Global Structure of Planar Quadratic Semi-Quasi-Homogeneous Polynomial Systems*

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Abstract This paper study the planar quadratic semi-quasi-homogeneous polynomial systems(short for PQSQHPS). By using the nilpotent singular points theorem, blow-up technique, Poincaré index formula, and Poincaré compaction method, the global phase portraits of such systems in canonical forms are discussed. Furthermore, we show that all the global phase portraits of PQSQHPS can be-classed into six topological equivalence classes.

Keywords Semi-quasi-homogeneous, quadratic system, singular point, global phase portraits.

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1. Introduction

In this paper, we consider the planar polynomial differential systems of the form

$$\dot{x} = P(x, y), \ \dot{y} = Q(x, y), \tag{1.1}$$

where P and Q are polynomials, $\dot{x}, \dot{y}$ are the first derivatives with regard to the time variable t. We call (1.1) polynomial differential system of degree n, if n is the maximum degree of P and Q.

The planar polynomial differential system (1.1) is called to be semi-quasi homogeneous, if there exist $s_1, s_2, d_1, d_2 \in \mathbb{N}^+$ and $d_1 \neq d_2$ such that for any arbitrary $\lambda \in \mathbb{R}^+$,

$$P(\lambda^{s_1} x, \lambda^{s_2} y) = \lambda^{s_1 - 1 + d_1} P(x, y), \ Q(\lambda^{s_1} x, \lambda^{s_2} y) = \lambda^{s_2 - 1 + d_2} Q(x, y). \tag{1.2}$$

We call $w = (s_1, s_2, d_1, d_2)$ the weight vector of system(1.1), s_1 and s_2 the weight exponents of system (1.1), and d_1, d_2 the weight degree with respect to weight exponents s_1 and s_2. In particular, system (1.1) is a semi-homogeneous system when $s_1 = s_2$, and, (1.1) is a quasi-homogeneous system when $d_1 = d_2$.

The weight vector $w = (s_1, s_2, d_1, d_2)$ of planar semi-quasi-homogeneous system (1.1) is not unique (see [19]) and a weight vector $w_m = (s_1^*, s_2^*, d_1^*, d_2^*)$ is called to be a minimal weight vector of (1.1) if any other weight vector $w = (s_1, s_2, d_1, d_2)$ of (1.1) satisfies $s_1^* \leq s_1, \ s_2^* \leq s_2, \ d_1^* \leq d_1, \ d_2^* \leq d_2$.

Obviously, the homogeneous and semi-homogeneous systems are respectively the special types of the quasi-homogeneous and semi-quasi homogeneous systems, which have attracted lots of interests due to their special properties. In fact, a large number of scholars have explored these two kinds of systems from different aspects, such as the centers [6, 7, 12], limit cycles [5], first integral [3], canonical forms [8], phase portraits of the systems [4, 15] and so on. The research on quasi-homogeneous systems has also become a hot topic since the 21st century. In the study of planar quasi-homogeneous polynomial systems, the identification of center from monodromy singularity, the number and location of limit cycles and various analytical first integrals (including polynomial first integral, rational first integral and Liouville first integral) are discussed. For example, the integrability of the planar quasi-homogeneous system was studied in the literature [1, 2, 10, 11], the bifurcation of limit cycles in quasi-homogeneous centers was discussed in [13, 14], the algorithm for obtaining the canonical forms of all given quasi-homogeneous but non-homogeneous systems was provided in [11]. Later, some authors used this algorithm to obtain the canonical form of quasi-homogeneous system of degree 2–6, and made further efforts to analyse the dynamic behavior of them. For instance, the authors of [17] analyzed the quintic quasi-homogeneous system, and, the global phase portraits and the first integral properties of such systems were obtained by further analysis after the canonical forms were given.

The semi-quasi-homogeneous polynomial differential systems are the generalizations of semi-homogeneous and quasi-homogeneous systems, and appear in many fields of the natural sciences. For example, in Newton's laws of motion when the mass of a particle is potion dependent, we have the motion equation $\ddot{x} - a\dot{x}^2 - bx^3 = 0$ which can be changed to the semi-quasi-homogeneous polynomial differential systems $\dot{x} = y, \dot{y} = ay^2 + bx^3$, with the weight vector $w_m = (2, 3, 2, 4)$, see [19] and the references therein.

