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Steady-state Solution for Reaction-diffusion Models with Mixed Boundary Conditions*

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Abstract In this paper, we deal with a diffusive predator-prey model with mixed boundary conditions, in which the prey population can escape from the boundary of the domain while predator population can only live in this area and can not leave. We first investigate the asymptotic behaviour of positive solutions and obtain a necessary condition ensuring the existence of positive steady state solutions. Next, we investigate the existence of positive steady state solutions by using maximum principle, the fixed point index theory, L_p-estimation, and embedding theorems, Finally, local stability and uniqueness are obtained by linear stability theory and perturbation theory of linear operators.

Keywords Mixed boundaries, local stability, uniqueness.

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1. Introduction

A number of biologists and mathematicians have devoted themselves to ecological mathematical models and have achieved many impressive results [1, 3, 5–7, 9, 12, 17, 18, 20, 22–25, 27, 28] since Lotka [11] and Volterra [26] established the following classical predator model

$$\begin{cases} \dfrac{\partial u}{\partial t} = r_1 u \left(1 - \dfrac{u}{k_1}\right) - cuv, \\ \dfrac{\partial v}{\partial t} = r_2 v \left(1 - \dfrac{v}{k_2}\right) + mcuv, \end{cases} \tag{1.1}$$

where u and v represent the densities of prey and predator, respectively, r_1, r_2 are the intrinsic growth rates of the prey and predator, respectively, $r_1 u(1 - \frac{u}{k_1})$ represents prey's natural growth rate, k_1 represents the maximum number of prey that the environment can support, $r_2 v(1 - \frac{v}{k_2})$ denotes predator's natural growth rate, k_2 is the maximum number of predator that the environment can support. Moreover, cu represents the number of prey that can be captured by the unit predator per

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unit time, which is also called the functional response function, and m is predator's transmission rate after capturing prey. This model has some obvious deficiencies and has been improved by using some suitable functional response function $f(u,v)$ instead of the simple function cu in different applications.

In this paper, we shall investigate a diffusive Lotka-Volterra model under the Neumann boundary condition combined with the third type of boundary condition: the prey species satisfy the Neumann boundary condition, while the predator species satisfy the third type of boundary condition. Let Ω be a bounded domain in $\mathbb{R}^n$ $(n \geqslant 1)$ with smooth boundary $\partial\Omega$, and ω be the outward unit normal vector on $\partial\Omega$. Define $\frac{mv}{\gamma+u^2}$ as the response function of the prey species, and $\frac{cu}{\gamma+u^2}$ as the response function of the predator species after predation. Let d_1 and d_2 be the diffusion coefficients of the prey and predator, respectively, which implies that when the population is unevenly distributed in the region, the species spontaneously return to a uniform state. Denote by a positive constant α the proportion of predators escaping from the regional boundary $\partial\Omega$. Thus, we shall investigate the following diffusive prey-predator model

$$\begin{cases} \dfrac{\partial u}{\partial t} - d_1\Delta u = au - u^2 - \dfrac{muv}{\gamma+u^2}, & (x,t) \in \Omega \times (0,\infty), \\ \dfrac{\partial v}{\partial t} - d_2\Delta v = bv - v^2 + \dfrac{cuv}{\gamma+u^2}, & (x,t) \in \Omega \times (0,\infty), \\ \dfrac{\partial u}{\partial \omega} = \dfrac{\partial v}{\partial \omega} + \alpha v = 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), & x \in \Omega, \\ v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.2}$$

where $u_0(x)$, $v_0(x) : \Omega \to \mathbb{R}^n$ are continuous initial functions. The steady state problem of (1.2) is

$$\begin{cases} -d_1\Delta u = au - u^2 - \dfrac{muv}{\gamma+u^2}, & x \in \Omega, \\ -d_2\Delta v = bv - v^2 + \dfrac{cuv}{\gamma+u^2}, & x \in \Omega, \\ \dfrac{\partial u}{\partial \omega} = \dfrac{\partial v}{\partial \omega} + \alpha v = 0, & x \in \partial\Omega. \end{cases} \tag{1.3}$$

One of our purposes is to investigate the existence of positive steady state solutions of system (1.2), which is equivalent to the existence of positive solutions of system (1.3). We first notice that (1.3) has a trivial solution $\mathbf{0} = (0,0)$ and a semi-trivial solutions $u_* = (a,0)$. We shall employ the in-cone fixed point index theory to calculate indexes at points $\mathbf{0}$ and u_*, and then make use of the maximum principle, L_p-estimation and embedding theorem to show that system (1.3) has at least one positive solution.

The paper is organized as follows: In Section 2, we give the necessary conditions ensuring the existence of positive steady state solutions and the asymptotic behaviour of the positive solution. In Section 3, we investigate the asymptotic behaviours of positive solutions of (1.2) and give some necessary conditions ensuring the existence of positive steady-state solutions of (1.2). Section 4 is devoted to the existence of positive steady state solutions of system (1.2). Section 5 is devoted to the local stability and uniqueness of the positive solution of system (1.3).

2. Preliminaries

In this section, we review some definitions and lemmas, which would be used in the subsequent analysis. Denote by $L^p(\Omega)$ $(p \geq 1)$ the Lebesgue space of integrable functions defined on Ω, and by $W^{k,p}(\Omega)$ $(k \geq 0,\ p \geq 1)$ the Sobolev space of the L^p-functions $f(x)$ defined on Ω whose derivatives $\frac{\mathrm{d}^n}{\mathrm{d}x^n} f$ $(n = 1, \dots, k)$ also belong to $L^p(\Omega)$. In particular, we rewrite $W^{k,2}(\Omega)$ as $H^k(\Omega)$.

