)$, on interval $[0, +\infty)$ with $u_0 = \varphi$. In addition, (2.5) generates a ω-periodic semiflow $\tilde{Q}_t(\cdot) := u_t(\cdot) : \tilde{C}^+ \to \tilde{C}^+$, i.e., $\tilde{Q}_t(\varphi) = u_t(\varphi)$ for $t \geq 0$ and $\tilde{Q} := \tilde{Q}_\omega$ has a global attractor in $\tilde{C}^+$.*

Proof. Clearly, $0 \leq u_{S_1}(t, x), u_1(t, x) \leq H(x), \forall t \in [0, +\infty)$.

Solve the next time-periodic system:

$$\begin{cases} \frac{\partial v(t,x)}{\partial t} = D_2 \triangle v(t, x) - d_2(t, x) v(t, x) + \mu_2(t, x), & \forall t > 0, x \in \Omega, \\ \frac{\partial v(t,x)}{\partial n} = 0, & \forall t > 0, x \in \partial\Omega. \end{cases} \tag{2.8}$$

It follows from [47, Lemma 2.1] that this system admits a unique positive ω-periodic solution $m^*(t, x)$ which is globally attractive in Y^+. Since the u_{S_2} equation in (2.5) is dominated by (2.8), by the comparison principle, there exists a time $t_1 = t(\varphi) > 0$ and a positive constant C_1 such that for any $\varphi \in \tilde{C}^+$, $u_{S_2}(t, x, \varphi) \leq m^*(t, x) \leq C_1$ when $t \geq t_1$.

Similarly,

$$\begin{cases} \dfrac{\partial u_2(t,x)}{\partial t} \leq D_2 \triangle u_2(t, x) - \tilde{d}_2 u_2(t, x) \\ \qquad\qquad + C_1 b \bar{\beta} \displaystyle\int_0^{\tau_2} f_2(a) \int_\Omega \Gamma_2(t, t-a, x, y) dy da, & t > t_1, x \in \Omega, \\ \frac{\partial u_2(t,x)}{\partial n} = 0, & t > t_1, x \in \partial\Omega, \end{cases}$$

where $\tilde{d}_2 = \min_{t \in [0,\omega], x \in \bar{\Omega}} d_2(t, x)$ and $\bar{\beta} = \max_{t \in [0,\omega], x \in \bar{\Omega}} \beta(t, x)$. There exists a time $t_2(\varphi) > t_1$ and a positive constant C_2 such that $u_2(t, x, \varphi) \leq C_2$ for $t \geq t_2(\varphi)$ and $\varphi \in \tilde{C}^+$.

Define $\tilde{Q}_t(\cdot) : \tilde{C}^+ \to \tilde{C}^+$ by $\tilde{Q}_t(\varphi) = u_t(\varphi), \forall \varphi \in \tilde{C}^+$. The proof of [47, Lemma 2.1] implies that $\{\tilde{Q}_t\}_{t\geq 0}$ is an ω-periodic semiflow on $\tilde{C}^+$. From the above analysis, we know that $\tilde{Q}_t$ is point disspative. By [44, Theorem 2.1.8], we have $\tilde{Q} := \tilde{Q}_\omega$ is compact. Following from [34, Theorem 2.9], one has that $\tilde{Q}$ has a global attractor in $\tilde{C}^+$. □

3. The basic reproduction number

In this section, the critical works are made. Now, we derive the basic reproduction number of the system (2.5) by the next generation operators method.

Let

$$\begin{aligned}
&C_\omega(\mathbb{R}\times\bar{\Omega},\mathbb{R}) := \{\phi \in C(\mathbb{R}\times\bar{\Omega},\mathbb{R}) : \quad \phi(t) = \phi(t+\omega),\ \forall t\in\mathbb{R}\},\\
&C_\omega^+ := \{\phi \in C_\omega : \quad \phi(t,x) \geq 0,\ \forall t\in\mathbb{R}, x\in\bar{\Omega}\},\\
&\mathbb{C}_\omega(\mathbb{R}\times\bar{\Omega},\mathbb{R}\times\mathbb{R}) := C_\omega(\mathbb{R}\times\bar{\Omega},\mathbb{R})\times C_\omega(\mathbb{R}\times\bar{\Omega},\mathbb{R}),\\
&\mathbb{C}_\omega^+ := \{\phi = (\phi_1,\phi_2) \in \mathbb{C}_\omega : \quad \phi_i(t,x) \geq 0,\ \forall t\in\mathbb{R}, x\in\bar{\Omega}, i=1,2\}.
\end{aligned}$$

Taking $u_1 = u_2 = 0$, we obtain the next equations for the density of susceptible population

$$\begin{cases}
\frac{\partial u_{S_i}(t,x)}{\partial t} = D_i \triangle u_{S_i}(t,x) - d_i(t,x)u_{S_i}(t,x) + \mu_i(t,x), \quad t>0, x\in\Omega, i=1,2,\\
\frac{\partial u_{S_i}(t,x)}{\partial n} = 0, \quad t>0, x\in\partial\Omega, i=1,2.
\end{cases}$$

It follows from [47, Lemma 2.1] that the top equations have a positive solution $u_{S_i}^*(t,x)(i=1,2)$, which is ω-periodic in t and globally asymptotically stable. Linearizing system (2.5) at $(u_{S_1}^*, u_{S_2}^*, 0, 0)$, we get the following system

$$\begin{cases}
\dfrac{\partial u_1(t,x)}{\partial t} = D_1\triangle u_1(t,x) - (d_1(t,x)+\rho)u_1(t,x) + \displaystyle\int_0^{\tau_1} f_1(a)\int_\Omega \Gamma_1(t,t-a,x,y)\\
\qquad\qquad \times \dfrac{c\beta(t-a,y)u_{S_1}^*(t-a,y)}{H(y)} u_2(t-a,y)dyda, \quad t>0, x\in\Omega,\\
\dfrac{\partial u_2(t,x)}{\partial t} = D_2\triangle u_2(t,x) - d_2(t,x)u_2(t,x) + \displaystyle\int_0^{\tau_2} f_2(a)\int_\Omega \Gamma_2(t,t-a,x,y)\\
\qquad\qquad \times \dfrac{bp\beta(t-a,y)u_{S_2}^*(t-a,y)}{lH(y)} u_1(t-a,y)dyda, \quad t>0, x\in\Omega,\\
\frac{\partial u_1(t,x)}{\partial n} = \frac{\partial u_2(t,x)}{\partial n} = 0, \quad t>0, x\in\partial\Omega.
\end{cases} \tag{3.1}$$