However, as far as we know, there are a few of results on the semi-quasi-homogeneous polynomial differential systems. Because they do not have such special properties as the semi-homogeneous and quasi-homogeneous systems that it is more difficult to be studied. Here we list two important results about the semi-quasi-homogeneous systems. The authors of [18] gave a criterion for the non-existence of a rational first integral of a semi-quasi-homogeneous system by using Kowalevsky exponent. Zhao studied the limit cycles of semi-quasi homogeneous systems in [21]. He gave some sufficient conditions for the nonexistence and existence of periodic orbits, and, gave a lower bound for the maximum number of limit cycles of such systems.

Very resent, inspired by [11], the authors of [19] established an algorithm to obtain all the semi-quasi homogeneous systems with a given degree and got all the canonical forms of semi-quasi homogeneous systems of degree 2 and 3. After obtaining the canonical forms of the planar semi-quasi-homogeneous systems, we naturally pay more attention to the global dynamic behavior of them. Therefore, in this paper, we will study the global structure of planar quadratic semi-quasi-homogeneous polynomial systems (short for PQSQHPS) on the basis of [19] and give the global phase portrait structures of PQSQHPS in the sense of topological equivalence.

2. Preliminaries and main results

In order to better analyze the phase portraits of PQSQHPS, it is necessary to introduce some preliminaries.

Lemma 2.1 ([21]). *If* (1.1) *is the planar semi-quasi-homogeneous system, and P and Q are coprime polynomials, then the origin is a unique finite singularity of system* (1.1).

Lemma 2.2 ([19]). *If* (1.1) *is a PQSQHPS, then system* (1.1) *has no center.*

The above lemma implies that, when we want to determine the global phase portraits of PQSQHPS, it suffices to analyse the singularities at the origin and at the infinity, as well as the limit cycle (such as the existence and the position). However, as we will see soon, the finite singularity of PQSQHPS is always degenerate and sometimes is of higher-order. Generally, the trajectory structure near the degenerate singularity is much more complicate than the elementary one. In this paper, we are going to use the classical methods, namely the nilpotent singularity theorem and the blow-up technique, to investigate the degenerate singularities, both at the origin and at the infinity, of PQSQHPS.

Let's first introduce the nilpotent singularity theorem and the blow-up technique.

Lemma 2.3 ([20]). *Consider the following system*

$$\dot{x} = P_2(x, y), \quad \dot{y} = y + Q_2(x, y), \tag{2.1}$$

where the origin $O(0,0)$ is an isolated singularity of the system (2.1)*, $P_2(x, y)$ and $Q_2(x, y)$ are analytic in the neighborhood of the point. Given that $y = \phi(x)$ is the solution of equation $y + Q_2(x, y) = 0$ in the neighborhood of the origin, and $\phi(0) = \phi'(0) = 0$. Let $\psi(x) = P_2(x, \phi(x)) = a_m x^m + [x]_{m+1}$, where $a_m \neq 0$, $m \geq 2$, $[x]_{m+1}$ represents the sum of those terms in $\psi(x)$ which degree is not less than $m+1$. Then the following holds:*

  (i) if m is odd and $a_m > 0$, then $O(0,0)$ is an unstable node;

  (ii) if m is odd and $a_m < 0$, then $O(0,0)$ is a saddle;

  (iii) if m is even, then $O(0,0)$ is a saddle-node.

The above lemma can be applied to study the singularity of system (1.1) when it's linear approximation system at this singularity has exactly one zero eigenvalue. When the linear approximation system of (1.1) at the singularity is not vanish but has two zero eigenvalues, we will use the following result.