To obtain the existence of positive solutions of equation (1.3), we shall calculate the index of a operator at the trivial solution and non-trivial solutions in order to apply the in-cone fixed point index theory. Similar to the method in [19] and [21], the indices of operators in spaces with mixed boundary conditions is defined as follows. Let $X = W_1 \times W_2$, $Y = C(\bar{\Omega}) \times C(\bar{\Omega})$, and $Z = C^2(\bar{\Omega}) \times C^2(\bar{\Omega})$, where

$$W_1 = \left\{ u \in C(\bar{\Omega}) \,\middle|\, \frac{\partial u}{\partial \omega} = 0, x \in \partial\Omega \right\}, \quad W_2 = \left\{ v \in C(\bar{\Omega}) \,\middle|\, \frac{\partial v}{\partial \omega} + \alpha v = 0, x \in \partial\Omega \right\}.$$

Let E be a closed convex subset in $P = \{(u,v) \in X \mid u, v \geqslant 0, x \in \Omega\}$ and $K : X \to X$ be a Fréchet differentiable compact operator satisfying $K(E) \subseteq E$. For a given $\phi \in X$ satisfying $K\phi = \phi$, define a wedge W_ϕ by

$$W_\phi = \text{closure}\left\{\psi \in X \mid \phi + s\psi \in E,\ s > 0\right\}.$$

Assume that X_ϕ is the biggest subspace in W_ϕ. If there exists a subspace Y_ϕ in X such that $X = X_\phi \oplus Y_\phi$, then computing the index of operator K at ϕ is equivalent to calculating the eigenvalues of the associated eigenvalue problems of $K'(\phi)$ in spaces X_ϕ and Y_ϕ, respectively, where $K'(\phi)$ denotes the Fréchet derivative operator at the point ϕ. If $K'(\phi)$ has no fixed points in W_ϕ, then the index of operator K at point ϕ, denoted by $\mathrm{Index}(K, \phi)$, exists. Let $T : X \to Y_\phi$ be a projection from X onto Y_ϕ along X_ϕ. If $TK'(\phi)$ has an eigenvalue greater than 1, then $\mathrm{Index}(K, \phi) = 0$. Otherwise,

$$\mathrm{Index}\,(K, \phi) = \mathrm{Index}_{X_\phi}\,(K'(\phi), 0) = (-1)^r,$$

where $\mathrm{Index}_{X_\phi}\,(K'(\phi), 0)$ denotes the index of linear operator K at the point 0 in space X_ϕ and r is the number of eigenvalues of $K'(\phi)$ restricted to the space X_ϕ satisfying $\lambda > 1$.

For convenience, let $\lambda_1(p) < \lambda_2(p) \leqslant \lambda_3(p) \leqslant \cdots$ be the eigenvalues of the following eigenvalue problem

$$\begin{cases} -\Delta\eta + p(x)\eta = \lambda\eta, & x \in \Omega, \\ \dfrac{\partial \eta}{\partial \omega} + \alpha\eta = 0, & x \in \partial\Omega, \end{cases}$$

where p is some suitable continuous function. In particular, set $\mu_i = \lambda_i(0)$ for $i \in \mathbb{N}$.

In addition, we also need the following three lemmas (see [2] for the detailed proof). Consider

$$\begin{cases} -d_2\Delta v + p(x)v = bv - v^2, & x \in \Omega, \\ \dfrac{\partial v}{\partial \omega} + \alpha v = 0, & x \in \partial\Omega. \end{cases} \tag{2.1}$$

Lemma 2.1 ([2])**.** *Suppose $\|p(x)\|_\infty \leqslant C$ for some positive constant C.*

(i) If $b \leqslant d_2\lambda_1\left(\frac{p(x)}{d_2}\right)$*, then* $v=0$ *is the unique non-negative solution of* (2.1).

(ii) If $b > d_2\lambda_1\left(\frac{p(x)}{d_2}\right)$*, then* (2.1) *has exactly one positive solution.*

Let θ_b be the unique positive solution of (2.1) with $p(x)\equiv 0$ in the case where $b > d_2\mu_1$, and define a linear operator L by $Lv = -d_2\Delta v + (2\theta_b - b)v$ for $v \in C^2(\Omega)\cap C(\bar{\Omega})$ subject to the following boundary condition $\frac{\partial v}{\partial \omega} + \alpha v = 0$ on $\partial\Omega$. Consider the following initial value problem

$$\begin{cases} \dfrac{\partial v}{\partial t} - d_2\Delta v = bv - v^2, & (x,t)\in\Omega\times(0,\infty),\\ \dfrac{\partial v}{\partial \omega} + \alpha v = 0, & (x,t)\in\partial\Omega\times(0,\infty),\\ v(x,0) = v_0(x), & x\in\Omega, \end{cases} \tag{2.2}$$

where $v_0(x)\geqslant 0$ and $v_0\not\equiv 0$. Now, we denote by $v_b(x,t)$ the unique positive solution of (2.2).

Lemma 2.2 ([2]). *Suppose that* $b > d_2\mu_1$*, then* $v_b(x,t)$ *converges to* θ_b *uniformly on* $\bar{\Omega}$ *as* $t\to\infty$*. If* $b < d_2\mu_1$*, then* (2.2) *has a globally asymptotically stable trivial solution* $\boldsymbol{0} = (0,0)$.

Lemma 2.3 ([2]). *(i) The mapping from* b *to* θ_b *is strictly increasing, and is continuously differentiable in* $(d_2\mu_1, +\infty)$*;*

(ii) On Ω*,* θ_b *tends to* 0 *as* b *approaches* $d_2\mu_1$*. Moreover,* $0<\theta_b<b$*;*

(iii) All the eigenvalues of operator L *are positive.*

3. Asymptotic behaviour

In the section, we investigate the asymptotic behaviour of the positive solutions.

Theorem 3.1. *Denote by* $(u(x,t),v(x,t))$ *the non-negative solution of* (1.2)*, if* $b + C\leqslant d_2\mu_1$ *and* $C = \frac{c}{2\sqrt{\gamma}}$*, then* $(u(x,t),v(x,t))\to(a,0)$ *as* $t\to\infty$.

Proof. The second equation of (1.2) satisfies $\frac{\partial v}{\partial t} - d_2\Delta v \leqslant (b+C)\,v - v^2$ for $(x,t)\in\Omega\times(0,\infty)$. It follows from $b+C\leqslant d_2\mu_1$ and Lemma 2.3 that $v(x,t)\to 0$ as $t\to\infty$, and hence that there exists $T>0$ such that $0\leqslant v(x,t)<\varepsilon<a$ for $t>T$. Note that the first equation of (1.2) can be rewritten as

$$\frac{\partial u}{\partial t} - d_1\Delta u = au - u^2 - \frac{muv}{\gamma+u^2} \leqslant au - u^2, \quad (x,t)\in\Omega\times(0,\infty).$$

Then we have $0\leqslant u(x,t)\leqslant u_a(x,t)$, where $u_a(x,t)$ is the solution of the following system

$$\begin{cases} \dfrac{\partial u}{\partial t} - d_1\Delta u = au - u^2, & (x,t)\in\Omega\times(0,\infty),\\ \dfrac{\partial u}{\partial \omega} = 0, & (x,t)\in\partial\Omega\times(0,\infty),\\ u(x,0) = u_0(x), & x\in\Omega. \end{cases} \tag{3.1}$$

Note that $u_a(x,t)\to a$ as $t\to\infty$, then we have $\limsup_{t\to\infty} u(x,t)\leqslant a$. In addition, the first equation of (1.2) becomes $\frac{\partial u}{\partial t} - d_1\Delta u \geqslant au - u^2 - \frac{m}{2\sqrt{\gamma}}\varepsilon$ for $(x,t)\in\Omega\times(T,\infty)$.