Define $F(t) : \mathbb{C}_\omega(\mathbb{R}\times\bar{\Omega},\mathbb{R}\times\mathbb{R}) \to \mathbb{C}_\omega(\mathbb{R}\times\bar{\Omega},\mathbb{R}\times\mathbb{R})$ by

$$F(t)(\phi_1,\phi_2)^T = \begin{pmatrix}
\int_0^{\tau_1} f_1(a)\int_\Omega \Gamma_1(t,t-a,x,y)\frac{c\beta(t-a,y)u_{S_1}^*(t-a,y)}{H(y)}\phi_2(-a,y)dyda\\
\int_0^{\tau_2} f_2(a)\int_\Omega \Gamma_2(t,t-a,x,y)\frac{bp\beta(t-a,y)u_{S_2}^*(t-a,y)}{lH(y)}\phi_1(-a,y)dyda
\end{pmatrix}.$$

Set $\tilde{V}(t,s) = \mathrm{diag}(\mathrm{V_3(t,s), V_2(t,s)})$. Suppose that $v(s,x) \in \mathbb{C}_\omega(\mathbb{R}\times\bar{\Omega},\mathbb{R}\times\mathbb{R})$ is the initial distribution of infectious individuals at time $s\in\mathbb{R}$ and location $x\in\bar{\Omega}$. For every given $s\geq 0$, $F(t-s)v(t-s+\cdot,x)$ represents the density of distribution of newly infected individuals at time $t-s(s<t)$ and location x. Then $\tilde{V}(t,t-s)F(t-s)v(t-s+\cdot,x)$ denotes the density distribution at location x of those infected individuals

who were newly infected at time $t-s$ and remained survive in the environment at time t for $t \geq s$. After that, the term

$$\int_0^{+\infty} \tilde{V}(t,t-s)F(t-s)v(t-s+\cdot,x)ds$$

states the distribution of the accumulative infective individuals of the two group at location x and time t for all previous time $t-s(s<t)$ when the time evolved from the previous time $t-s$ to t.

Now we define two linear operators by

$$[Lv](t) := \int_0^{+\infty} \tilde{V}(t,t-s)F(t-s)v(t-s+\cdot,\cdot)ds, \qquad \forall t \in \mathbb{R}, v \in \mathbb{C}_\omega,$$

and

$$[\hat{L}v](t) := F(t)\left(\int_0^{+\infty} \tilde{V}(t+\cdot,t-s+\cdot)v(t-s+\cdot,\cdot)ds\right), \qquad \forall t \in \mathbb{R}, v \in \mathbb{C}_\omega.$$

Set

$$[Av](t) := \int_0^{+\infty} \tilde{V}(t,t-s)v(t-s,\cdot)ds, \qquad \forall t \in \mathbb{R}, v \in \mathbb{C}_\omega,$$
$$[Bv](t) := F(t)v(t+\cdot,\cdot)ds, \qquad \forall t \in \mathbb{R}, v \in \mathbb{C}_\omega.$$

Then $L = A \circ B$ and $\hat{L} = B \circ A$. As a consequence, L and $\hat{L}$ have the same spectral radius, that is, $r(L) = r(\hat{L})$. Next, we use the spectral radius of L to depict the basic reproduction number for (2.5), namely,

$$R_0 := r(L) = r(\hat{L}).$$

As the previous result, there exists constants $M > 1$ and $c_i \in \mathbb{R}$ such that

$$\| V_i(t,s) \| \leq Me^{c_i(t-s)}, \qquad \forall t \geq s,\ t,s \in \mathbb{R},\ i = 2,3.$$

One has that $c_i^* := \omega(V_i) \leq c_i$, where

$$\omega(V_i) = \inf\{\omega \mid \exists M \geq 1 \ s.t. \ \| V_i(t,s) \| \leq Me^{\omega t}, \quad \forall t \geq s,\ t,s \in \mathbb{R}\}.$$

Let $r(V_i(\omega,0))$ be the spectral radius of the operator $V_i(\omega,0)$. Due to the compactness and strong positive of $V_i(t,s)$ on Y^+, it follows from the Krein-Rutman theorem ([18, Theorem 7.2]) that $r(V_i(\omega,0)) > 0\,(i = 2,3)$. By [18, Lemma 14.2], we further have $r(V_i(\omega,0)) < 1$. Hence, the conclusion that $c_i^* < 0\,(i = 2,3)$ is obtained by [19, Proposition A.2].

Define $\hat{P} : C([-\tau,0],Y \times Y) \to C([-\tau,0],Y \times Y)$ by $\hat{P}(\phi) = v_\omega(\phi)$, where

$$v_t(\phi)(\theta,x) = v(t+\theta,x,\phi) = (v_1(t+\theta,x,\phi), v_2(t+\theta,x,\phi)),$$
$$\forall t \geq 0, \forall(\theta,x) \in [-\tau,0] \times \bar{\Omega}.$$

Refering to [46, Theorem 3.5] and [27, Theorem 3.7], we get

Lemma 3.1. *$R_0 - 1$ has the same sign as $r(\hat{P}) - 1$.*

Let $P : C_2 \times C_1 \to C_2 \times C_1$ and $P(\varphi) = \bar{v}_\omega(\varphi)$. Here

$$\bar{v}_t(\varphi)(\theta, x) = (\bar{v}_1(t + \theta_1, x, \varphi), \bar{v}_2(t + \theta_2, x, \varphi)),$$
$$\forall t \geq 0,\ x \in \bar{\Omega},\ \theta = (\theta_1, \theta_2) \in [-\tau_2, 0] \times [-\tau_1, 0].$$

It follows from [39, Section 5.3] that $\bar{v}(t, x, \varphi) \gg 0$ for any $t > 2\tau$, $x \in \bar{\Omega}$ and $\varphi \in C_2^+ \times C_1^+ \setminus \{0\}$. Moreover, $\bar{v}_t$ is compact for all $t > 2\tau$. Hence, $P^n(n\omega > 2\tau)$ is compact and strongly positive. By using [28, Lemma 3.1], we admit that the spectral radius $r(P)$ is a unique simple eigenvalue of P having a strongly positive eigenvector.

