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Qualitative Analysis of Crossing Limit Cycles in Discontinuous Liénard-Type Differential Systems*

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Abstract In this paper, we investigate qualitative properties of crossing limit cycles for a class of discontinuous nonlinear Liénard-type differential systems with two zones separated by a straight line. Firstly, by applying left and right Poincaré mappings we provide two criteria on the existence, uniqueness and stability of a crossing limit cycle. Secondly, by geometric analysis we estimate the position of the unique limit cycle. Several lemmas are given to obtain an explicit upper bound for the amplitude of the limit cycle. Finally, a predator-prey model with nonmonotonic functional response is studied, and Matlab simulations are presented to show the agreement between theoretical results and numerical analysis.

Keywords Discontinuous Liénard-type differential system, crossing limit cycle, existence, uniqueness, stability, position.

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1. Introduction

In the real world, many problems from mechanics, physics and engineering are discontinuous in nature, such as forcing terms of electro-magnetic field and controls in engineering. The mathematical modeling can be described by differential equations with discontinuous right-hand sides, called as discontinuous differential systems. Up to now, there have been rich achievements on the basic properties of solutions and the stability theory (see monograph [6] for example). The main topics have been become the analysis of existence and uniqueness of periodic orbits, number and bifurcation of limit cycles in qualitative theory of planar systems of ordinary differential equations. An important type of planar systems is of the form

$$\frac{d^2x}{dt^2} + f(x)\frac{dx}{dt} + g(x) = 0, \tag{1.1}$$

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called as Liénard differential equation which exhibits notably rich dynamics. The equivalent system of (1.1)

$$\frac{dx}{dt} = y - F(x), \qquad \frac{dy}{dt} = -g(x) \tag{1.2}$$

with $F(x) = \int_0^x f(s)ds$, and the more general form

$$\frac{dx}{dt} = h(y) - F(x), \qquad \frac{dy}{dt} = -g(x) \tag{1.3}$$

are both called as Liénard-type systems. Many planar models in physical applications, chemical reaction and biological models can be transformed into the Liénard forms (1.2) or (1.3). The problem on limit cycles of (1.2) and (1.3), such as the (non)existence, uniqueness, number and relative position have been widely studied, see [1,4,10,11,15-16,18-22] for example. The existence is often proved by the well known Poincaré-Bendixson theorem by constructing a trapping zone where a limit cycle is located. The following conditions appear frequently in the literatures of the study of limit cycles of (1.2) and (1.3):

- (i) $xg(x) > 0$ for $x \neq 0$, and $G(x) = \int_0^x g(s)ds$ with $G(\pm\infty) = +\infty$;
- (ii) there exist $x_1 < 0 < x_0$ such that $F(x_1) = F(0) = F(x_0) = 0$, $xF(x) > 0$ for $x \in (x_0, +\infty) \cup (-\infty, x_1)$ and $xF(x) < 0$ for $x \in (x_1, x_0)$;
- (iii) $F'(x) > 0$ for $x \in (x_0, +\infty) \cup (-\infty, x_1)$, and $F(\pm\infty) = \pm\infty$;
- (iv) $h'(y) > 0$ for $y \in \mathbb{R}$, and $h(\pm\infty) = \pm\infty$.

Note that the theory of smooth systems cannot be directly applied to the discontinuous case. In fact, there may be appearing sliding solutions, grazing solutions or impact solutions. Hence it is necessary to study the discontinuous differential systems in theory and applications. About the investigation of existence, uniqueness, number and bifurcation of limit cycles, one can see [2-3,5-6,7-9,12-14] for example. In this paper, we consider a Liénard-type system (1.3) allowing for discontinuity, where the discontinuities occur along a straight line (called as a discontinuity line). The analysis shows that the system has no sliding solutions. Therefore, we focus on the existence, uniqueness and stability of a crossing limit cycle. Here the crossing limit cycle is defined as a limit cycle with trajectory intersecting the discontinuity line at finite points, and the time parameter representation of this kind of limit cycles is usually continuous but possesses discontinuous derivative on the discontinuity line.

Another classical topic in the qualitative theory of planar systems of ordinary differential equations is to estimate the position and amplitude of limit cycles. However, there are few papers involving the upper bound of amplitude of limit cycles for Liénard-type systems [1,3-4,15-16,18]. Recently, Yang and Zeng [18] studied an upper bound of the amplitude of a unique limit cycle for (1.2) with symmetry under some conditions as follows

- (i) $f(x)$ has a unique positive zero $a_1 > 0$ and $f(x)(x - a_1) > 0$ for $x > 0, x \neq a_1$;
- (ii) $f(x)/g(x)$ is monotone increasing on $(a_1, +\infty)$;
- (iii) $\int_0^x F(s)g(s)ds > 0$ for sufficiently large $x > 0$.

The condition (iii) is quite natural to guarantee that the integral $\int_0^x F(s)g(s)ds$ has a unique positive zero, which gives an upper bound for the amplitude of a limit cycle. For the same symmetric system as in [18], Chen, Han and Xia [3] also studied the amplitude of limit cycles in discontinuous case. In section 4 of this paper, by considering the upper bounds of amplitude of the unique crossing limit cycle on left and right half plane respectively, we estimate the position of the limit cycle on the plane. Here the amplitude of a limit cycle on the left/right half plane $\{(x, y) \in \mathbb{R}^2 : x < 0\}/\{(x, y) \in \mathbb{R}^2 : x > 0\}$ is defined as the minimal/maximal value of the x-coordinate on the limit cycle, and the amplitude of a limit cycle on the plane is defined as the maximum absolute value of the x-coordinate on the limit cycle. Physically the amplitude represents the maximal deviation of the oscillation from the equilibrium state.

This paper is organized as follows. In section 2, we present some relevant preliminaries. In section 3, we state and prove the existence, uniqueness and stability of a crossing limit cycle. In section 4, we give an explicit upper bound for the amplitude of the unique limit cycle on the plane. In section 5, applications are given to illustrate the obtained results, and Matlab simulations are presented to show the agreement between theoretical results and numerical analysis. Concluding remarks are outlined in section 6.

2. Preliminaries

Consider the discontinuous Liénard-type system

$$\begin{cases} \dfrac{dx}{dt} = h(y) - F(x), \\ \dfrac{dy}{dt} = -g(x), \end{cases} \tag{2.1}$$

where $F, h \in C(\mathbb{R}, \mathbb{R})$ and g is given by

$$g(x) = \begin{cases} g_+(x), & x > 0, \\ g_-(x), & x < 0. \end{cases} \tag{2.2}$$

Let Σ_0 denote the discontinuity line and $\Sigma_\pm$ the two subregions of $\mathbb{R}^2$ separated by Σ_0 as follows

$$\Sigma_0 = \{x = 0, y \in \mathbb{R}\}, \quad \Sigma_+ = \{(x, y) \in \mathbb{R}^2 : x > 0\}, \quad \Sigma_- = \{(x, y) \in \mathbb{R}^2 : x < 0\}.$$

Then $\mathbb{R}^2 = \Sigma_+ \cup \Sigma_0 \cup \Sigma_-$ and the unit normal vector to Σ_0 is taken to be $\mathbf{n}^T = (1, 0)$.