Let $\varepsilon \to 0^+$, using the same method as above, we obtain $\liminf_{t\to\infty} u(x,t) \geqslant a$. Thus, $(u(x,t), v(x,t)) \to (a,0)$ as $t \to \infty$.

4. Existence of steady-state solutions

Assume that (u,v) is a positive solution of system (1.3), then v satisfies

$$\begin{cases} bv - v^2 < -d_1\Delta v \leqslant (b+C)v - v^2, & x \in \Omega, \\ \dfrac{\partial v}{\partial \omega} + \alpha v = 0, & x \in \partial\Omega. \end{cases} \tag{4.1}$$

If $b > d_1\mu_1$, it follows from Lemma 2.1 that $\theta_b < v \leqslant \theta_{b+C} < b + C$. Thus, the first equation of system (1.3) can be rewritten as

$$\begin{cases} -d_1\Delta u + \dfrac{m\theta_b u}{\gamma + u^2} \leqslant au - u^2, & x \in \Omega, \\ \dfrac{\partial u}{\partial \omega} = 0, & x \in \partial\Omega. \end{cases} \tag{4.2}$$

It follows from the existence of positive solutions that $a > d_1\lambda_1\left(\frac{m\theta_b}{d_1\gamma}\right)$. From Harnack's inequality and the maximum principle, it follows that $u < a$. Hence, we have the following result.

Theorem 4.1. *If system* (1.3) *with* $b > d_1\mu_1$ *has a positive solution* (u,v)*, then* $a > d_1\lambda_1\left(\frac{m\theta_b}{d_1\gamma}\right)$, $0 < u < a$ *and* $\theta_b < v \leqslant \theta_{b+C} < b + C$.

System (1.3) is equivalent to the following system

$$\begin{cases} u = K_1(u,v) \triangleq (M - d_1\Delta)^{-1}\left(au - u^2 - \dfrac{muv}{\gamma + u^2} + Mu\right), \\ v = K_2(u,v) \triangleq (M - d_2\Delta)^{-1}\left(bv - v^2 + \dfrac{cuv}{\gamma + u^2} + Mv\right). \end{cases}$$

Define a differentiable compact operator $K\colon X \to X$ as $K(u,v) = (K_1(u,v), K_2(u,v))$. By the fixed point theory, we shall calculate the indices of the operator K at the trivial solution and semi-trivial solution, respectively. If the sum of these indexes is not equal to 1, then there exists a constant $R > 0$ such that K has a positive fixed point in the spherical area $B_R(0)$, which is different from the trivial and semi-trivial solutions. Therefore, to investigate the existence of positive solutions of (1.3), it suffices to solve the fixed point problem of operator K.

First of all, we choose a closed convex set

$$F = \left\{(u,v) \in P \,\middle|\, u + \frac{mv}{\gamma + u^2} \leqslant a + M, v \leqslant b + M\right\},$$

and its subset

$$E = \left\{(u,v) \in P \,\middle|\, u + \gamma^{-1}mv \leqslant a + M, v \leqslant b + M\right\},$$

which is also a closed convex set. To prove $K(F) \subseteq F$, we only need to prove the following proposition.

Proposition 4.1. $K(E) \subseteq E$.

Proof. For $(u,v) \in E$ and $(p,q) = K(u,v)$, we have

$$\begin{cases} (-d_1\Delta + M)\,p = (a+M)\,u - u^2 - \dfrac{muv}{\gamma+u^2}, & x \in \Omega, \\ (-d_2\Delta + M)\,q = (b+M)\,v - v^2 + \dfrac{cuv}{\gamma+u^2}, & x \in \Omega, \\ \dfrac{\partial p}{\partial \omega} = \dfrac{\partial q}{\partial \omega} + \alpha q = 0, & x \in \partial\Omega. \end{cases} \tag{4.3}$$

To obtain $(p,q) \in E$, it suffices to prove $\tilde{p} \geqslant 0$ and $\tilde{q} \geqslant 0$, where $\tilde{p} = a + M - \gamma^{-1}mq - p$ and $\tilde{q} = b + M - q$. It follows from (4.3) that

$$\begin{cases} (-d_1\Delta + M)\,(a+M-\gamma^{-1}mq-\tilde{p}) = (a+M)\,u - u^2 - \dfrac{muv}{\gamma+u^2}, & x \in \Omega, \\ (-d_2\Delta + M)\,(b+M-\tilde{q}) = (b+M)\,v - v^2 + \dfrac{cuv}{\gamma+u^2}, & x \in \Omega. \end{cases}$$

and hence that

$$\begin{cases} d_1\gamma^{-1}m\Delta q + d_1\Delta\tilde{p} + M\left(a+M-\gamma^{-1}mq-\tilde{p}\right) = (a+M)u - u^2 - \dfrac{muv}{\gamma+u^2}, & x \in \Omega, \\ d_2\Delta\tilde{q} + M\,(b+M-\tilde{q}) = (b+M)v - v^2 + \dfrac{cuv}{\gamma+u^2}, & x \in \Omega. \end{cases}$$

This together with the second equation of (4.3) implies that

$$\begin{cases} -d_1\Delta\tilde{p} + M\tilde{p} = \dfrac{d_2}{d_1}m\left[v^2 - \left(b+M+\dfrac{cu}{\gamma+u^2}\right)v\right] + u^2 \\ \qquad - \left(a+M-\dfrac{mv}{\gamma+u^2}\right)u + \left(a+M-\dfrac{d_2-d_1}{d_2}\gamma^{-1}mq\right)M, & x \in \Omega, \\ -d_2\Delta\tilde{q} + M\tilde{q} = v^2 - \left(b+M+\dfrac{cu}{\gamma+u^2}\right)v + M\,(b+M)\,, & x \in \Omega, \\ \dfrac{\partial\tilde{p}}{\partial\omega} = \alpha mq \geqslant 0, & x \in \partial\Omega, \\ \dfrac{\partial\tilde{q}}{\partial\omega} + \alpha\tilde{q} = \alpha\,(M+b) > 0, & x \in \partial\Omega. \end{cases} \tag{4.4}$$

Choose M sufficiently large such that $(b+M)\,M - \frac{1}{4}(b+C+M)^2 > 0$, then we have

$$\begin{aligned} & v^2 - \left(b+M+\frac{cu}{\gamma+u^2}\right)v + (b+M)\,M \\ & \geqslant v^2 - (b+M+C)\,v + (b+M)\,M \\ & \geqslant (b+M)\,M - \frac{1}{4}(b+C+M)^2 > 0. \end{aligned}$$

In view of the second equation of (4.4) and the maximum principle, we obtain $\tilde{q} > 0$, which is equivalent to $q \leqslant b + M$.