Lemma 2.4 ([9]). *Consider the system of form*

$$\dot{x} = y + A(x, y), \quad \dot{y} = B(x, y), \tag{2.2}$$

where the origin $O(0,0)$ is an isolated singularity of system (2.2)*, $A(x, y)$ and $B(x, y)$ are analytic in a neighborhood of the origin and $j_1A(0,0,) = j_1B(0,0) = 0$. Given that $y = f(x)$ is the solution of equation $y + A(x, y) = 0$ in a neighborhood of the origin, and let $F(x) = B(x, f(x)), G(x) = (\partial A/\partial x + \partial B/\partial y)(x, f(x))$. If $F(x) = a_m x^m + o(x^m)$ and $G(x) = b_n x^n + o(x^n)$ for $m, n \in N$, $m \geq 2$, $n \geq 1$, $a_m \neq 0$, $b_n \neq 0$, then we have*

(1) if m is odd and $a_m > 0$ then the origin is a saddle;

(2) if m is odd, $a_m < 0$ and

> *(2.1) Either $m < 2n + 1$, or $m = 2n + 1$ and $b_n^2 + 4(n+1)a_m < 0$, then the origin is a center or a focus;*
>
> *(2.2) n is odd and either $m > 2n+1$, or $m = 2n+1$ and $b_n^2 + 4(n+1)a_m \geq 0$, then the phase portrait of the origin consists of one hyperbolic and one elliptic sector;*
>
> *(2.3) n is even and either $m > 2n+1$, or $m = 2n+1$ and $b_n^2 + 4(n+1)a_m \geq 0$, then the origin is a node;*

(3) if m is even, and

> *(3.1) $m < 2n + 1$, then the origin is a cusp;*
>
> *(3.2) $m > 2n + 1$, then the origin is a saddle-node.*

Lemma 2.4 is a part of the conclusion of the nilpotent singularity theorem. The readers are referred to the references [9] (pages 116–117) for more details.

Next, we introduce briefly the blow-up technique (see [16] pages 157–171) which can be employed to study the high order degenerate singularity of system (1.1), i.e., the linear approximation system of (1.1) at the singularity is identically zero. Making the following transformation

$$\begin{cases} x = \mu^\alpha \bar{x}, \\ y = \mu^\beta \bar{y}, \end{cases} \tag{2.3}$$

where $0 \leq \mu \ll 1$, with some suitable positive integers α and β, the high order degenerate singularity of system (1.1) can be blown up into several elementary singularities, see the literature [9] for more details and for the method to search the suitable α and β.

It is well known that, if a limit cycle surround a unique isolate singularity, then the index of this singularity is 1. In order to determine the existence of the limit cycles for PQSQHPS, we need the following result, which can be found in many monograph of ODE such as [9, 20].

Lemma 2.5. *(Poincaré Index Formula) Let Q ba an isolated singular point having the finite sectorial decomposition property. Let e, h and p denote the number of elliptic, hyperbolic, and parabolic sectors of Q, respectively, and suppose that $e + h + p > 0$. Then the index of Q equals to $(e - h)/2 + 1$.*

In the end of this section, we would like to introduce the main results of this paper, which are obtained on the basis of the following lemma.

Lemma 2.6 ([19]). *After the appropriate reversible linear transformations, all the planar quadratic semi-quasi-homogeneous but non-semi-homogeneous systems with the minimal weight vector w_m, can be written as one of the following forms*

(A_1) $\dot{x} = x^2$, $\dot{y} = y^2 + x$, *with* $w_m = (2, 1, 3, 2)$.
(A_{21}) $\dot{x} = y^2 + x$, $\dot{y} = ay^2 + x$, *with* $a(a-1) \neq 0, w_m = (2, 1, 1, 2)$.
(A_{22}) $\dot{x} = y^2 + x$, $\dot{y} = y^2$, *with* $w_m = (2, 1, 1, 2)$.
(A_{23}) $\dot{x} = y^2 + x$, $\dot{y} = x$, *with* $w_m = (2, 1, 1, 2)$.
(A_{24}) $\dot{x} = y^2$, $\dot{y} = y^2 + x$, *with* $w_m = (2, 1, 1, 2)$.
(A_{25}) $\dot{x} = x$, $\dot{y} = y^2 + x$, *with* $w_m = (2, 1, 1, 2)$.