Denote the vector field of (2.1)-(2.2) by

$$V(x, y) = \begin{cases} V_-(x, y), & (x, y) \in \Sigma_-, \\ V_+(x, y), & (x, y) \in \Sigma_+, \end{cases} \tag{2.3}$$

where $V_\pm(x, y) = (h(y) - F(x),\ -g_\pm(x))^T$. Since the second component of (2.3) is discontinuous on $\mathbb{R}^2$, by using the Filippov theory (for more details see [6-7] for example) to define orbits of the system when they intersect Σ_0 such that the orbits can be concatenated in a natural way. We give the following definition similar to Definition 2.1 in [7].

Definition 2.1. For any given $(0, y) \in \Sigma_0$, if

$$[\mathbf{n}^T V(0^-, y)] \cdot [\mathbf{n}^T V(0^+, y)] \leq 0,$$

$(0, y)$ is called as a *sliding point*. A set of sliding points is a *sliding set* denoted by Σ_s. We say $(0, y)$ is a *crossing point*, if

$$[\mathbf{n}^T V(0^-, y)] \cdot [\mathbf{n}^T V(0^+, y)] > 0.$$

A set of crossing points is called as a *crossing set*.

It follows that the origin $O(0,0)$ is the unique sliding point on Σ_0. So we focus on crossing periodic orbits of (2.1)-(2.2) in the next text. Here *a crossing periodic orbit* is defined as a closed orbit but it does not share points with Σ_s.

In this paper, we assume the following hypotheses for (2.1)-(2.2).

$(H1)$ $F \in C(\mathbb{R}, \mathbb{R})$, and there are $x_1 < 0 < x_0$ such that $F(0) = F(x_0) = F(x_1) = 0$, $xF(x) > 0$ for $x \in (x_0, +\infty) \cup (-\infty, x_1)$ and $xF(x) < 0$ for $x \in (x_1, x_0)$.

$(H2)$ $g_+ \in C^1([0, +\infty), [0, +\infty))$ and $g_- \in C^1((-\infty, 0], (-\infty, 0])$.

$(H3)$ $h \in C(\mathbb{R}, \mathbb{R})$ satisfying $yh(y) > 0$ for $y \neq 0$, and there are $0 < \delta < M < +\infty$ such that

$$h(y) \geq |h(-y)| \text{ for } 0 < y < \delta, \qquad h(y) \leq |h(-y)| \text{ for } y > M. \tag{2.4}$$

$(H4)$ $F \in C^1(\mathbb{R}\backslash\{0\}, \mathbb{R})$ satisfying $F'(x) > 0$ for $x \in (-\infty, x_1) \cup (x_0, +\infty)$, $F'(0^+)$ and $F'(0^-)$ exist, and $F(\pm\infty) = \pm\infty$.

$(H5)$ h is monotone increasing satisfying $h(\pm\infty) = \pm\infty$.

Remark 2.1. As we will see, $(H3)$ is mainly used to prove Lemma 3.3 in section 3 which is key to establish the existence result of crossing periodic orbits of (2.1)-(2.2).

By $(H1)$-$(H3)$, the possible singular point of (2.1)-(2.2) is the origin $O(0,0)$. If the system has a crossing periodic orbit Γ, by the Poincaré-Bedixson theorem [see Theorem 4.6 in Chapter 1, 21], the interior of bounded region limited by Γ must contain the origin.

3. Existence and uniqueness

In this section, we study the existence, uniqueness and stability of a crossing limit cycle surrounding the origin. From [6], one can easily verify the following result.

Lemma 3.1. *For any* $(\overline{x}_0, \overline{y}_0) \in \mathbb{R}^2$, *there exists a unique solution pair* $x(t) = x(t; \overline{x}_0, \overline{y}_0)$, $y(t) = y(t; \overline{x}_0, \overline{y}_0)$ *of* (2.1)-(2.2) *satisfying the initial value condition* $x(0) = \overline{x}_0, y(0) = \overline{y}_0$. *Further the solution* $(x(t), y(t))$ *is continuously dependent on* $(\overline{x}_0, \overline{y}_0)$.

Let $P(x_P, y_P) \in \mathbb{R}^2\backslash\{O\}$, and denote by L_P^+, L_P^-, L_P the positive, negative and the whole crossing orbit of (2.1)-(2.2) passing through the point P respectively. Let

$$\Sigma_0^+ = \{(x, y) : x = 0, y > 0\}, \qquad \Sigma_0^- = \{(x, y) : x = 0, y < 0\},$$
$$\Sigma_F = \{(x, y) \in \mathbb{R}^2 : F(x) = h(y)\}.$$

The following lemma gives some basic properties of solutions of (2.1)-(2.2), whose proof is similar to Lemma 3.1 in [9].

Lemma 3.2. *Let* $(H1)$-$(H5)$ *hold. Then we have the following three statements.*

1. *If* $P \in \Sigma_F$*, there exists* $T_+ > 0$ *(*$T_- < 0$*) such that* L_P^+ *(*L_P^-*) intersects with* $\Sigma_0 \backslash \{O\}$ *for the first time when* $t = T_+$ *(*$t = T_-$*).*
2. *If* $P \in \Sigma_0$*, there exists* $T_+ > 0$ *(*$T_- < 0$*) such that* L_P^+ *(*L_P^-*) intersects with* $\Sigma_F \backslash \{O\}$ *for the first time when* $t = T_+$ *(*$t = T_-$*).*
3. *For any given* $P \in \mathbb{R}^2 \backslash \{O\}$*, the corresponding solution passing through* P *exists for* $t \in (-\infty, +\infty)$*.*

The results in Lemma 3.2 imply that all orbits of (2.1)-(2.2) cross Σ_0 transversally in a clockwise fashion. Inspired by [11], we make a similar definition as follows.

Definition 3.1. For any $(0, y) \in \Sigma_0^+$, let $(0, -p_R(y)) \in \Sigma_0^-$ be the first intersection point of the orbit starting from $(0, y)$ and Σ_0. Then the planar mapping $P_R : (0, y) \to (0, -p_R(y))$ is called as a *right Poincaré mapping.* Similarly, for $(0, -y) \in \Sigma_0^-$, let $(0, p_L(y)) \in \Sigma_0^+$ be the first intersection point of the orbit starting from $(0, -y)$ and Σ_0, and we call $P_L : (0, -y) \to (0, p_L(y))$ as a *left Poincaré mapping.*

By the continuous dependence of solutions on initial values, P_L and P_R are continuous, and consequently the component functions p_L and p_R are also continuous. For the orbit of (2.1)-(2.2) starting from a given $(0, y) \in \Sigma_0^+$, by Definition 3.1 it intersects Σ_0^- at $(0, -p_R(y))$ and Σ_0^+ then at $(0, p_L(p_R(y)))$. Clearly the orbit is periodic if and only if $y = p_L(p_R(y))$.