Lemma 3.2. *Let $\mu = \frac{\ln r(P)}{\omega}$. Then there exists a positive ω-periodic function $v^*(t, x)$ such that $e^{\mu t} v^*(t, x)$ is a solution of system* (3.1)*.*

Proof. Denote $\bar{\varphi}$ is the eigenvector corresponding to $r(P)$. Then $\bar{\varphi} \in \mathrm{int}(\mathrm{C}_2^+ \times \mathrm{C}_1^+)$. Let $\bar{v}(t, x, \bar{\varphi}) = (\bar{v}_1(t, x, \bar{\varphi}), \bar{v}_2(t, x, \bar{\varphi}))$ be the solution of (3.1) satisfying $\bar{v}_0(\bar{\varphi}) = \bar{\varphi}$. Clearly, $\bar{v}_t(\bar{\varphi}) \gg 0$ for all $t \geq 0$. Define

$$v_1^*(t, x) = e^{-\mu t} \bar{v}_1(t, x, \bar{\varphi}),\ \ t \geq -\tau_2, x \in \bar{\Omega},$$
$$v_2^*(t, x) = e^{-\mu t} \bar{v}_2(t, x, \bar{\varphi}),\ \ t \geq -\tau_1, x \in \bar{\Omega}.$$

Thus $v^*(t, x) = (v_1^*(t, x), v_2^*(t, x)) \gg 0, \forall t \geq -\dot{\tau}\,(\dot{\tau} = \min\{\tau_1, \tau_2\}), x \in \bar{\Omega}$. And $v^*(t, x)$ satisfies the following system with parameter μ:

$$\begin{cases} \dfrac{\partial v_1^*}{\partial t} = D_1 \triangle v_1^*(t, x) - (d_1(t, x) + \rho + \mu) v_1^*(t, x) + \displaystyle\int_0^{\tau_1} f_1(a) \int_\Omega \Gamma_1(t, t-a, x, y) \\ \qquad \times \dfrac{c\beta(t-a, y)}{H(y)} u_{S_1}^*(t-a, y) e^{-\mu a} v_2^*(t-a, y) dy da, \\ \dfrac{\partial v_2^*}{\partial t} = D_2 \triangle v_2^*(t, x) - (d_2(t, x) + \mu) v_2^*(t, x) + \displaystyle\int_0^{\tau_2} f_2(a) \int_\Omega \Gamma_2(t, t-a, x, y) \\ \qquad \times \dfrac{bp\beta(t-a, y)}{lH(y)} u_{S_2}^*(t-a, y) e^{-\mu a} v_1^*(t-a, y) dy da, \\ \frac{\partial v_1^*}{\partial n} = \frac{\partial v_2^*}{\partial n} = 0, \\ v_0^*(\theta, x) = (v_1^*(\theta_1, x), v_2^*(\theta_2, x)) = (e^{-\mu\theta_1} \bar{\varphi}_1(\theta_1, x), e^{-\mu\theta_2} \bar{\varphi}_2(\theta_2, x)). \end{cases} \tag{3.2}$$

Here $v_t^*(\theta, x)$ is defined by

$$\begin{aligned} v_t^*(\theta, x) &= (v_1^*(t + \theta_1, x), v_2^*(t + \theta_2, x)) \\ &= (e^{-\mu(t+\theta_1)} \bar{\varphi}_1(t + \theta_1, x, \bar{\varphi}), e^{-\mu(t+\theta_2)} \bar{\varphi}_2(t + \theta_2, x, \bar{\varphi})), \end{aligned}$$
$$\forall t \geq 0,\ x \in \Omega,\ \theta = (\theta_1, \theta_2) \in [-\tau_2, 0] \times [-\tau_1, 0].$$

Since

$$\begin{aligned} v_1^*(\omega + \theta_1, x) &= e^{-\mu(\omega+\theta_1)} v_1(\omega + \theta_1, x, \bar{\varphi}) = e^{-\mu(\omega+\theta_1)} (P(\bar{\varphi}))_1(\theta_1, x) \\ &= e^{-\mu(\omega+\theta_1)} r(P) \bar{\varphi}_1(\theta_1, x) = e^{-\mu\theta_1} \bar{\varphi}_1(\theta_1, x) \\ &= v_1^*(\theta_1, x), \qquad\qquad \forall \theta_1 \in [-\tau_2, 0], x \in \bar{\Omega}, \\ v_2^*(\omega + \theta_2, x) &= e^{-\mu(\omega+\theta_2)} v_2(\omega + \theta_2, x, \bar{\varphi}) = e^{-\mu(\omega+\theta_2)} (P(\bar{\varphi}))_2(\theta_2, x) \\ &= e^{-\mu(\omega+\theta_2)} r(P) \bar{\varphi}_2(\theta_2, x) = e^{-\mu\theta_2} \bar{\varphi}_2(\theta_2, x) \end{aligned}$$

$$= v_2^*(\theta_2, x), \qquad \forall \theta_2 \in [-\tau_1, 0], x \in \bar{\Omega},$$

we have $v_0^*(\theta, x) = v_\omega^*(\theta, x)$ for all $\theta = (\theta_1, \theta_2) \in [-\tau_2, 0] \times [-\tau_1, 0]$ and $x \in \bar{\Omega}$. So,

$$v_1^*(t, x) = v_1^*(t + \omega, x), \qquad \forall t \geq -\tau_2, x \in \bar{\Omega},$$
$$v_2^*(t, x) = v_2^*(t + \omega, x), \qquad \forall t \geq -\tau_1, x \in \bar{\Omega}.$$

Therefore, $v^*(t, x)$ is an ω-periodic solution of (3.2) and $e^{\mu t} v^*(t, x)$ is also a solution of (3.1). □

Lemma 3.3. $r(\hat{P}) = r(P)$.

Proof. Set $\mathcal{A} := C([-\tau, 0], Y \times Y)$ and $\mathcal{B} := C_2 \times C_1$. Assume $v(t, s, \varphi)$ is a solution of (2.5) on $\mathcal{A}$ and $u(t, s, \phi) = (u_1(t, s, \phi), u_2(t, s, \phi))$ is a solution of (2.5) on $\mathcal{B}$. It follows from the uniqueness of solutions that $v(t, s, \varphi) = u(t, s, \phi)$ with $\varphi = \phi$ for all $t \geq s$.