Denote $\delta = \frac{1}{2} - \frac{2d_2-d_1}{d_2\gamma}m > 0$ with $0 < m < 1$ and suppose M is large enough such that $\delta M \geqslant \frac{d_2-d_1}{d_2}mb$ and hence that

$$M\left(a+M-\frac{d_2-d_1}{d_2\gamma}mq\right) \geqslant M\left[a - \frac{d_2-d_1}{d_2\gamma}mb + \left(1-\frac{d_2-d_1}{d_2\gamma}m\right)M\right] \geqslant M\,(a+\delta M)\,.$$

Set

$$G(x,y)=\frac{d_2}{d_1\gamma}mx^2+\frac{mxy}{\gamma+y^2}+y^2-\frac{d_2}{d_1}m\left(b+C+M\right)x-\left(a+M\right)y+M\left(a+\delta M\right)$$

for $(x,y)\in R$, where $R=\{(x,y)\,|\,g_i(x,y)\geqslant 0, i=1,2,3,4\}$ is a feasible region of the following quadratic programming problem

$$\begin{cases}\min G(x,y),\\ g_1(x,y)=x\geqslant 0,\\ g_2(x,y)=y\geqslant 0,\\ g_3(x,y)=b+M-x\geqslant 0,\\ g_4(x,y)=a+M-mx-y\geqslant 0.\end{cases}\tag{4.5}$$

Similarly to the proof of [19], by the Kuhn-kucller theory for quadratic programming problems, we obtain that for arbitrary $(x,y)\in R$, there exists $\min G(x,y)\geqslant 0$. From the first equation in (4.4) and the maximum principle, it follows that $\tilde{p}\geqslant 0$, that is, $a+M-mq\geqslant p$. Thus, when M is large enough, we have $K(E)\subseteq E$, i.e., $K(F)\subseteq F$. Thus, the proof is completed. □

Next, we shall calculate the indexes of operator K at the points 0 and u_*, respectively. Here, the method for calculating eigenvalues and eigenvectors is similar to that in [16].

Proposition 4.2. **(i)** $\mathrm{Index}(K,0)=0$*;*

(ii) *If* $b>d_2\lambda_1\left(-\frac{ca}{d_2(\gamma+a^2)}\right)$*, then* $\mathrm{Index}(K,u_*)=0$*;*

(iii) *If* $b<d_2\lambda_1\left(-\frac{ca}{d_2(\gamma+a^2)}\right)$*, then* $\mathrm{Index}(K,u_*)=1$*.*

Proof. We start with the calculation of $\mathrm{Index}(K,0)$. At the point $\phi=\mathbf{0}$, we have $W_0=P$, $X_0=0$, $Y_0=X$, $T_0=1$. Let $K'(0)$ be the Fréchet derivative operator of K at $\mathbf{0}$. If $(\xi,\eta)\in W_0-\{0\}$ is an eigenvector of $K'(0)$ associated with eigenvalue λ, then

$$\begin{cases}-d_1\Delta\xi=a\xi+\dfrac{1-\lambda}{\lambda}\left(a+M\right)\xi & x\in\Omega,\\ -d_2\Delta\eta=b\eta+\dfrac{1-\lambda}{\lambda}\left(b+M\right)\eta, & x\in\Omega,\\ \dfrac{\partial\xi}{\partial\omega}=\dfrac{\partial\eta}{\partial\omega}+\alpha\eta=0, & x\in\partial\Omega.\end{cases}\tag{4.6}$$

Note that $(1,0)$ is a solution of (4.6), then we have $a+\frac{1-\lambda}{\lambda}\left(a+M\right)=0$, which means $\lambda>1$. Hence, $\mathrm{Index}(K,0)=0$.

At point $\phi=u_*$, we have $W_{u_*}=\{(u,v)|v\geqslant 0\}$, $X_{u_*}=\{(u,0)|u\in W_1\}$, $Y_{u_*}=\{(0,v)|v\in W_2\}$, $T_{u_*}:(u,v)\to(0,v)$, where the definitions of W_1 and W_2 are the same as them in Section 2. Let $K'(u_*)$ be the Fréchet derivative operator of K at point u_*. If $(\xi,\eta)\in W_{u_*}-\{0\}$ is a fixed point for $K'(u_*)$, then (ξ,η) satisfies

$$\begin{cases}-d_1\Delta\xi=-a\xi-\dfrac{ma}{\gamma+a^2}\eta, & x\in\Omega,\\ -d_2\Delta\eta=b\eta+\dfrac{ca}{\gamma+a^2}\eta, & x\in\Omega,\\ \dfrac{\partial\xi}{\partial\omega}=\dfrac{\partial\eta}{\partial\omega}+\alpha\eta=0, & x\in\partial\Omega.\end{cases}\tag{4.7}$$

Note that $a > 0$ and $b > 0$, then it follows from the second equation of (4.7) that $\eta = 0$, and hence $\xi = 0$. This is a contradiction, and so $\mathrm{Index}(K, u_*)$ exists.

If λ is the eigenvalue of $T_{u_*}K'(u_*)$ with an associated eigenvector $(0, \eta)$, then η satisfies

$$\begin{cases} -d_2\Delta\eta + \left[\dfrac{\lambda-1}{\lambda}\left(b + M + \dfrac{ca}{\gamma + a^2}\right) - \dfrac{ca}{\gamma + a^2}\right]\eta = b\eta, & x \in \Omega, \\ \dfrac{\partial \eta}{\partial \omega} + \alpha\eta = 0, & x \in \partial\Omega. \end{cases} \tag{4.8}$$

It follows that

$$b = d_2\lambda_i\left(\frac{\lambda - 1}{\lambda d_2}\left(b + M + \frac{ca}{\gamma + a^2}\right) - \frac{ca}{d_2(\gamma + a^2)}\right)$$

for some $i \in \mathbb{N}$. Thus, if

$$b > d_2\lambda_1\left(-\frac{ca}{d_2(\gamma + a^2)}\right),$$

then $\lambda > 1$ and hence $\mathrm{Index}(K, u_*) = 0$. If

$$b < \lambda_1\left(-\frac{ca}{d_2(\gamma + a^2)}\right)d_2,$$

then $\lambda < 1$ and hence $\mathrm{Index}(K, u_*) = \mathrm{Index}_{X_{u_*}}(K'(u_*), 0) = (-1)^r$.