Theorem 2.1. *The global phase portraits of all the systems in Lemma 2.6 are shown below.*
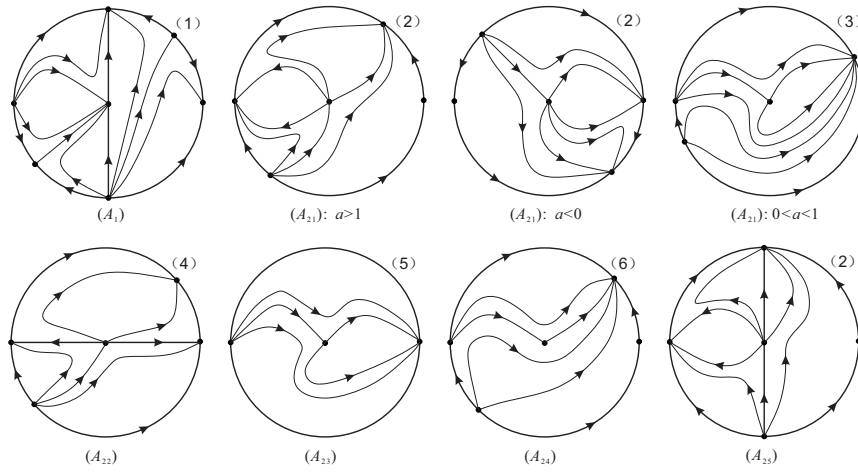
Figure 1. The global phase portraits of PQSQHPS

It is not hard to check that, in Figure 1, the global phase portraits with the same number in the upper right corner are topologically equivalent. Therefore, we can get the following Theorem 2.2 directly from Theorem 2.1.

Theorem 2.2. *The set of all global phase portraits has six topological equivalence classes in PQSQHPS (where the semi-homogeneous and the quasi-homogeneous systems have been excluded), as shown in Figure 2.*
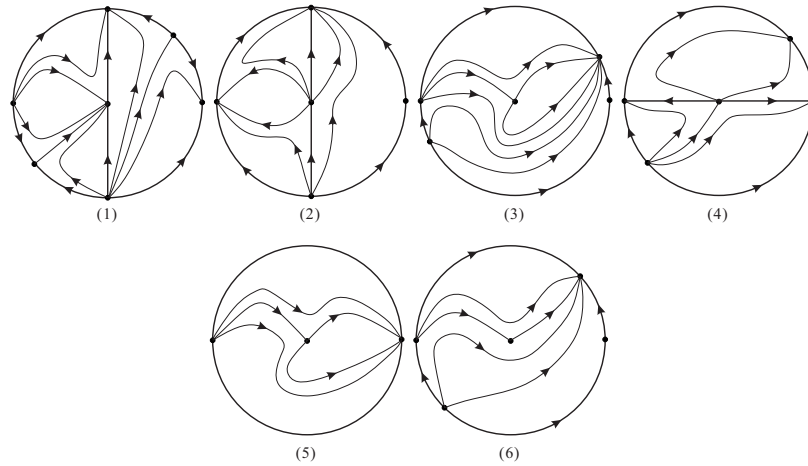
Figure 2. Topological equivalent classes of global phase portraits for PQSQHPS

The following result can be obtained immediately from Theorem 2.2 and Lemma 2.5.

Corollary 2.1. *For any PQSQHPS, the index of the origin (as the unique finite singularity of the system) is zero.*

3. The proof of Theorem 2.1

According to Lemma 2.6, all the canonical forms of PQSQHPS are the systems (A_1), (A_{21})-(A_{25}). Therefore, we will discuss all the global phase portraits of these systems one by one.

Proof.

(1) The phase portrait of (A_1).