Define

$$v_t(s, \varphi)(\theta) = v(t + \theta, s, \varphi), \quad v_s(s, \varphi) = \varphi, \qquad \forall t \geq s, \forall \theta \in [-\tau, 0],$$
$$u_t(s, \phi)(\theta) = (u_1(t + \theta_1, s, \phi), u_2(t + \theta_2, s, \phi)), \quad u_s(s, \phi) = \phi,$$
$$\forall t \geq s, \forall \theta = (\theta_1, \theta_2) \in [-\tau_2, 0] \times [-\tau_1, 0].$$

Let $\hat{U}(t, s)$ and $U(t, s)$ are the evolution operators of (2.5) on $\mathcal{A}$ and $\mathcal{B}$ respectively. And they satisfy that

$$\hat{U}(t, s)\varphi = v_t(s, \varphi), \qquad \forall \varphi \in \mathcal{A}, \ \forall t \geq s,$$
$$U(t, s)\phi = u_t(s, \phi), \qquad \forall \phi \in \mathcal{B}, \ \forall t \geq s.$$

Note $\omega(\hat{U})$ and $\omega(U)$ are the exponent growth bound of $\hat{U}(t, s)$ and $U(t, s)$ respectively. Then the next claims are true.

Claim 1: $\omega(U) \leq \omega(\hat{U}) + \delta, \ \ \forall \delta > 0$.

The definition of the exponent growth bound tell us there exist $M_\delta > 1$ such that

$$\| \hat{U}(t + s, s)\varphi \|_{\mathcal{A}} \leq M_\delta e^{(\omega(\hat{U}) + \delta)t} \| \varphi \|_{\mathcal{A}}, \qquad \forall t \geq 0, \forall s \in \mathbb{R}, \forall \varphi \in \mathcal{A}.$$

Then

$$\| v(t + s, s, \varphi) \|_{Y \times Y} \leq M_\delta e^{(\omega(\hat{U}) + \delta)t} \| \varphi \|_{\mathcal{A}}, \qquad \forall t \geq 0, \forall s \in \mathbb{R}, \forall \varphi \in \mathcal{A}.$$

For any given $\phi = (\phi_1, \phi_2) \in \mathcal{B}$, we define $\hat{\phi} = (\hat{\phi}_1, \hat{\phi}_2)$ by

$$\hat{\phi}_1(\theta) = \begin{cases} \phi_1(-\tau_2), & -\tau \leq \theta \leq -\tau_2, \\ \phi_1(\theta_1), & -\tau_2 \leq \theta \leq 0, \end{cases}$$
$$\hat{\phi}_2(\theta) = \begin{cases} \phi_2(-\tau_1), & -\tau \leq \theta \leq -\tau_1, \\ \phi_2(\theta_2), & -\tau_1 \leq \theta \leq 0. \end{cases}$$

It is easy seen that $\hat{\phi} \in \mathcal{A}$. The uniqueness of solutions implies that $u(t, s, \phi) = v(t, s, \hat{\phi})$ for $t \geq s, s \in \mathbb{R}$.

Since $\| \hat{\phi} \|_{\mathcal{A}} = \| \phi \|_{\mathcal{B}}$, we get

$$\begin{aligned} \| u(t+s,s,\phi) \|_{Y\times Y} &= \| v(t+s,s,\hat{\phi}) \|_{Y\times Y} \\ &\leq M_\delta e^{(\omega(\hat{U})+\delta)t} \| \hat{\phi} \|_{\mathcal{A}} \\ &= M_\delta e^{(\omega(\hat{U})+\delta)t} \| \phi \|_{\mathcal{B}}, \qquad \forall t\geq 0, \forall s\in\mathbb{R}. \end{aligned}$$

Hence, there have $\tilde{M}_\delta \geq M_\delta$ such that

$$\| U(t+s,s)\phi \|_{\mathcal{B}} = \| u(t+s,s,\phi) \|_{\mathcal{B}} \leq \tilde{M}_\delta e^{(\omega(\hat{U})+\delta)t} \| \phi \|_{\mathcal{B}}, \\ \forall t\geq 0,\ \forall s\in\mathbb{R},\ \forall \phi\in\mathcal{B}.$$

Therefore,

$$\| U(t+s,s) \| \leq \tilde{M}_\delta e^{(\omega(\hat{U})+\delta)t}, \qquad \forall t\geq 0, s\in\mathbb{R}.$$

By the definition of $\omega(U)$, one has

$$\omega(U) \leq \omega(\hat{U}) + \delta.$$

Claim 2: $\omega(\hat{U}) \leq \omega(U) + \delta, \ \ \forall \delta > 0.$

According to the definition of $\omega(U)$, there exist $K_\delta > 1$ such that

$$\| U(t+s,s)\phi \|_{\mathcal{B}} \leq K_\delta e^{(\omega(U)+\delta)t} \| \phi \|_{\mathcal{B}}, \qquad \forall t\geq 0,\ \forall s\in\mathbb{R},\ \forall\phi\in\mathcal{B}.$$

Then

$$\| u(t+s,s,\phi) \|_{Y\times Y} \leq K_\delta e^{(\omega(U)+\delta)t} \| \phi \|_{\mathcal{B}}, \qquad \forall t\geq 0,\ \forall s\in\mathbb{R},\ \forall\phi\in\mathcal{B}.$$

For every $\varphi\in\mathcal{A}$, we assume $\tau=\tau_1$ and define $\hat{\varphi}=(\hat{\varphi}_1,\varphi_2)$, where

$$\hat{\varphi}_1(\theta_1) = \varphi_1(\theta_1), \qquad -\tau_2 \leq \theta_1 \leq 0.$$

Clearly, $\hat{\varphi}\in\mathcal{B}$. When $\tau=\tau_2$, we obtain the same result. Therefore, $u(t,s,\hat{\varphi}) = v(t,s,\varphi)$, $\forall t\geq s, s\in\mathbb{R}$.