Assume that λ_* is the eigenvalue of $K'(u_*)$, and (ξ_*, η_*) is the eigenvector in X_{u_*} associated with the eigenvalue λ_*, then $\eta_* = 0$ and $\xi_* \neq 0$ satisfies

$$\begin{cases} -d_1\Delta\xi_* + \dfrac{a}{\lambda_*}\xi_* = \dfrac{1-\lambda_*}{\lambda_*}M\xi_*, & x \in \Omega, \\ \dfrac{\partial \xi_*}{\partial \omega} = 0, & x \in \partial\Omega, \end{cases} \tag{4.9}$$

and hence

$$\frac{(1-\lambda_*)M - a}{\lambda_*}M = \mu_i \geq 0,\ i \geqslant 1.$$

which implies that $\lambda_* < 1$ and hence that the number of eigenvalues larger than one is zero. Thus, $r = 0$ and

$$\mathrm{Index}(K, u_*) = \mathrm{Index}_{X_{u_*}}(K'(u_*), 0) = (-1)^r = 1.$$

This proves conclusions (ii) and (iii) and hence completes the proof. □

Now, we can state the existence of positive solution of (1.3).

Theorem 4.2. *Assume that*

$$b > d_2\mu_1 \text{ and } a > d_1\lambda_1\left(\frac{m\theta_b}{d_1\gamma}\right),$$

or

$$d_2\lambda_1\left(-\frac{ca}{(\gamma + a^2)\,d_2}\right) < b \leqslant d_2\mu_1,$$

then there exists at lest one positive solution of system (1.3).

Proof. Let M be a sufficiently large positive constant and consider the following system

$$\begin{cases} -d_1\Delta u + t\,(M-1)\,u + u = t\left[(M-1)\,u + f_1(u,v)\right], & x\in\Omega,\\ -d_2\Delta v + tMv = t\left[Mv + f_2(u,v)\right], & x\in\Omega,\\ \dfrac{\partial u}{\partial\omega} = \dfrac{\partial v}{\partial\omega} + \alpha v = 0, & x\in\partial\Omega,\end{cases} \tag{4.10}$$

where

$$f_1(u,v) = \begin{cases} (a+1)u - u^2 - \dfrac{muv}{\gamma+u^2}, & u\geqslant 0, v\geqslant 0,\\ (a+1)u - u^2, & u\geqslant 0, v<0,\\ 0 & u<0,\end{cases}$$

and

$$f_2(u,v) = \begin{cases} bv - v^2 + \dfrac{cuv}{\gamma+u^2}, & v\geqslant 0, u\geqslant 0,\\ bv - v^2, & v\geqslant 0, u<0,\\ 0 & v<0.\end{cases}$$

Let $\Omega_1 = \{x|x\in\Omega, u(x)<0\}$, $\Omega_2 = \{x|x\in\Omega, v(x)<0\}$, and (u,v) be a solution of (4.10). In Ω_1, the solution u of (4.10) are equivalent to

$$\begin{cases} -d_1\Delta u + u = 0, & x\in\Omega_1,\\ u<0, & x\in\Omega_1,\\ \dfrac{\partial u}{\partial\omega} = 0, & x\in\partial\Omega\cap\partial\Omega_1.\end{cases} \tag{4.11}$$

Integrating system (4.11) on Ω_1, we have

$$0 = d_1\int_{\partial\Omega_1\cap\Omega}\frac{\partial u}{\partial\omega}dx = \int_{\Omega_1} u dx$$

and hence $\Omega_1 = \emptyset$. In Ω_2, the solution v of (4.10) can be expressed as

$$\begin{cases} -d_2\Delta v = 0, & x\in\Omega_2,\\ v<0, & x\in\Omega_2,\\ \dfrac{\partial v}{\partial\omega} + \alpha v = 0, & x\in\partial\Omega\cap\partial\Omega_2.\end{cases} \tag{4.12}$$

Integrating (4.12) on Ω_2 results in

$$0 = \int_{\Omega_2}\Delta v dx = \int_{\partial\Omega_2\cap\Omega}\frac{\partial v}{\partial\omega}dx = \alpha\int_{\partial\Omega_2\cap\partial\Omega} v dx.$$

and so $v|_{\partial\Omega_2} = 0$. According to the maximum principle, it is easy to see that $v|_{\Omega_2} = 0$, which contradicts the fact $v<0$ on Ω_2. Hence, $\Omega_2 = \emptyset$. Thus, if (u,v) is a solution of (4.10), then it is a non-negative solution to the following equations:

$$\begin{cases} -d_1\Delta u + t\,(M-1)\,u + u = t\left[(M+a)\,u - u^2 - \dfrac{muv}{\gamma+u^2}\right], & x\in\Omega,\\ -d_2\Delta v + tMv = t\left[(M+b)\,v - v^2 + \dfrac{cuv}{\gamma+u^2}\right], & x\in\Omega,\\ \dfrac{\partial u}{\partial\omega} = \dfrac{\partial v}{\partial\omega} + \alpha v = 0, & x\in\partial\Omega.\end{cases} \tag{4.13}$$

By the maximum principle and the fact that $u \geqslant 0$ and $v \geqslant 0$, we have

$$\max_{\Omega} |u| \leqslant a+1, \quad \max_{\Omega} |v| \leqslant b + \frac{cu}{\gamma + u^2} \leqslant b + C.$$

By the L_p-estimation and the embedding theory, we have

$$|u|_{1+\alpha}, |v|_{1+\alpha} \leqslant C_2 \left(\|u\|_{2,p} + \|v\|_{2,p} \right) \leqslant C \left(\|f_1(u,v)\|_p + \|f_2(u,v)\|_p + \|u\|_p + \|v\|_p \right).$$

This implies there exists $R > 0$ such that every solution (u,v) to (4.10) satisfies $\|(u,v)\|_\infty < R$, and that equation (4.10) has no solution on the boundary $\partial B_R(0)$ for all $t \in [0,1]$.