The following lemma is a key step to establish the existence result of crossing periodic orbits of (2.1)-(2.2).

Lemma 3.3. *Let* $(H1)$-$(H5)$ *hold. Then the function* $q_z(y) \equiv p_z(y) - y$ *satisfies*

$$q_z(y) > 0 \quad \text{for} \quad y \in (0, \delta_z), \qquad q_z(y) < 0 \quad \text{for} \quad y \in (M_z, +\infty),$$

where $0 < \delta_z < M_z < +\infty$ *and* $z \in \{L, R\}$*.*

Proof. We only prove the statement for the case of $q_R(y)$, and the proof for $q_L(y)$ is similar. Consider the trajectory arc of (2.1)-(2.2) on Σ_+, which starts from any $(0, y) \in \Sigma_0^+$. When P_R applies, it intersects Σ_0^- at $(0, -p_R(y)) \in \Sigma_0^-$. Let

$$\lambda(x, y) = H(y) + G(x), \tag{3.1}$$

where $G(x) = \int_0^x g(s)ds$ with $G(0) = 0$ and $H(y) = \int_0^y h(s)ds$ with $H(0) = 0$. Then along the trajectory arc, one has that

$$\frac{d\lambda(x(t), y(t))}{dt} = -g_+(x(t))F(x(t)). \tag{3.2}$$

By Lemma 3.2, for the point $C^+(x_0, 0) \in \Sigma_F$ there must exist $A^+(0, y^+) \in \Sigma_0^+$ such that $L_{C^+}^-$ intersects with Σ_0^+ at the point A^+ for the first time. Then according to $0 < y < y^+$ and $y > y^+$, there are two possible cases to consider.
(i) Trajectory arcs starting from the point $(0, y) \in \Sigma_0^+$ with $y < y^+$ are completely contained in the strip region $\{(x, y) \in \mathbb{R}^2 : 0 < x < x_0\}$.

Choose any $A_0(0, y_{A_0})$ with $0 < y_{A_0} < y^+$, consider the trajectory arc $\widehat{A_0C_0B_0}$ starting from A_0 and P_R maps it to $B_0(0, -y_{B_0})$ with $y_{B_0} > 0$ (see Figure 1). Since $F(x) < 0$ for $0 < x < x_0$, it follows from (3.2) that

$$\lambda(B_0) - \lambda(A_0) = \int_{\widehat{A_0B_0}} F(x)dy = \int_{y_{A_0}}^{-y_{B_0}} F(x(y))dy > 0.$$

This together with (3.1) mean that $H(-y_{B_0}) - H(y_{A_0}) > 0$. Next we show that when $y_{A_0} > 0$ is small,

$$q_R(y_{A_0}) = p_R(y_{A_0}) - y_{A_0} > 0. \tag{3.3}$$

Choose a point $\overline{C}(0, -y_{\overline{C}})$ with $y_{\overline{C}} > 0$ such that $H(-y_{\overline{C}}) = H(y_{A_0})$. Then $H(-y_{\overline{C}}) < H(-y_{B_0})$. Thus we have that $y_{B_0} > y_{\overline{C}}$. Note that

$$\int_0^{y_{A_0}} h(s)ds = \int_0^{y_{\overline{C}}} |h(-t)|dt,$$

it follows from (2.4) that $y_{\overline{C}} \geq y_{A_0}$ and $p_R(y_{A_0}) = y_{B_0} > y_{\overline{C}} \geq y_{A_0}$ for $y_{A_0} > 0$ small. Hence (3.3) holds.

(ii) Trajectory arcs starting from any $(0, y) \in \Sigma_0^+$ with $y > y^+$ are not completely contained in $\{(x, y) \in \mathbb{R}^2 : 0 < x < x_0\}$.

Choose any $A_1(0, y_{A_1})$ with $y_{A_1} > y^+$, consider the trajectory arc $\widehat{A_1C_1D_1B_1}$ starting from A_1 and P_R maps it to $B_1(0, -y_{B_1})$ with $y_{B_1} > 0$ (see Figure 1). With the similar analysis to Lemma 3.3 in [9], we have that $H(-y_{B_1}) - H(y_{A_1}) < 0$ for y_{A_1} large enough. Next we show that when $y_{A_1} > 0$ is large,

$$q_R(y_{A_1}) = p_R(y_{A_1}) - y_{A_1} < 0. \tag{3.4}$$

Choose a point $\widetilde{C}(0, -y_{\widetilde{C}})$ with $y_{\widetilde{C}} > 0$ such that $H(-y_{\widetilde{C}}) = H(y_{A_1})$. Then $H(-y_{B_1}) < H(-y_{\widetilde{C}})$, and so $y_{B_1} < y_{\widetilde{C}}$. Note that

$$\int_0^{y_{A_1}} h(s)ds = \int_0^{y_{\widetilde{C}}} |h(-t)|dt,$$

it follows from (2.4) that $y_{A_1} \geq y_{\widetilde{C}}$ and $p_R(y_{A_1}) = y_{B_1} < y_{\widetilde{C}} \leq y_{A_1}$ for y_{A_1} large enough. Hence (3.4) holds. The proof is completed. □

Remark 3.1. Note that Lemma 3.3 is reduced to the corresponding one in [9] for $h(y) = y$ (see Lemma 3.3 in [9]). Hence our result is a generalization.

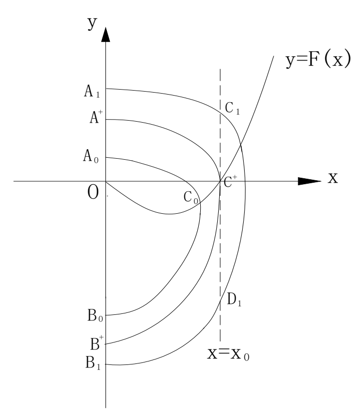

Figure 1. Trajectory arcs of (2.1)-(2.2) on Σ_+.

Now we present the existence result of crossing periodic orbits of (2.1)-(2.2).

Theorem 3.1. *Let* $(H1)$*-*$(H5)$ *hold. Then there exists at least one crossing periodic orbit surrounding the origin on* $\mathbb{R}^2$.

Proof. The proof is similar to the one of Theorem 3.4 in [9] by Lemma 3.3 and then omitted. □

Remark 3.2. By Lemma 3.2, the crossing periodic orbit of (2.1)-(2.2) intersects Σ_F only once on Σ_+ (*resp.* Σ_-), and the x-component of the intersection point is the maximal (*resp.* minimal) value on the periodic orbit.