Since $\| \hat{\varphi} \|_{\mathcal{B}} \leq \| \varphi \|_{\mathcal{A}}$, we know

$$\begin{aligned} \| v(t+s,s,\varphi) \|_{Y\times Y} &= \| u(t+s,s,\hat{\varphi}) \|_{Y\times Y} \\ &\leq K_\delta e^{(\omega(U)+\delta)t} \| \hat{\varphi} \|_{\mathcal{B}} \\ &\leq K_\delta e^{(\omega(U)+\delta)t} \| \varphi \|_{\mathcal{A}}, \qquad \forall t\geq 0, \forall s\in\mathbb{R}. \end{aligned}$$

Further, there exists $\tilde{K}_\delta \geq K_\delta$ such that

$$\| \hat{U}(t+s,s)\varphi \|_{\mathcal{A}} = \| v_t(s,\varphi) \|_{\mathcal{A}} \leq \tilde{K}_\delta e^{(\omega(U)+\delta)t} \| \varphi \|_{\mathcal{A}} \\ \forall t\geq 0,\ \forall s\in\mathbb{R},\ \forall\varphi\in\mathcal{A}.$$

Hence

$$\| \hat{U}(t+s,s) \| \leq \tilde{K}_\delta e^{(\omega(U)+\delta)t}, \qquad \forall t\geq 0, s\in\mathbb{R}.$$

By the definition of $\omega(\hat{U})$, we obtain that

$$\omega(\hat{U}) \leq \omega(U) + \delta.$$

Let $\delta\to 0^+$. The claim 1 and claim 2 imply that $\omega(\hat{U}) = \omega(U)$. However, $\hat{P} = \hat{U}(\omega,0)$ and $P = U(\omega,0)$. By refering to [19, Proposition A.2], it is showed that $r(P) = r(\hat{P})$. □

4. Threshold dynamics

In this section, we establish a threshold result on the extinction and uniform persistence of the disease in terms of R_0. First, the next lemma holds.

Lemma 4.1. *Assume* $(u_{S_1}(t,x,\varphi), u_{S_2}(t,x,\varphi), u_1(t,x,\varphi), u_2(t,x,\varphi))$ *is a solution of* (2.5) *with* $\varphi \in \tilde{C}^+$. *Then the following results hold, namely,*

(1) *If there exists some* $t_0 \geq 0$ *such that* $u_i(t_0,\cdot,\varphi) \not\equiv 0\,(i=1,2)$, *then*

$$u_i(t,x,\varphi) > 0, \qquad \forall t > t_0,\ x \in \bar{\Omega},\ i=1,2;$$

(2) *For any* $\varphi \in \tilde{C}^+$, *we always have* $u_{S_i}(t,\cdot,\varphi) > 0\,(i=1,2)$, $\forall t > 0$, *and*

$$\liminf_{t\to+\infty} u_{S_i}(t,x,\varphi) \geq C, \quad i=1,2,$$

uniformly for $x \in \bar{\Omega}$, *where* C *is a positive constant.*

Proof. The proof of the lemma is similar to those of [5, Lemma 9] or [47, Lemma 4.2], here is it omited. □

Theorem 4.1. *Assume* $u(t,x,\phi)$ *is a solution of system* (2.5) *satisfying initial value* $u_0 = \phi \in \tilde{C}^+$. *Then the following states is true.*

(i) If $R_0 < 1$, *then the disease free* ω*-periodic solution* $(u^*_{S_1}, u^*_{S_2}, 0, 0)$ *is globally attractive in* $\tilde{C}^+$;

(ii) If $R_0 > 1$, *then there exists a constant* $\eta > 0$ *such that for any* $\phi \in \tilde{C}^+$ *with* $\phi_1(0,\cdot) \not\equiv 0$ *or* $\phi_2(0,\cdot) \not\equiv 0$, *we have*

$$\liminf_{t\to+\infty} u_i(t,\cdot,\phi) \geq \eta, \ \ i=1,2.$$

Proof. (i) If $R_0 < 1$, then $r(P) < 1$ by lemma 3.1 and lemma 3.3. Hence $\mu = \frac{\ln r(P)}{\omega} < 0$.

Consider the next equation with parameter $\epsilon > 0$:

$$\begin{cases} \dfrac{\partial \bar{u}_1^\epsilon(t,x)}{\partial t} = D_1 \triangle \bar{u}_1^\epsilon(t,x) - (d_1(t,x)+\rho)\bar{u}_1^\epsilon(t,x) + \displaystyle\int_0^{\tau_1} f_1(a) \int_\Omega \Gamma_1(t,t-a,x,y) \\ \qquad\qquad \times \dfrac{c\beta(t-a,y)}{H(y)}(u^*_{S_1}(t-a,y)+\epsilon)\bar{u}_2^\epsilon(t-a,y)dyda, \quad t>0, x\in\Omega, \\ \dfrac{\partial \bar{u}_2^\epsilon(t,x)}{\partial t} = D_2 \triangle \bar{u}_2^\epsilon(t,x) - d_2(t,x)\bar{u}_2^\epsilon(t,x) + \displaystyle\int_0^{\tau_2} f_2(a) \int_\Omega \Gamma_2(t,t-a,x,y) \\ \qquad\qquad \times \dfrac{bp\beta(t-a,y)}{lH(y)}(u^*_{S_2}(t-a,y)+\epsilon)\bar{u}_1^\epsilon(t-a,y)dyda, \quad t>0, x\in\Omega, \\ \frac{\partial \bar{u}_1^\epsilon(t,x)}{\partial n} = \frac{\partial \bar{u}_2^\epsilon(t,x)}{\partial n} = 0, \quad t>0, x\in\partial\Omega. \end{cases} \tag{4.1}$$

For any $\phi \in C_2 \times C_1$, let $\bar{u}^\epsilon(t,x,\phi) = (\bar{u}_1^\epsilon(t,x,\phi), \bar{u}_2^\epsilon(t,x,\phi))$ be the solution of (4.1) with $\bar{u}_0^\epsilon(\phi)(\theta,x) = (\phi_1(\theta_1,x), \phi_2(\theta_2,x))$, where

$$\begin{aligned} &\bar{u}_t^\epsilon(\phi)(\theta,x) = (\bar{u}_1^\epsilon(t+\theta_1,x,\phi), \bar{u}_2^\epsilon(t+\theta_2,x,\phi)), \\ &\forall t\geq 0, x\in\bar{\Omega}, \theta=(\theta_1,\theta_2)\in[-\tau_2,0]\times[-\tau_1,0]. \end{aligned}$$

Define $P^\epsilon : C_2^+ \times C_1^+ \to C_2^+ \times C_1^+$ by $P^\epsilon(\phi) = \bar{u}_\omega^\epsilon(\phi)$. Let $r(P^\epsilon)$ be the spectral radius of P^ϵ. Since $\lim_{\epsilon\to 0^+} r(P^\epsilon) = r(P) < 1$, we can obtain a sufficiently small number $\epsilon_0 > 0$ such that $r(P^\epsilon) < 1$ for $\epsilon \in [0, \epsilon_0)$. Hence, $\bar{\mu}^\epsilon = \frac{\ln r(P^\epsilon)}{\omega} < 0$ for given $\epsilon \in [0, \epsilon_0)$. According to lemma 3.2, there exists a positive ω-periodic function $\bar{v}^\epsilon(t,x)$ such that $e^{\bar{\mu}^\epsilon t}\bar{v}^\epsilon(t,x)$ is a solution of (4.1).