Denote the operator as

$$K_t(u,v) = \left((-d_1\Delta + t(M-1)+1)^{-1} \left((M-1)u + f_1\right) t, (-d_2\Delta + tM)^{-1} \left(Mv + f_2\right) t \right).$$

Obviously, $K_1 = K$ at $t = 1$. It follows that $\mathrm{Index}\,(K_1, B_R(0)) = \mathrm{Index}\,(K, B_R(0))$. Note that for arbitrary $t \in [0,1]$, there is no solution of (4.10) on the boundary $\partial B_R(0)$, that is, there is no fixed point for K_t on $\partial B_R(0)$. According to the homotopy invariance of the index, we have $\mathrm{Index}\,(K_1, B_R(0)) = \mathrm{Index}\,(K, B_R(0)) = \mathrm{Index}\,(K_0, B_R(0))$. An easy calculation yields that

$$K_0(u,v) = \left((-d_1\Delta + 1)^{-1}, (-d_2\Delta)^{-1} \right).$$

By Lemma 2.1, if $b \leqslant \mu_1 d_2$, $u_* = (a,0)$ is the unique semi-trivial solution of (1.3). If

$$b > \mu_1 d_2 \text{ and } a > d_1\lambda_1 \left(\frac{m\theta_b}{d_1\gamma} \right),$$

then we have

$$0 = \mathrm{Index}(K,0) + \mathrm{Index}(K,u_*) = \mathrm{Index}(K,B_R(0)) = \mathrm{Index}(K_0,B_R(0)) = 1,$$

which is a contradiction. Thus, there exists a positive fixed point of K in $B_R(0)$, that is, system (1.3) has positive solutions. Assume that

$$d_2\lambda_1 \left(-\frac{ca}{(\gamma + a^2)d_2} \right) < b \leqslant d_2\mu_1,$$

then the semi-trivial solution $v_* = (0,\theta_b)$ does not exist and

$$0 = \mathrm{Index}(K,0) + \mathrm{Index}(K,u_*) = \mathrm{Index}(K,B_R(0)) = \mathrm{Index}(K_0,B_R(0)) = 1,$$

which is a contradiction as well. Therefore, there exists a positive fixed point of K in $B_R(0)$, that is, (1.3) has positive solutions. This completes the proof. □

5. Local stability and uniqueness

Theorem 5.1. *If $b > d_2\mu_1$ and there exists a constant $\delta_0 > 0$ such that*

$$d_1\lambda_1 \left(\frac{m\theta_b}{d_1\gamma} \right) < a < d_1\lambda_1 \left(\frac{m\theta_b}{d_1\gamma} \right) + \delta_0,$$

then the unique positive solution of (1.3) *is locally stable.*

Proof. We first prove the uniqueness of the positive solution of (1.3). According to the Crandall-Rabinowitz bifurcation theory [4], the bifurcation point of (1.3) is $(a,u,v)=(a^*,0,\theta_b)$, where

$$a^* = d_1\lambda_1\left(\frac{m\theta_b}{d_1\gamma}\right).$$

Within the neighbourhood of the bifurcation point there exits exactly one positive solution curve of (1.3), which can be expressed as

$$(a,u,v)=(a(s),u(s),v(s))=(a(s),s(\xi_0+\Phi(s)),\theta_b+s(\eta_0+\Psi(s)))$$

for $0<s\ll 1$, where $\eta_0=(-d_2\Delta+2\theta_b-b)^{-1}\left(\frac{c\theta_b}{\gamma}\xi_0\right)$, ξ_0 is a positive eigenvector associated with the eigenvalue a^* such that $\int_\Omega \xi_0^2 dx=1$, $a(s)$, $\Phi(s)$ and $\Psi(s)$ satisfy $a(0)=a^*$, $\Phi(0)=0$ and $\Psi(0)=0$ in C^1.

To prove the uniqueness of the solution to (1.3), we only need to prove that for every sequence $\{a_i\}$ converging to a^* as $i\to\infty$, the solution (u_i,v_i) of (1.3) with $a=a_i$ converges to $(0,\theta_b)$ in Z. Assume on the contrary that the sequence $\{(u_i,v_i)\}$ has a sub-sequence, still denoted by $\{(u_i,v_i)\}$, converging to $(u_0,v_0)\in Z$, and $(u_0,v_0)\neq(0,\theta_b)$. Then (u_0,v_0) is a non-negative solution of

$$\begin{cases} -d_1\Delta u_0 = d_1\lambda_1\left(\dfrac{m\theta_b}{d_1}\right)u_0 - u_0^2 - \dfrac{m u_0 v_0}{\gamma+u_0^2}, & x\in\Omega,\\ -d_2\Delta v_0 = b v_0 - v_0^2 + \dfrac{c u_0 v_0}{\gamma+u_0^2}, & x\in\Omega,\\ \dfrac{\partial u_0}{\partial\omega} = \dfrac{\partial v_0}{\partial\omega}+\alpha v_0 = 0, & x\in\partial\Omega. \end{cases} \tag{5.1}$$

If $(u_0,v_0)=(0,0)$ or $(u_0,v_0)=(a,0)$, then using a similar method to the study of the operator index $\mathrm{Index}(K,v_*)$, we can have a contradiction. If (u_0,v_0) is a positive solution of (5.1), then $a=a^*$ and u_0 satisfies

$$\begin{cases} -d_1\Delta u_0 + \left(u_0+\dfrac{m v_0}{\gamma+u_0^2}\right)u_0 = a^* u_0, & x\in\Omega,\\ \dfrac{\partial u_0}{\partial\omega}=0, & x\in\partial\Omega. \end{cases}$$

By Lemma 2.1, we have

$$\lambda_1\left(\frac{m\theta_b}{d_1\gamma}\right)=\lambda_1\left(\frac{u_0}{d_1}+\frac{m v_0}{d_1(\gamma+u_0^2)}\right).$$

By Theorem 4.1, we have $v_0>\theta_b$, $0<u_0<a$, and hence

$$\lambda_1\left(\frac{m\theta_b}{d_1\gamma}\right)\neq\lambda_1\left(\frac{u_0}{d_1}+\frac{m v_0}{d_1(\gamma+u_0^2)}\right),$$

which yields a contradiction. Therefore, system (1.3) has a unique positive solution.

Next we shall discuss the stability of the unique positive solution of (1.3) by linear stability theory (see [8] for more details). Suppose that the linearisation operators of (1.3) at point $(a,u,v)=(a,0,\theta_b)$ and point $(a,u,v)=(a(s),u(s),v(s))$ are $T_1=T\left(a,0,\theta_b\right)$ and $T_2=T\left(a(s),u(s),v(s)\right)$, respectively. Then $T_1: X\cap Z\to Y$

is continuous. When $a = a(0) = a^*$, $T_1 = T(a, 0, \theta_b) = T(a^*, 0, \theta_b)$, and 0 is an i-simple eigenvalue, which means that there are two functions in the neighbourhood of point $(a^*, 0, \theta_b)$: One is a continuous differentiable mapping $a \to (\alpha(a), \kappa(a))$ from the neighbourhood of the bifurcation point a^* into $R \times Z$; The other is a continuous differentiable mapping $s \to (\beta(s), \chi(s))$ from the neighbourhood of 0 into $R \times Z$. Both of them satisfy the following conditions (see [15] for more details):

(a) $\alpha(a^*) = \beta(0) = 0$, $\kappa(a^*) = \chi(0) = (\xi_0, \eta_0)$;

(b) $T_1\kappa(a) = \alpha(a)\kappa(a)$ with $|a - a^*| \ll 1$;

(c) $T_2\chi(s) = \beta(s)\chi(s)$ with $0 < |s| \ll 1$;

(d) $\alpha'(a^*) \neq 0$ and the symbol of $sa'(s)\alpha'(a^*)$ is opposite to that of $\beta(s)$;

(e) if $s \to 0$, then $\frac{sa'(s)\alpha'(a^*)}{\beta(s)} \to -1$ with $s \neq 0$ and $\beta(s) \neq 0$.