Note that there exists $A^+(0, y^+) \in \Sigma_0^+$ (see Figure 1) such that all positive orbits of (2.1)-(2.2) starting from the point $(0, y)$ with $y > y^+$ will cross transversally the line $\{(x, y) : x = x_0\}$ and the curve Σ_F for $x > x_0$. Similarly, on Σ_- there exists $A^-(0, -y^-) \in \Sigma_0^-$ such that the positive orbits starting from $(0, -y)$ with $y > y^-$ will cross transversally the line $\{(x, y) : x = x_1\}$ and the curve Σ_F for $x < x_1$. Then we give the following lemma which is key to obtain the uniqueness and stability of the crossing periodic orbit of (2.1)-(2.2).

Lemma 3.4. *Consider any orbits of* (2.1)-(2.2) *crossing* Σ_F *for* $x \in (x_0, +\infty) \cup (-\infty, x_1)$. *Then the following two statements hold.*

(*i*) *On* Σ_+, *the function* $y \mapsto H(-p_R(y)) - H(y)$ *is strictly decreasing for* $y > y^+$.

(*ii*) *On* Σ_-, *the function* $y \mapsto H(-y) - H(p_L(y))$ *is strictly increasing for* $y > y^-$.

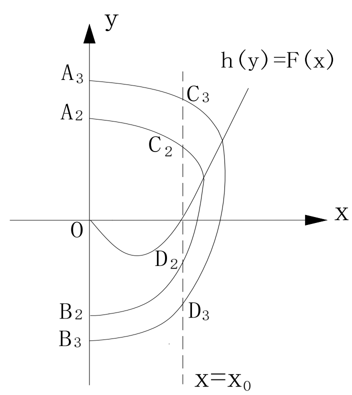

Figure 2. Any two trajectory arcs of (2.1)-(2.2) crossing Σ_F for $x > x_0$.

Proof. Choose any $A_2(0, y_{A_2})$ and $A_3(0, y_{A_3})$ with $y_{A_3} > y_{A_2} > y^+$, consider the trajectory arcs $\widehat{A_2C_2D_2B_2}$ and $\widehat{A_3C_3D_3B_3}$ starting from A_2 and A_3 respectively, and crossing Σ_F for $x > x_0$ on Σ_+ (see Figure 2). With the similar analysis to Lemma 3.3 in [9], one has that

$$\lambda(B_3) - \lambda(A_3) < \lambda(B_2) - \lambda(A_2).$$

It is equivalent to $H(-p_R(y_{A_3})) - H(y_{A_3}) < H(-p_R(y_{A_2})) - H(y_{A_2})$. Hence the function $y \mapsto H(-p_R(y)) - H(y)$ is strictly decreasing for $y > y^+$.

For the trajectory arc on Σ_-, it starts from any $(0, -y)$ with $y > y^-$ and crosses Σ_F for $x < x_1$. When P_L applies, it intersects with Σ_0^+ at $(0, p_L(y))$. Similarly, the function $y \mapsto H(-y) - H(p_L(y))$ is strictly increasing for $y > y^-$. The proof is completed. □

Now we are ready to state and prove the uniqueness and stability of the crossing periodic orbit of (2.1)-(2.2).

Theorem 3.2. *Let* $(H1)$-$(H5)$ *hold. Then* (2.1)-(2.2) *have exactly one crossing limit cycle and it is stable.*

Proof. By Theorem 3.1, (2.1)-(2.2) have a crossing periodic orbit denoted by $\widetilde{L}$. We next show that it is unique and stable.

Suppose on the contrary that (2.1)-(2.2) have two crossing periodic orbits $\widetilde{L}, \overline{L}$, and assume that $\overline{L}$ locates in the interior of $\widetilde{L}$. Let $(0, \widetilde{y}_R)$ and $(0, -\widetilde{y}_L)$ denote the intersection points of $\widetilde{L}$ intersecting Σ_0^+ and Σ_0^- respectively, $(0, \overline{y}_R)$ and $(0, -\overline{y}_L)$ denote the intersection points of $\overline{L}$ intersecting Σ_0^+ and Σ_0^- respectively. Then $p_R(\overline{y}_R) = \overline{y}_L$, $p_L(\overline{y}_L) = \overline{y}_R$, $p_R(\widetilde{y}_R) = \widetilde{y}_L$ and $p_L(\widetilde{y}_L) = \widetilde{y}_R$. Moreover, by the properties of planar autonomous systems, $\widetilde{y}_R > \overline{y}_R > 0$ and $\widetilde{y}_L > \overline{y}_L > 0$.

By Lemma 3.4, $H(-p_R(\widetilde{y}_R)) - H(\widetilde{y}_R) < H(-p_R(\overline{y}_R)) - H(\overline{y}_R)$. Namely $H(-\widetilde{y}_L) - H(p_L(\widetilde{y}_L)) < H(-\overline{y}_L) - H(p_L(\overline{y}_L))$, this is a contradiction. Hence $\widetilde{L}$ is the unique crossing limit cycle surrounding the origin.

Now we show the stability. By (3.1), for $x \in (0, x_0) \cup (x_1, 0)$ one has that

$$\frac{d\lambda(x(t), y(t))}{dt} = -g(x(t))F(x(t)) > 0.$$

So $\widetilde{L}$ is stable from the interior, and it must encircle $(x_1, 0)$ and $(x_0, 0)$ as the interior points. We only show that $\widetilde{L}$ is also stable from the exterior.

Let $\widehat{L}$ denote any orbit of (2.1)-(2.2), and it locates in the exterior of $\widetilde{L}$. Assume that $\widehat{L}$ starts from $(0, \widehat{y}_R) \in \Sigma_0^+$, and P_R maps it to $(0, -\widehat{y}_L) \in \Sigma_0^-$. Then $\widehat{y}_L = p_R(\widehat{y}_R)$. We claim that

$$p_L(\widehat{y}_L) < \widehat{y}_R. \tag{3.5}$$

In fact, by $\widehat{y}_R > \widetilde{y}_R > 0$ one has that

$$\begin{aligned} H(-\widehat{y}_L) - H(\widehat{y}_R) &= H(-p_R(\widehat{y}_R)) - H(\widehat{y}_R) \\ &< H(-p_R(\widetilde{y}_R)) - H(\widetilde{y}_R) = H(-\widetilde{y}_L) - H(p_L(\widetilde{y}_L)). \end{aligned} \tag{3.6}$$

On the other hand, it follows from $\widehat{y}_L > \widetilde{y}_L > 0$ that

$$H(p_L(\widehat{y}_L)) - H(-\widehat{y}_L) < H(p_L(\widetilde{y}_L)) - H(-\widetilde{y}_L). \tag{3.7}$$

From (3.6)-(3.7), we have that $H(\widehat{y}_R) > H(p_L(\widehat{y}_L))$. Hence (3.5) holds. The proof is completed. □

4. Amplitude of the limit cycle

In this section, by analyzing the upper bounds for amplitude of the unique crossing limit cycle of (2.1)-(2.2) on Σ_+ and Σ_- respectively, we estimate the position of the limit cycle on the plane. Here the amplitude of a limit cycle on Σ_+ (*resp.* Σ_-) is defined as the maximal (*resp.* minimal) value of the x-coordinate on the limit cycle, and the amplitude of a limit cycle on the plane is defined as the maximum absolute value of the x-coordinate on the limit cycle. Physically the amplitude represents the maximal deviation of the oscillation from the equilibrium state.