Since $u_{S_i}(i = 1, 2)$ in (2.5) can be dominated by the next system:

$$\begin{cases} \frac{\partial u_{S_i}(t,x)}{\partial t} \leq D_i \triangle u_{S_i}(t,x) - d_i(t,x)u_{S_i}(t,x) + \mu_i(t,x), & t > 0, x \in \Omega, i = 1, 2, \\ \frac{\partial u_{S_i}(t,x)}{\partial n} = 0, \quad t > 0, x \in \partial\Omega, i = 1, 2. \end{cases} \tag{4.2}$$

Then it follows from [47, Lemma 2.1] and the comparison principle that exists some integer $n_1 > 0$ such that

$$u_{S_i}(t,x) \leq u^*_{S_i}(t,x) + \epsilon, \qquad \forall t \geq n_1\omega, x \in \bar{\Omega}, i = 1, 2.$$

Hence

$$\begin{cases} \dfrac{\partial u_1(t,x)}{\partial t} \leq D_1 \triangle u_1(t,x) - (d_1(t,x) + \rho)u_1(t,x) + \displaystyle\int_0^{\tau_1} f_1(a) \int_\Omega \Gamma_1(t, t-a, x, y) \\ \qquad \times \dfrac{c\beta(t-a,y)}{H(y)}(u_{S_1^*}(t-a,y) + \epsilon)u_2(t-a,y)dyda, \quad t > n_1\omega, x \in \Omega, \\ \dfrac{\partial u_2(t,x)}{\partial t} = D_2 \triangle u_2(t,x) - d_2(t,x)u_2(t,x) + \displaystyle\int_0^{\tau_2} f_2(a) \int_\Omega \Gamma_2(t, t-a, x, y) \\ \qquad \times \dfrac{bp\beta(t-a,y)}{lH(y)}(u_{S_2}(t-a,y) + \epsilon)u_1(t-a,y)dyda, \quad t > n_1\omega, x \in \Omega, \\ \frac{\partial u_1(t,x)}{\partial n} = \frac{\partial u_2(t,x)}{\partial n} = 0, \quad t > n_1\omega, x \in \partial\Omega. \end{cases}$$

So, for any $\phi \in \tilde{C}^+$, there exists some $\alpha_1 > 0$ such that

$$(u_1(t,x,\phi), u_2(t,x,\phi)) \leq \alpha_1(e^{\bar{\mu}^\epsilon t}\bar{v}_1^\epsilon(t,x), e^{\bar{\mu}^\epsilon t}\bar{v}_2^\epsilon(t,x)), \quad \forall t \in [n_1\omega, n_1\omega + \tau],\ x \in \bar{\Omega}.$$

By [35, Proposition 3], we get

$$(u_1(t,\cdot,\phi), u_2(t,\cdot,\phi)) \leq \alpha_1(e^{\bar{\mu}^\epsilon t}\bar{v}_1^\epsilon(t,\cdot), e^{\bar{\mu}^\epsilon t}\bar{v}_2^\epsilon(t,\cdot)), \quad \forall t \geq n_1\omega.$$

However, $\bar{\mu}^\epsilon < 0$. Hence, $\lim_{t\to+\infty} u_i(t,\cdot,\psi) = 0(i = 1, 2)$. Then, the equation u_{S_i} $(i = 1, 2)$ in (2.5) is asymptotic to

$$\begin{cases} \frac{\partial v_i(t,x)}{\partial t} = D_i \triangle v_i(t,x) - d_i(t,x)v_i(t,x) + \mu_i(t,x), & t > 0, x \in \Omega, i = 1, 2, \\ \frac{\partial v_i(t,x)}{\partial n} = 0, \quad t > 0, x \in \partial\Omega, i = 1, 2, \end{cases} \tag{4.3}$$

[47, Lemma 2.1] implys (4.3) has an unique ω-periodic solution $u^*_{S_i}(t,x)$ $(i = 1, 2)$ which is globally attractive in Y^+. Next, we prove that $\lim_{t\to+\infty} u_{S_i}(t,x,\phi) = u^*_{S_i}(t,x,\phi)$ $(i = 1, 2)$ uniformly for $x \in \bar{\Omega}$.

Let $\tilde{Q} := \tilde{Q}(\omega)$. Note $J = \bar{\omega}(\phi)$ is the omega limit set for $\phi \in \tilde{C}^+$, namely,

$$J = \{\phi = (\phi^*_{S_1}, \phi^*_{S_2}, \phi^*_1, \phi^*_2) \in \tilde{C}^+ : \text{ŽæÔÚ}\{n_k\} \to +\infty, \text{ s.t.} \\ \lim_{k\to+\infty} \tilde{Q}^{n_k}(\phi) = \lim_{k\to+\infty} \tilde{Q}(n_k\omega)(\phi) = (\phi^*_{S_1}, \phi^*_{S_2}, \phi^*_1, \phi^*_2)\}.$$

By [49], we admit that J is an internally chain transitive set for $\tilde{Q}$. Because $\lim_{t\to+\infty} u_i(t,x,\phi) = 0(i = 1, 2)$, we have $J = J_1 \times \{\hat{0}\}$. Here $\hat{0} = (\hat{0}_1, \hat{0}_2)$,

$$\hat{0}_1(\theta_1, \cdot) = 0, \qquad \forall \theta_1 \in [-\tau_2, 0],$$

$$\hat{0}_2(\theta_2, \cdot) = 0, \qquad \forall \theta_2 \in [-\tau_1, 0].$$

According to lemma 4.1, one has $\hat{0} \notin J_1$.