Hence, to investigate the stability of $(u(s), v(s))$ with $0 < s \ll 1$, it suffices to determine the symbol of $sa'(s)\alpha'(a^*)$. Since the eigen-functions of operator T_1 take the form of $(\xi, 0)$ and $(0, \eta)$, the elements in the spectral set $\sigma(T_1)$ are real, and $\sigma(T_1)$ can be given by

$$\sigma(T_1) = \sigma\left(-d_1\Delta - a + \frac{m\theta_b}{\gamma}\right) \cup \sigma(-d_2\Delta - b + 2\theta_b).$$

Let $\kappa(a) = (\kappa_1(a), \kappa_2(a))$, then we have $(\kappa_1(a^*), \kappa_2(a^*)) = (\xi_0, \eta_0)$. It follows from the condition (b) that

$$\begin{cases} -d_1\Delta\kappa_1(a) + \left(\dfrac{m\theta_b}{\gamma} - a\right)\kappa_1(a) = \alpha(a)\kappa_1(a), & x \in \Omega, \\ \dfrac{\partial\kappa_1(a)}{\partial\omega} = 0, & x \in \partial\Omega. \end{cases}$$

Note that ξ_0 is an interior point of $W_1^+ = \{u \in W_1 | u \geqslant 0, x \in \Omega\}$, then $\kappa(a) > 0$ for all a satisfying $|a - a^*| \ll 1$. It follows that

$$\alpha(a) = \lambda_1\left(\frac{m\theta_b - a\gamma}{d_1\gamma}\right)d_1 = a^* - a,$$

and hence that $\alpha'(a^*) = -1$.

Substituting $a = a(s)$, $u = s(\xi_0 + \Phi(s))$ and $v = \theta_b + s(\eta_0 + \Psi(s))$ into system (1.3) yields

$$\begin{cases} -d_1\Delta[s(\xi_0 + \Phi(s))] = a(s)[s(\xi_0 + \Phi(s))] - [s(\xi_0 + \Phi(s))]^2 \\ \qquad\qquad - \dfrac{m[s(\xi_0 + \Phi(s))][\theta_b + s(\eta_0 + \Psi(s))]}{\gamma + [s(\xi_0 + \Phi(s))]^2}, & x \in \Omega, \\ \dfrac{\partial[s(\xi_0 + \Phi(s))]}{\partial\omega} = 0, & x \in \partial\Omega. \end{cases} \tag{5.2}$$

Dividing by s both sides of system (5.2), differentiating it with respect to s at $s = 0$, and noticing that $a(0) = a^*$, $\Phi(0) = 0$, and $\Psi(0) = 0$, we have

$$\begin{cases} -d_1\Delta\Phi'(0) = \left(a^* - \dfrac{m\theta_b}{\gamma}\right)\Phi'(0) + a'(0)\xi_0 - \xi_0^2 - \dfrac{m\xi_0\eta_0}{\gamma}, & x \in \Omega, \\ \dfrac{\partial\Phi'(0)}{\partial\omega} = 0, & x \in \partial\Omega. \end{cases} \tag{5.3}$$

Multiplying by ξ_0 both sides of system (5.3) and integrating it over Ω, we have

$$-d_1\int_\Omega \Delta\Phi'(0)\xi_0 dx=\int_\Omega\left(a^*-\frac{m\theta_b}{\gamma}\right)\Phi'(0)\xi_0 dx+a'(0)\int_\Omega \xi_0^2 dx-\int_\Omega\left(\frac{m\eta_0}{\gamma}+\xi_0\right)\xi_0^2 dx.$$

It follows that

$$\int_\Omega\left[-d_1\Delta\xi_0+\frac{m\theta_b\xi_0}{\gamma}\right]\Phi'(0)dx=a^*\int_\Omega \xi_0\Phi'(0)dx+a'(0),$$

and hence that

$$a'(0)=\int_\Omega\left(\frac{m\eta_0}{\gamma}+\xi_0\right)\xi_0^2 dx>0.$$

This together with the conclusion $\alpha'(a^*)=-1$ implies that $\beta(s)>0$ and hence that the unique positive solution of (1.3) is locally linearly stable when $a^*<a<a^*+\delta_0$. This completes the proof.

Theorem 5.2. *Assume that*

$$a>d_1\lambda_1\left(\frac{m\theta_b}{d_1\gamma}\right), b>d_2\mu_1 \text{ and } 0<c\ll 1$$

or

$$a>d_1\lambda_1\left(\frac{m\theta_b}{d_1\gamma}\right), b>d_2\mu_1 \text{ and } \gamma\gg 1,$$

then (1.3) *has exactly one positive solution, which is linearly stable.*

Proof. Here, we only discuss the first case because the second can be dealt with analogously. First, we shall prove the uniqueness of the positive solution of (1.3). When $c=0$, system (1.3) becomes

$$\begin{cases}-d_1\Delta u=au-u^2-\dfrac{muv}{\gamma+u^2}, & x\in\Omega,\\ -d_2\Delta v=bv-v^2, & x\in\Omega,\\ \dfrac{\partial u}{\partial\omega}=\dfrac{\partial v}{\partial\omega}+\alpha v=0, & x\in\partial\Omega,\end{cases}\tag{5.4}$$

which has exactly one positive solution (u^*,θ_b) when $a>\lambda_1\left(\frac{m\theta_b}{d_1\gamma}\right)d_1$ and $b>d_2\mu_1$. Define a function F: $R^+\times\left(C^2(\bar\Omega)\cap W_1(\bar\Omega)\right)\times\left(C^2(\bar\Omega)\cap W_2(\bar\Omega)\right)\to C(\bar\Omega)\times C(\bar\Omega)$ as

$$F(c,u,v)=\left(d_1\Delta u+au-u^2-\frac{muv}{\gamma+u^2}, d_2\Delta v+bv-v^2+\frac{cuv}{\gamma+u^2}\right).$$