By $(H1)$ and the Rolle's theorem, there are $\underline{x} \in (x_1, 0)$ and $\overline{x} \in (0, x_0)$ such that $F'(\overline{x}) = F'(\underline{x}) = 0$. Further we make the following hypothesis for (2.1)-(2.2).

$(H6)$ $\frac{g_+(x)}{F'(x)}$ and $\frac{g_-(x)}{F'(x)}$ are monotone decreasing on $(\overline{x}, +\infty)$ and $(-\infty, \underline{x})$ respectively.

Consider an orbit L_A of (2.1)-(2.2), which starts from $A(0, y_A) \in \Sigma_0^+$ and crosses Σ_F transversally for $x > x_0$ and $x < x_1$ (see Figure 3). By Lemma 3.2, it intersects with Σ_F and Σ_0^- at $B(x_B, y_B)$ and $C(0, y_C)$ respectively for the first time. Later it continues to enter Σ_- and intersect again Σ_F and Σ_0^+ at $D(-x_D, y_D)$ and $E(0, y_E)$ respectively. Let $\triangle_R(x_B) = H(y_A) - H(y_C), \triangle_L(x_D) = H(y_C) - H(y_E)$ and $\triangle(x) = H(y_A) - H(y_E)$. If $y_A = y_E$ (*i.e.* $\Delta(x) = 0$), the orbit is a crossing periodic orbit, and the amplitude is exactly $\max\{x_B, x_D\}$. If $y_E < y_A$ (*i.e.* $\Delta(x) > 0$), there exists at least one crossing periodic orbit contained in Ω, and $\max\{x_B, x_D\}$ is an upper bound of the amplitude. Here Ω denotes a closed region encircled by $\widehat{ABCDE} \cup \overline{EA}$. In fact, the origin is a source singular point and Ω is a positive invariant region. So by the Poincaré-Bendixson theorem, (2.1)-(2.2) have at least one crossing periodic orbit in Ω. If $y_A < y_E$ (*i.e.* $\Delta(x) < 0$), Ω is encircled by a crossing periodic orbit, and $\max\{x_B, x_D\}$ is a low bound of the amplitude. Therefore, we will aim to find $\max\{x_B, x_D\}$ as small as possible such that $\Delta(x) \geq 0$.

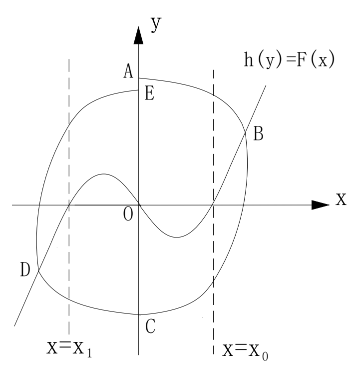

Figure 3. An orbit L_A of (2.1)-(2.2) crossing Σ_F transversally for $x > x_0$ and $x < x_1$.

In the following, we consider the part of L_A on Σ_+ (*i.e.* the arc $\widehat{ABC}$). Denote arcs $\widehat{AB}$ and $\widehat{BC}$ by the graphs of functions $y = \overline{y}(x)$ and $y = \underline{y}(x)$ for $x \in [0, x_B]$ respectively (see Figure 4). Let $\overline{v}(x) = h(\overline{y}) - F(x)$ and $\underline{v}(x) = F(x) - h(\underline{y})$. Then for $x \in (0, x_B)$, one has that

(1) $\overline{y}(x)$ is strictly monotone decreasing, and $h(\overline{y}(x_B)) = F(x_B)$.

(2) $\underline{y}(x) < h^{-1}(F(x)) < \overline{y}(x)$, and $\underline{y}(0) = y_C < 0 < \overline{y}(0) = y_A$.

(3) $\underline{y}(x)$ is strictly monotone increasing, and $h(\underline{y}(x_B)) = F(x_B)$.

(4) $\overline{v}(x) > 0, \underline{v}(x) > 0$ and $\overline{v}(x_B) = \underline{v}(x_B) = 0$.

We give three lemmas to determine an upper bound of amplitude of the unique crossing limit cycle of (2.1)-(2.2) on Σ_+.

Lemma 4.1. *Let $x_B > x_0$, then*

$$H(y_A) > \frac{1}{\overline{v}(x_0)} \int_0^{x_B} F(x) g_+(x) dx + G(x_B) + H(y_B).$$

Proof. Differentiating $\lambda(x,y) = H(y) + G(x)$ in x and taking $y = \overline{y}(x)$ then

$$\frac{d}{dx}[H(\overline{y}(x)) + G(x)] = -\frac{g_+(x)F(x)}{\overline{v}(x)}.$$

Take integral for the above equality from $x = 0$ to $x = x_B$, one has that

$$H(y_B) = -\int_0^{x_B} \frac{g_+(x)F(x)}{\overline{v}(x)} dx - G(x_B) + H(y_A).$$

Since $F(x)(x - x_0) > 0$ for $x \in (0, x_B) \setminus \{x_0\}$, it follows from the property (1) that

$$\begin{aligned} \overline{v}(x) &= h(\overline{y}) - F(x) > \overline{v}(x_0), \quad x \in [0, x_0), \\ \overline{v}(x) &= h(\overline{y}) - F(x) < \overline{v}(x_0), \quad x \in (x_0, x_B]. \end{aligned}$$

Hence $-\frac{g_+(x)F(x)}{\overline{v}(x)} < -\frac{g_+(x)F(x)}{\overline{v}(x_0)}$ for $x \in (0, x_B] \setminus \{x_0\}$. The proof is completed. □

Lemma 4.2. *Let $(H6)$ hold. Then $\underline{v}(x)$ is monotone decreasing on $x \in [0, x_B]$.*

Proof. By $(H1)$ and the Rolle's theorem, there exists at least one $\overline{x} \in (0, x_0)$ such that $F'(\overline{x}) = 0$. Since $\frac{F'(x)}{g_+(x)}$ is monotone increasing on $(\overline{x}, +\infty)$, $\overline{x}$ is the unique zero of $F'(x) = 0$ on $(0, +\infty)$ satisfying $F'(x) < 0$ for $x \in [0, \overline{x})$ and $F'(x) > 0$ for $x \in (\overline{x}, +\infty)$.

Note that $\underline{v}(x)$ satisfies the differential equation

$$\frac{d\underline{v}(x)}{dx} = F'(x) - h'(\underline{y}) \frac{g_+(x)}{\underline{v}(x)}.$$

From the property (3), it is obvious that $\frac{d\underline{v}(x)}{dx} < 0$ for $x \in [0, \overline{x}]$.