For every $\varphi \in C_1^+ \times C_2^+$, we assume $w(t, x, \varphi(0, \cdot)) = (w_1(t, x, \varphi(0)), w_2(t, x, \varphi(0)))$ is a solution of (4.3) with $w(0, x) = \varphi(0, x)$ in $C_1^+ \times C_2^+$. Define

$$w_{1t}(\theta_2, x, \varphi) = \begin{cases} w_1(t + \theta_2, x, \varphi(0)), & t + \theta_2 > 0, t > 0, \theta_2 \in [-\tau_1, 0], \\ \varphi_1(t + \theta_2, x), & t + \theta_2 \le 0, t > 0, \theta_2 \in [-\tau_1, 0], \end{cases}$$

$$w_{2t}(\theta_1, x, \varphi) = \begin{cases} w_2(t + \theta_1, x, \varphi(0)), & t + \theta_1 > 0, t > 0, \theta_1 \in [-\tau_2, 0], \\ \varphi_2(t + \theta_1, x), & t + \theta_1 \le 0, t > 0, \theta_1 \in [-\tau_2, 0]. \end{cases}$$

Then $w_t = (w_{1t}, w_{2t})$ is a solution semiflow of (4.3) on $C_1^+ \times C_2^+$. Let $\bar{P}(\varphi) = w_\omega(\varphi)$. It follows from [47, Lemma 2.1] that $\omega(\varphi) = \{u_0^* = (u_{S_1,0}^*, u_{S_2,0}^*)\}$, where

$$u_{S_1,0}^*(\theta_2, \cdot) = u_{S_1}^*(\theta_2, \cdot), \qquad \theta_2 \in [-\tau_1, 0],$$
$$u_{S_2,0}^*(\theta_1, \cdot) = u_{S_2}^*(\theta_1, \cdot), \qquad \theta_1 \in [-\tau_2, 0].$$

Since $\tilde{Q}(J) = J$ and $u_i(t, x, (\phi_{S_1}, \phi_{S_2}, \hat{0}_1, \hat{0}_2)) \equiv 0 (i = 1, 2)$, we obtain that $\tilde{Q}(J) = \bar{P}(J_1) \times \{\hat{0}\}$. Hence, $\bar{P}(J_1) = J_1$. Therefore, J_1 is an internally chain transitive set for $\bar{P}$. Through [47, Lemma 2.1], we get u_0^* is globally attractive in $C_1^+ \times C_2^+$. And [47, Lemma 3.1] prove that $\bar{J} = \{u_0^*\}$. So, $J = \{(u_{S_1,0}^*, u_{S_2,0}^*, \hat{0}_1, \hat{0}_2)\}$. By the definition of J, one has

$$\lim_{t \to +\infty} (u_{S_1}(t, \cdot, \phi), u_{S_2}(t, \cdot, \phi), u_1(t, \cdot, \phi), u_2(t, \cdot, \phi)) = (u_{S_1}^*, u_{S_2}^*, 0, 0).$$

(ii) If $R_0 > 1$, then $r(P) > 1$. Hence $\mu = \frac{\ln r(P)}{\omega} > 0$. Let

$$W_0 = \{\phi \in \tilde{C}^+ : \phi_1(0, \cdot) \not\equiv 0, \text{ or } \phi_2(0, \cdot) \not\equiv 0\},$$
$$\partial W_0 = \tilde{C}^+ \setminus W_0 = \{\phi \in \tilde{C}^+ : \phi_1(0, \cdot) \equiv 0, \text{ and } \phi_2(0, \cdot) \equiv 0\}.$$

Lemma 4.1 indicates that $u_1(t, x, \phi) > 0$ for all $t > 0$ and $x \in \bar{\Omega}$ if $\phi \in W_0$ and $\phi_1(0, \cdot) \not\equiv 0$. Further, we have $u_1(t, x, \phi) > 0$ for $t > \tau_2$ and $x \in \bar{\Omega}$. If $\phi \in W_0$ and $\phi_2(0, \cdot) \not\equiv 0$, then the similar result holds. As a consequence, $u_i(t, x, \phi) > 0, \forall t > \tau, x \in \bar{\Omega}, i = 1, 2$. We obtain that there exists some $n_0 \in N$ such that $\tilde{Q}^n(W_0) \subset W_0$ for $n > n_0$.

Set

$$M_\partial := \{\psi \in \partial C_0 : Q^n(\psi) \in \partial C_0, \forall n \in N\},$$

and $\omega(\phi)$ be the omega limit set of the orbit $\gamma^+(\phi) := \{\tilde{Q}^n(\phi) : \forall n \in N\}$. Let $M = (u_{S_1}^*, u_{S_2}^*, \hat{0}_1, \hat{0}_2)$. For any $\phi \in M_\partial$, $\tilde{Q}^n(\phi) \in \partial W_0$, $\forall n \in N$. So, for every $\phi \in M_\partial$, $u_i(t, x, \phi) \equiv 0\,(i = 1, 2)$, $\forall t \ge 0, x \in \bar{\Omega}$. Hence,

$$\lim_{t \to +\infty} u_{S_i}(t, \cdot, \phi) = u_{S_i}(t, \cdot), \qquad i = 1, 2.$$

Finally, $\omega(\phi) = \{M\}$, $\forall \phi \in M_\partial$.

Now, we consider the linear equation with parameter $\delta(>0)$:

$$\begin{cases} \dfrac{\partial v_1^\delta}{\partial t} = D_1 \triangle v_1^\delta(t,x) - (d_1(t,x)+\rho) v_1^\delta(t,x) + \displaystyle\int_0^{\tau_1} f_1(a) \int_\Omega \Gamma_1(t,t-a,x,y) \\ \qquad \times \dfrac{cl\beta(t-a,y)}{(p-l)\delta + lH(y)} (u_{S_1}^*(t-a,y)-\delta) v_2^\delta(t-a,y) dy da, \qquad t>0, x\in\Omega, \\ \dfrac{\partial v_2^\delta}{\partial t} = D_2 \triangle v_2^\delta(t,x) - d_2(t,x) v_2^\delta(t,x) + \displaystyle\int_0^{\tau_2} f_2(a) \int_\Omega \Gamma_2(t,t-a,x,y) \\ \qquad \times \dfrac{bp\beta(t-a,y)}{(p-l)\delta + lH(y)} (u_{S_2}^*(t-a,y)-\delta) v_1^\delta(t-a,y) dy da, \qquad t>0, x\in\Omega, \\ \frac{\partial v_1^\delta}{\partial n} = \frac{\partial v_2^\delta}{\partial n} = 0, \qquad t>0, x\in\partial\Omega. \end{cases} \tag{4.4}$$

Let $v^\delta(t,x,\phi) = (v_1^\delta(t,x,\phi), v_2^\delta(t,x,\phi))$ is a solution of (4.4) with $v_0^\delta(\phi) = \phi \in C_2 \times C_1$, where

$$v_t^\delta(\phi)(\theta,x) = (v_1^\delta(t+\theta_1,x,\phi), v_2^\delta(t+\theta_2,x,\phi)),$$
$$\forall t\geq 0, x\in\bar\Omega, \theta=(\theta_1,\theta_2)\in[-\tau_2,0]\times[-\tau_1,0].$$

Define $P^\delta : C_2\times C_1 \to C_2\times C_1$, *i.e.* $P^\delta(\phi) = v_\omega^\delta(\phi)$. $r(P^\delta)$ represents the spectral radius of P^δ. Since $\lim_{\delta\to 0^+} r(P^\delta) = r(P) > 1$, there exists a sufficient small number $\delta_0 > 0$ such that $r(P^\delta) > 1$ for $\delta\in[0,\delta_0)$. For given $\delta\in(0,\delta_0)$, by the continuous dependence of solutions on initial value, there is $\delta^*\in(0,\delta_0)$ such that

$$\| \tilde Q(t)\phi - \tilde Q(t) M \| < \delta, \qquad \forall t\in[0,\omega],$$

for $\| \phi - M \| < \delta^*$. Next, we prove the following claim.