Solving solutions of (1.3) is equivalent to solve $F(c,u,v)=0$. Moreover, it is easy to see that (u^*,θ_b) is a unique positive solution of $F(0,u,v)=0$ when $a>\lambda_1\left(\frac{m\theta_b}{d_1\gamma}\right)d_1$ and $b>d_2\mu_1$. The linearized operator of F at $(0,u^*,\theta_b)$, denoted by

$$G=D_{(u,v)}F(0,u^*,\theta_b):\left(C^2(\bar\Omega)\cap W_1(\bar\Omega)\right)\times\left(C^2(\bar\Omega)\cap W_2(\bar\Omega)\right)\to C(\bar\Omega)\times C(\bar\Omega),$$

is defined by

$$G(\xi,\eta)=\left(d_1\Delta\xi+\left(a-2u^*-\frac{m\theta_b}{\gamma+u^{*2}}+\frac{2m\theta_b u^{*2}}{(\gamma+u^{*2})^2}\right)\xi-\frac{mu^*}{\gamma+u^{*2}}\eta, d_2\Delta\eta+(b-2\theta_b)\eta\right).$$

Suppose that there exists $0 < \varepsilon_0 \ll 1$ such that for every sequence $\{(c_i, u_i, v_i)\}$ of zero points of F satisfying $c_i \to 0$ as $i \to \infty$, we have $\|u_i - u^*\|_\infty + \|v_i - \theta_b\|_\infty \geqslant \varepsilon_0$. Then, it is easy to see that $\theta_b < v_i < \theta_{(b+C_i)}$ and

$$C_i = \max\left\{\frac{c_i}{2a}, \frac{c_i}{2\sqrt{\gamma}}\right\} \to 0 \ as \ i \to \infty.$$

Thus, $v_i \to \theta_b$ uniformly on $\bar{\Omega}$ as $i \to \infty$ and so $\|u_i - u^*\|_\infty \geqslant \frac{\varepsilon_0}{2}$. Note that

$$\begin{cases} -d_1\Delta u_i = au_i - u_i^2 - \dfrac{mu_iv_i}{\gamma + u_i^2}, & x \in \Omega, \\ \dfrac{\partial u_i}{\partial \omega} = 0, & x \in \partial\Omega. \end{cases}$$

According to the elliptic equation regular theory, there exists a subsequence in $C^2(\bar{\Omega})$, still denoted by $\{u_i\}$, such that $u_i \to u_0$ and $u_0 \geqslant 0$. Obviously,

$$\begin{cases} -d_1\Delta u_0 = au_0 - u_0^2 - \dfrac{mu_0\theta_b}{\gamma + u_0^2}, & x \in \Omega, \\ \dfrac{\partial u_0}{\partial \omega} = 0, & x \in \partial\Omega. \end{cases} \tag{5.5}$$

Let $\bar{u}_i = u_i / \|u_i\|_\infty$. If $u_0 = 0$, then $\bar{u}_i \to q$ in $C^2(\bar{\Omega})$ with $q \geqslant 0$ and $q \not\equiv 0$ satisfying the following equation

$$\begin{cases} -d_1\Delta q = aq - \dfrac{m\theta_b q}{\gamma}, & x \in \Omega, \\ \dfrac{\partial q}{\partial \omega} = 0, & x \in \partial\Omega. \end{cases}$$

It follows that $a = d_1\lambda_1\left(\frac{m\theta_b}{d_1\gamma}\right)$, which contradicts the assumptions. Hence, u_0 is the non-negative non-trivial solution of system (5.5). From Harnack's inequality, it follows that u_0 is a positive solution in (5.5). The uniqueness of the positive solution of (5.4) implies that $u_0 = u^*$, which contradicts $\|u_i - u^*\|_\infty \geqslant \frac{\varepsilon_0}{2}$. Therefore, for every sequence $\{(c_i, u_i, v_i)\}$ of zero points of F satisfying $c_i \to 0$ as $i \to \infty$, we have $\|u_i - u^*\|_\infty + \|v_i - \theta_b\|_\infty \to 0$ as $i \to \infty$.

Assume that $G(\xi, \eta) = (0, 0)$, we have

$$\begin{cases} -d_1\Delta\xi + \left(2u^* + \dfrac{m\theta_b}{\gamma + u^{*2}} - \dfrac{2m\theta_b u^{*2}}{(\gamma + u^{*2})^2}\right)\xi = a\xi - \dfrac{mu^*}{\gamma + u^{*2}}\eta, & x \in \Omega, \\ -d_2\Delta\eta + (2\theta_b - b)\eta = 0, & x \in \Omega, \\ \dfrac{\partial \xi}{\partial \omega} = \dfrac{\partial \eta}{\partial \omega} + \alpha\eta = 0, & x \in \partial\Omega. \end{cases}$$

According to Lemma 2.2, $[-d_2\Delta + (2\theta_b - b)]^{-1}$ exists and so $\eta = 0$. The above system can be rewritten as

$$\begin{cases} -d_1\Delta\xi + \left(2u^* + \dfrac{m\theta_b}{\gamma + u^{*2}} - \dfrac{2m\theta_b u^{*2}}{(\gamma + u^{*2})^2}\right)\xi = a\xi, & x \in \Omega, \\ \dfrac{\partial \xi}{\partial \omega} = 0, & x \in \partial\Omega. \end{cases}$$

Suppose $\xi \neq 0$, then there is some $i \in \mathbb{N}$ such that

$$a = d_1\lambda_i\left(\frac{2u^*}{d_1} + \frac{m\theta_b}{d_1(\gamma+u^{*2})} - \frac{2m\theta_b u^{*2}}{d_1\left(\gamma+u^{*2}\right)^2}\right) \neq d_1\lambda_1\left(\frac{u^*}{d_1} + \frac{m\theta_b}{d_1(\gamma+u^{*2})}\right). \tag{5.6}$$

Besides, u^* is a positive solution of the following equations

$$\begin{cases} -d_1\Delta u + \left(u^* + \dfrac{m\theta_b}{\gamma+u^{*2}}\right)u = au, & x\in\Omega, \\ \dfrac{\partial u}{\partial\omega} = 0, & x\in\partial\Omega. \end{cases}$$

Therefore, we have

$$a = d_1\lambda_1\left(\frac{u^*}{d_1} + \frac{m\theta_b}{d_1(\gamma+u^{*2})}\right),$$

which contradicts (5.6). Therefore, $\xi = 0$. This implies that the linear operator G is invertible and hence F has a unique zero point (c, u, v) in the neighborhood of the point $(0, u^*, \theta_b)$. Note that (u^*, θ_b) is a unique stable positive solution of system (1.3) with $c = 0$, then we conclude that system (1.3) with $0 < c \ll 1$ has a unique positive solution, which is linearly stable. This completes the proof. □

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