Due to $\underline{v}(x) > 0$ and $F'(x) > 0$ for $x \in (\overline{x}, x_B]$, in order to obtain $\frac{d\underline{v}(x)}{dx} < 0$ on $(\overline{x}, x_B]$ it suffices to show that the curve $y = \underline{v}^*(x)$ with $\underline{v}^*(x) = \frac{\underline{v}(x)}{h'(\underline{y})}$ does not meet the isocline $y = \frac{g_+(x)}{F'(x)}$ on such interval. It follows that $\frac{g_+(x)}{F'(x)} > 0$ for $x \in (\overline{x}, +\infty)$ with $\lim_{x \to \overline{x}^+} \frac{g_+(x)}{F'(x)} = +\infty$, and $\frac{g_+(x)}{F'(x)}$ is monotone decreasing on $(\overline{x}, +\infty)$. There are following two possible cases to consider.

Case I. If the curve $y = \underline{v}^*(x)$ touches tangentially the isoline $y = \frac{g_+(x)}{F'(x)}$ at $x_2 \in (\overline{x}, x_B)$, *i.e.* $\underline{v}^*(x_2) = \frac{g_+(x_2)}{F'(x_2)}$ but $\underline{v}^*(x) < \frac{g_+(x)}{F'(x)}$ for $x \in (\overline{x}, x_B] \setminus \{x_2\}$. It is obvious.

Case II. If $y = \underline{v}^*(x)$ crosses transversally $y = \frac{g_+(x)}{F'(x)}$ for the first time at $x_2 \in (\overline{x}, x_B)$. Then $\underline{v}^*(x) > \frac{g_+(x)}{F'(x)}$, *i.e.* $\frac{d\underline{v}(x)}{dx} > 0$ for $x \in U^+(x_2)$, where $U^+(x_2)$ denotes a right-neighborhood of x_2. Since $\frac{g_+(x)}{F'(x)}$ is monotone decreasing on $(\overline{x}, +\infty)$, the curve $y = \underline{v}^*(x)$ will not meet the isocline again. This contradicts with $\underline{v}(x_B) = 0$. Hence $\underline{v}(x)$ is monotone decreasing for $x \in [0, x_B]$. The proof is completed. □

Lemma 4.3. *Let $x_B > x_0$, then*

$$H(y_C) \leq -\frac{1}{\underline{v}(x_0)} \int_0^{x_B} F(x) g_+(x) dx + G(x_B) + H(y_B).$$

Proof. By the similar way to Lemma 4.1, it follows that

$$\frac{d}{dx}[H(\underline{y}(x)) + G(x)] = \frac{g_+(x)F(x)}{\underline{v}(x)}.$$

Taking integral from $x = 0$ to $x = x_B$ in the above equality gives then

$$H(y_C) = G(x_B) + H(y_B) - \int_0^{x_B} \frac{g_+(x)F(x)}{\underline{v}(x)} dx.$$

By Lemma 4.2 and $F(x)(x - x_0) > 0$ for $x \in (0, x_B) \setminus \{x_0\}$, one has that

$$\frac{-g_+(x)F(x)}{\underline{v}(x)} \le \frac{-g_+(x)F(x)}{\underline{v}(x_0)}, \qquad x \in (0, x_B].$$

Hence the conclusion follows. The proof is completed. □

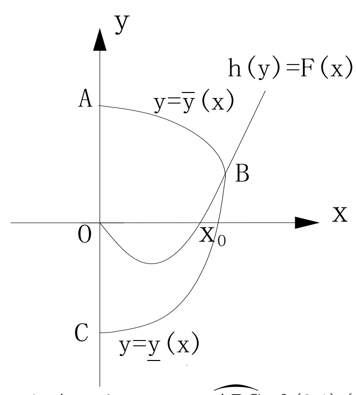

Figure 4. A trajectory arc $\widehat{ABC}$ of (2.1)-(2.2) on Σ_+.

Similarly, for the part of L_A on Σ_- (see the arc $\widehat{CDE}$ in Figure 3), we present the arcs $\widehat{CD}$ and $\widehat{DE}$ by the graphs of functions $y = \underline{y}(x)$ and $y = \overline{y}(x)$ for $x \in [-x_D, 0]$ respectively, and let $\overline{v}(x) = h(\overline{y}) - F(x)$ and $\underline{v}(x) = F(x) - h(\underline{y})$. Then for $x \in (-x_D, 0)$, one has that

(1) $\overline{y}(x)$ is strictly monotone increasing and $h(\overline{y}(-x_D)) = F(-x_D)$.

(2) $\underline{y}(x) < h^{-1}(F(x)) < \overline{y}(x)$ and $\underline{y}(0) = y_C < 0 < \overline{y}(0) = y_E$.

(3) $\underline{y}(x)$ is strictly monotone decreasing and $h(\underline{y}(-x_D)) = F(-x_D)$.

(4) $\overline{v}(x) > 0$, $\underline{v}(x) > 0$ and $\overline{v}(-x_D) = \underline{v}(-x_D) = 0$.

With the similar analysis, we have the following three lemmas to obtain an upper bound for the amplitude of the unique crossing limit cycle of (2.1)-(2.2) on Σ_-.

Lemma 4.4. *Let $x_D > -x_1$, then*

$$H(y_C) > \frac{1}{\underline{v}(x_1)} \int_{-x_D}^{0} F(x)g_-(x)dx + G(-x_D) + H(y_D).$$

Lemma 4.5. *Let $(H6)$ holds. Then $\overline{v}(x)$ is monotone increasing for $x \in [-x_D, 0]$.*

Lemma 4.6. *If $x_D > -x_1$, then*

$$H(y_E) \le -\frac{1}{\overline{v}(x_1)} \int_{-x_D}^{0} F(x)g_-(x)dx + G(-x_D) + H(y_D).$$

Now we are ready to state and prove the result on an explicit upper bound for the amplitude of the unique crossing limit cycle of (2.1)-(2.2) on the plane.

Theorem 4.1. *Let (H1)-(H6) hold. Then there is an upper bound $x^* = \max\{x_B, x_D\}$ for the amplitude of the unique crossing limit cycle of (2.1)-(2.2) such that the limit cycle locates in $\{(x,y) \in \mathbb{R}^2 : |x| < x^*\}$, where x_B and x_D are uniquely determined by $\int_0^{x_B} g_+(x)F(x)dx = 0$ and $\int_{-x_D}^0 g_-(x)F(x)dx = 0$ respectively.*

Proof. We first show that there is a unique x_R^* such that $\int_0^{x_R^*} g_+(x)F(x)dx = 0$.