Claim: $\limsup_{n\to+\infty} \| \tilde Q^n(\phi) - M \| \geq \delta^*$, $\forall \phi\in W_0$.

Suppose, by contradiction, that there is some $\phi_0\in W_0$ such that

$$\limsup_{n\to\infty} \| \tilde Q^n(\phi_0) - M \| < \delta^*.$$

Then there exists $n_2\in N$ such that

$$\| \tilde Q^n(\phi_0) - M \| < \delta^*, \qquad \forall n\geq n_2.$$

By the continuous dependence of solutions on initial value,

$$u_{S_i}(t,x,\phi_0) > u_{S_i}^*(t,x) - \delta, \qquad t\geq n_2\omega, x\in\bar\Omega, i=1,2,$$
$$0 < u_i(t,x,\phi_0) < \delta, \qquad t\geq n_2\omega, x\in\bar\Omega, i=1,2.$$

Hence $u_1(t,x,\phi_0)$ and $u_2(t,x,\phi_0)$ satisfy

$$\begin{cases} \dfrac{\partial u_1}{\partial t} \geq D_1 \triangle u_1(t,x) - (d_1(t,x)+\rho) u_1(t,x) + \displaystyle\int_0^{\tau_1} f_1(a) \int_\Omega \Gamma_1(t,t-a,x,y) \\ \qquad \times \dfrac{cl\beta(t-a,y)}{(p-l)\delta + lH(y)} (u_{S_1}^*(t-a,y)-\delta) u_2(t-a,y) dy da, \quad t\geq n_2\omega+\tau, x\in\Omega, \\ \dfrac{\partial u_2}{\partial t} \geq D_2 \triangle u_2(t,x) - d_2(t,x) u_2(t,x) + \displaystyle\int_0^{\tau_2} f_2(a) \int_\Omega \Gamma_2(t,t-a,x,y) \\ \qquad \times \dfrac{bp\beta(t-a,y)}{(p-l)\delta + lH(y)} (u_{S_2}^*(t-a,y)-\delta) u_1(t-a,y) dy da, \quad t\geq n_2\omega+\tau,\ x\in\Omega, \\ \frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n} = 0, \qquad t\geq n_2\omega+\tau,\ x\in\partial\Omega. \end{cases} \tag{4.5}$$

Since $u_i(t,x,\phi_0) \gg 0$ for all $\forall t > \tau$, $x \in \bar{\Omega}$ and $i = 1,2$, there exists some $\alpha_2 > 0$ sucn that

$$(u_1(t,x,\phi_0), u_2(t,x,\phi_0)) \geq \alpha_2 e^{\mu^\delta t} v^\delta(t,x), \qquad \forall t \in [n_2\omega, n_2\omega + \tau], x \in \bar{\Omega}.$$

Here, $\mu^\delta = \frac{\ln r(P^\delta)}{\omega}$ and $e^{\mu^\delta t} v^\delta(t,x)$ is a solution of (4.4). By (4.5) and the comparison principle,

$$(u_1(t,x,\phi_0), u_2(t,x,\phi_0)) \geq \alpha_2 e^{\mu^\delta t} v^\delta(t,x), \qquad \forall t \geqslant n_2\omega + \tau, x \in \bar{\Omega}.$$

Combining $\mu^\delta > 0$, we get

$$\lim_{t\to+\infty} u_i(t,\cdot,\phi_0) = +\infty, \qquad i = 1,2.$$

This leads to a contradiction.

The above claim states that $W^s(M) \bigcap W_0 = \varnothing$ and $M \subset W_0$ is an isolated invariant set. Here, $W^s(M) := \{\phi \in \tilde{C}^+ : \lim_{n\to+\infty} d(\tilde{Q}(\phi), M) = 0\}$. By [49, Theorem 1.3.1 and Remark 1.3.1], we know that $\tilde{Q} : \tilde{C}^+ \to \tilde{C}^+$ is uniformly persistent with respect to $(W_0, \partial W_0)$. It is easy seen that $\tilde{Q}^n = \tilde{Q}(n\omega)$ is compact and point dissipative. According to [34, Theorem 2.9], $\tilde{Q} : W_0 \to W_0$ admits a global attractor A_0 in W_0.

Define a continuous function $p : \tilde{C}^+ \to \mathbb{R}_+$ by

$$p(\phi) = \min\{\min_{x\in\bar{\Omega}} \phi_1(0,x), \ \min_{x\in\bar{\Omega}} \phi_2(0,x)\}, \qquad \forall \phi \in \tilde{C}^+.$$

Since $A_0 = \tilde{Q}(A_0)$, it follows that

$$\phi_i(0,x) > 0, \qquad \forall \phi \in A_0, \ i = 1,2.$$

Let $\mathbf{B}_0 := \bigcup_{t\in[0,\omega]} \tilde{Q}(t)A_0$. Then $\mathbf{B}_0 \subset W_0$ and

$$\lim_{t\to+\infty} d(\tilde{Q}(t)\phi, \mathbf{B}_0) = 0, \qquad \forall \phi \in W_0.$$

Moreover, $\mathbf{B}_0 \subset W_0$ is a compact subset. Hence, $\min_{\phi\in\mathbf{B}_0} p(\phi) > 0$. According to lemma 4.1, there exists some $\eta^* > 0$ such that

$$\liminf_{t\to+\infty} u_i(t,\cdot,\phi) \geq \eta^*, \qquad i = 1,2.$$

Further, there exists some constant $\eta \in (0,\eta^*)$ such that

$$\liminf_{t\to+\infty} u_i(t,\cdot,\phi) \geq \eta, \qquad \forall \phi \in W_0, \ i = 1,2.$$

This completes the proof. □

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