By $F(x)(x-x_0) > 0$ for $x \in (0,+\infty)\setminus\{x_0\}$, $\int_0^x g_+(x)F(x)dx < 0$ for $0 < x \le x_0$. Since $\int_0^x g_+(x)F(x)dx$ is monotone increasing for $x > x_0$, from $F(+\infty) = +\infty$ there exists sufficiently large x such that $\int_0^x g_+(x)F(x)dx > 0$. Hence by the property of strictly monotone increasing, there is a unique x_R^* such that $\int_0^{x_R^*} g_+(x)F(x)dx = 0$ and $\int_0^x g_+(x)F(x)dx > 0$ for $x > x_R^*$. Moreover, from Lemma 3.4 there is a unique x_B^* such that $\Delta_R(x_B^*) = 0$ and $\Delta_R(x_B) > 0$ for $x_B > x_B^*$. By Lemmas 4.1 and 4.3, one has that

$$\Delta_R(x_B) = H(y_A) - H(y_C) > \left(\frac{1}{\overline{v}(x_0)} + \frac{1}{\underline{v}(x_0)}\right) \int_0^{x_B} F(x)g_+(x)dx. \tag{4.1}$$

Let $x_B = x_R^*$ such that $\int_0^{x_B} F(x)g_+(x)dx = 0$ and then $\Delta_R(x_R^*) > 0$. Namely x_R^* is an upper bound of the amplitude of the unique crossing limit cycle on Σ_+.

Similarly, on Σ_- there exists a unique x_L^* such that $\int_{-x_L^*}^0 g_-(x)F(x)dx = 0$, and from Lemmas 4.4 and 4.6 one has that

$$\Delta_L(x_D) = H(y_C) - H(y_E) > \left(\frac{1}{\overline{v}(x_1)} + \frac{1}{\underline{v}(x_1)}\right) \int_{-x_D}^{0} F(x)g_-(x)dx. \tag{4.2}$$

Let $x_D = x_L^*$ such that $\Delta_L(x_L^*) > 0$. Hence (4.1)-(4.2) imply that the unique limit cycle of (2.1)-(2.2) locates in $\{(x,y) \in \mathbb{R}^2 : |x| < x^*\}$ with $x^* = \max\{x_B, x_D\}$. The proof is completed. □

5. Applications

In this section, two examples including the celebrated nonlinear differential equation model of predator-prey system are presented to illustrate the obtained results.

Example 5.1. Xiao and Ruan [17] studied dynamics of the following predator-prey system with nonmonotonic functional response

$$\begin{cases} \dfrac{dx}{dt} = rx(1 - \dfrac{x}{K}) - \dfrac{xy}{c + x^2}, \\ \dfrac{dy}{dt} = y(\dfrac{\mu x}{c + x^2} - D), \end{cases} \tag{5.1}$$

for $x > 0, y > 0$, where r, c, μ, K and D are positive parameters. Clearly (5.1) has a unique positive equilibrium point $(x_1, y_1) = (\frac{\mu-\sqrt{\mu^2-4cD^2}}{2D}, r(1-\frac{x_1}{K})(c+x_1^2))$ for

$$\mu^2 > \frac{16}{3}cD^2, \qquad x_2 > K > x_3,$$

and

$$x_2 = \frac{\mu + \sqrt{\mu^2 - 4cD^2}}{2D}, \qquad x_3 = \frac{2\mu - \sqrt{\mu^2 - 4cD^2}}{2D}.$$

The authors gave some conditions such that (5.1) has a unique limit cycle in $\{(x, y) : 0 < x < K,\ 0 < y < +\infty\}$ (see Figure 5), and by the variable transformation (5.1) can be transformed into the Liénard-type form.

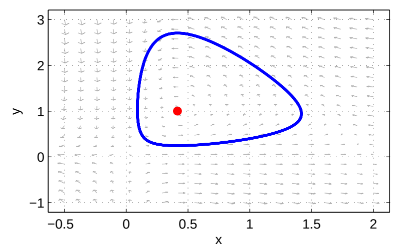

Figure 5. A unique limit cycle of (5.1) and the red dot denotes the equilibrium point (x_1, y_1), where $r = \frac{1}{8-5\sqrt{2}}, c = 1, \mu = 2\sqrt{2}, K = 2$ and $D = 1$.

Now we consider the discontinuous Liénard-type system as follows

$$\begin{cases} \dfrac{dx}{dt} = h(y) - F(x), \\ \dfrac{dy}{dt} = -g(x), \end{cases} \tag{5.2}$$

where $F(x) = \frac{x(x^2+(5-3\sqrt{2})x+14-10\sqrt{2})}{16-10\sqrt{2}}$,

$$g(x) = \begin{cases} \frac{x(x+2)}{\sqrt{2}-1-x}, & 0 \le x < \sqrt{2}-1, \\ \frac{x-1}{\sqrt{2}-1-x}, & \sqrt{2}-3 < x < 0, \end{cases} \qquad h(y) = \begin{cases} e^y - 1, & y \ge 0, \\ y, & -\delta \le y \le 0, \\ ye^{-(y+\delta)}, & y \le -\delta, \end{cases} \tag{5.3}$$

and $\delta > 0$ is a constant.

Obviously, $\Sigma_0 = \{x = 0, y \in \mathbb{R}\}$, $\Sigma_- = \{\sqrt{2} - 3 < x < 0, y \in \mathbb{R}\}$ and $\Sigma_+ = \{0 < x < \sqrt{2} - 1, y \in \mathbb{R}\}$. By computations, there are $x_0 = \frac{3\sqrt{2}-5+\sqrt{10\sqrt{2}-13}}{2} > 0$ and $x_1 = \frac{3\sqrt{2}-5-\sqrt{10\sqrt{2}-13}}{2} < 0$ such that $F(x_0) = F(x_1) = F(0) = 0$, $F(x) > 0$ for $x \in (x_1, 0) \cup (x_0, \sqrt{2} - 1)$, $F(x) < 0$ for $x \in (\sqrt{2} - 3, x_1) \cup (0, x_0)$, and $F'(x) > 0$ for $x \in (x_0, \sqrt{2}-1) \cup (\sqrt{2}-3, x_1)$; $g^- \in C^1((\sqrt{2}-3, 0], \mathbb{R}), g^+ \in C^1([0, \sqrt{2}-1), \mathbb{R})$ with $g(0) = 0 = g(0^+), g(0^-) = \frac{-1}{\sqrt{2}-1}$ and $xg(x) > 0$ for $x \in (\sqrt{2} - 3, 0) \cup (0, \sqrt{2} - 1)$; $h \in C(\mathbb{R}, \mathbb{R})$ satisfying $yh(y) > 0$ for $y \neq 0$, $h'(y) > 0$ for $y \in (-\infty, -\delta) \cup (-\delta, +\infty)$ and $h'(-\delta^+) = 1, h'(-\delta^-) = 1 + \delta$, and $h_+(y) > |h_-(-y)|$ for $0 < y < \delta$, $h_+(y) <$

$|h_-(-y)|$ for $y > M$ with $M > \max\{e^\delta, \delta\}-1$. Hence $(H1)$-$(H5)$ hold. By Theorems 3.1-3.2, (5.2)-(5.3) have a unique stable crossing limit cycle surrounding the origin. Indeed Matlab simulation shows the result shown in Figure 6. From the phase portrait, we observe that the crossing limit cycle is continuous but non-smooth at the intersection points of the limit cycle and Σ_0.

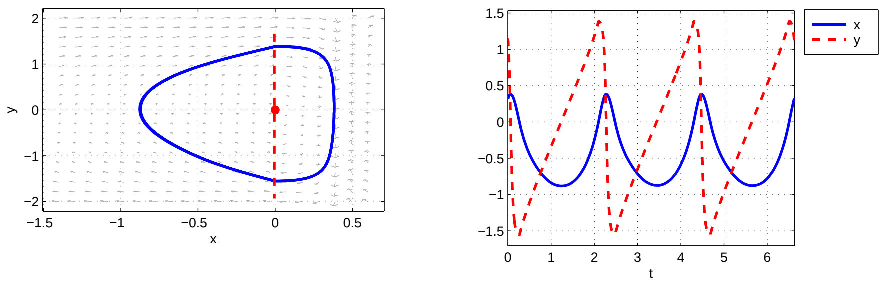

Figure 6. Left: a crossing limit cycle of (5.2)-(5.3), red dot denotes the equilibrium point $O(0,0)$ and red dash line denotes Σ_0; Right: time series $(t, x(t))$ and $(t, y(t))$.

Remark 5.1. Example 5.1 shows that the unique limit cycle of (5.1) is preserved under discontinuity perturbations. This indicates that the oscillatory nature of the predator-prey model is preserved.

Example 5.2. Consider a discontinuous Liénard-type system with the discontinuity line $\Sigma_0 = \{x = 0\}$ as follows

$$\begin{cases} \dfrac{dx}{dt} = h(y) - F(x), \\ \dfrac{dy}{dt} = -g(x), \end{cases} \tag{5.4}$$

where functions F, h and g are of the form

$$F(x) = \begin{cases} -x^2 - 2x, & x < 0, \\ x^2 - x, & x \geq 0, \end{cases} \qquad h(y) = \begin{cases} y^2 + y, & y \geq 0, \\ y, & -1 \leq y \leq 0, \\ ye^{-(y+1)}, & y \leq -1, \end{cases} \tag{5.5}$$

$$g(x) = \begin{cases} 2x - 1, & x < 0, \\ x, & x \geq 0. \end{cases} \tag{5.6}$$

It is easy to see that $F, h \in C(\mathbb{R}, \mathbb{R})$ and $g^- \in C^1((-\infty, 0], (-\infty, 0)), g^+ \in C^1([0, +\infty), (0, +\infty))$ satisfying $g(0^+) = 0, g(0^-) = -1$. There are $x_0 = 1, x_1 = -2$ such that $F(-2) = F(1) = F(0) = 0$, $F(x) > 0$ for $x \in (-2, 0) \cup (1, +\infty)$, $F(x) < 0$ for $x \in (-\infty, -2) \cup (0, 1)$, $F'(x) > 0$ for $x \in (\frac{1}{2}, +\infty) \cup (-\infty, -1)$, $F'(x) < 0$ for $x \in (-1, 0) \cup (0, \frac{1}{2})$ and $F'(0^+) = -1, F'(0^-) = -2$; $yh(y) > 0$ for $y \neq 0$, $h'(y) > 0$ for $y \in (-\infty, -1) \cup (-1, +\infty)$ and $h'(-1^+) = 1, h'(-1^-) = 2$, and we can choose $\delta = 1$ and $M = 2$ such that (2.4) holds. Hence $(H1)$-$(H5)$ hold. By Theorems 3.1-3.2, (5.4)-(5.6) have a unique stable crossing limit cycle surrounding the origin. Indeed Matlab simulation shows the result shown in Figure 7.

Moreover, one can easily verify that $\frac{g_+(x)}{F'(x)}$ and $\frac{g_-(x)}{F'(x)}$ are monotone decreasing for $x > 0$ and $x < 0$ respectively. On Σ_+, it follows that

$$\int_0^x F(s)g_+(s)ds = x^3(\frac{1}{4}x - \frac{1}{3}) = 0.$$

Hence $x = \frac{4}{3}$ is the right upper bound for the amplitude of the unique crossing limit cycle. Similarly, on Σ_- it follows from

$$\int_x^0 F(x)g_-(x)dx = x^2(\frac{1}{2}x^2 + x - 1) = 0$$

that $x = -\sqrt{3}-1$ is the left upper bound of the amplitude. Hence by Theorem 4.1, the unique crossing limit cycle of (5.4)-(5.6) locates in $\{(x,y) \in \mathbb{R}^2 : |x| < \sqrt{3}+1\}$.

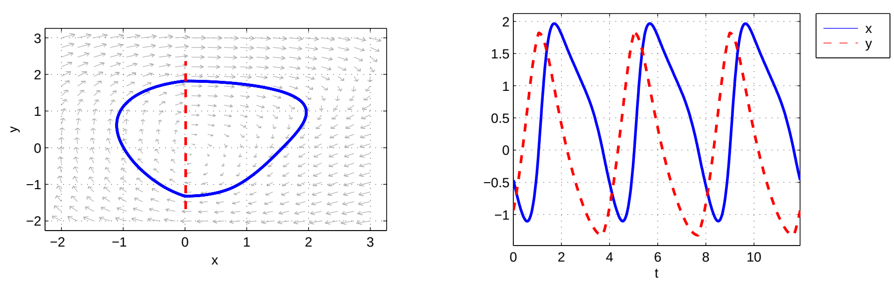

Figure 7. Left: a crossing limit cycle of (5.4)-(5.6) and red dash line denotes Σ_0; Right: time series $(t, x(t))$ and $(t, y(t))$.

6. Concluding remarks

In the above sections, we have mainly investigated the existence, uniqueness, stability and relative position of a crossing limit cycle for a discontinuous nonlinear Liénard-type differential system with two zones separated by Σ_0. By adopting the Filippov theory to define orbits of the system when they intersect Σ_0 such that the orbits can be concatenated in a natural way. Firstly, we presented the properties of solutions and the left and right Poincaré mappings as well. Then by Poincaré mapping method and analysis techniques, we provided two criteria on the existence, uniqueness and stability of a crossing limit cycle surrounding the origin. Secondly, by geometric analysis we further studied the position of the unique crossing limit cycle. By considering the upper bounds of the amplitude of the limit cycle on Σ_+ and Σ_- respectively, we gave several lemmas to obtain the explicit upper bound. Finally, two examples including an application to the predator-prey model are presented to illustrate the obtained results, and Matlab simulations are also presented to show the agreement between theoretical results and numerical analysis.

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