Let's first study the singularity of system (A_1) at the origin by applying Lemma 2.4 (with the change of coordinates $x \leftrightarrow y$). Let $A(x,y) = x^2$ and $B(x,y) = y^2$. By $x + B(x,y) = 0$, we get $x = f(y) = -y^2$, and

$$F(y) = A(f(y), y) = y^4, \quad G(y) = \left(\frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2 + x) \right) \Big|_{x=f(y)} = 2y - 2y^2.$$

From Lemma 2.4, we know that the origin is a saddle-node. Notice that $x = 0$ is the invariant straight line of the system, it is easy to get the trajectory direction near the origin, which is shown in Figure 3 (a).
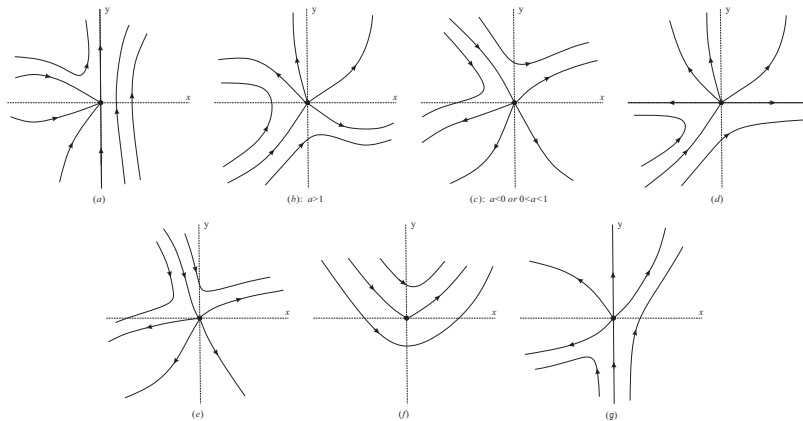
Figure 3. Phase portraits of singularities of systems (A_1), (A_{21})-(A_{25}) at the origin

Next consider the singularities of system (A_1) at the infinity. After making the Poincaré transformation $(x,y) \to (1/z, u/z)$ and the time transformation $d\tau = dt/z$, system (A_1) becomes

$$\begin{cases} \frac{dz}{d\tau} = -z, \\ \frac{du}{d\tau} = u^2 + z - u. \end{cases} \tag{3.1}$$

System (3.1) has two singularities at $A(0,0)$ and at $B(0,1)$ respectively on the u-axis. It is easy to see that A is a stable node and B is a saddle. In order to study the singularity at the infinity on the y-axis direction, by making the Poincaré

transformation $(x, y) \rightarrow (v/z, 1/z)$ and making the time scale $d\tau = dt/z$ for the system (A_1), we get

$$
\begin{cases}
\frac{dz}{d\tau} = -z - z^2 v, \\[2mm]
\frac{dv}{d\tau} = -v + v^2 - zv^2.
\end{cases}
\tag{3.2}
$$

System (3.2) has a singularity at $C(0, 0)$. Moreover, the singularity is a stable node.

On the other hand, since system (A_1) has an invariant straight line $x = 0$, it follows that system (A_1) has no limit cycle. Thus we can get the global phase portrait of (A_1) easily, which was shown in Figure 1.

(2) The phase portraits of (A_{21}).

In order to study the singularity at the origin, we take the transformation $x = \bar{y}, \ y = \bar{x} + \bar{y}$, which change the system (A_{21}) to

$$
\begin{cases}
\dot{\bar{x}} = (a - 1)(\bar{x}^2 + 2\bar{x}\bar{y} + \bar{y}^2), \\[2mm]
\dot{\bar{y}} = \bar{y} + \bar{x}^2 + 2\bar{x}\bar{y} + \bar{y}^2.
\end{cases}
\tag{3.3}
$$

System (3.3) is a special case of system (2.1) with $P_2(\bar{x}, \bar{y}) = (a-1)(\bar{x}^2 + 2\bar{x}\bar{y} + \bar{y}^2)$ and $Q_2(\bar{x}, \bar{y}) = \bar{x}^2 + 2\bar{x}\bar{y} + \bar{y}^2$. By $\bar{y} + Q_2(\bar{x}, \bar{y}) = 0$, we get $\bar{y} = \phi(\bar{x}) = -\bar{x}^2 + 2\bar{x}^3 + \cdots$ and $\psi(\bar{x}) = P_2(\bar{x}, \phi(\bar{x})) = (a-1)\bar{x}^2 + [\bar{x}]_3$. From Lemma 2.3, we know that the origin of system (3.3) is a saddle-node. Since the transformation is non-degenerate and keeps the origin unchanged, the origin of system (A_{21}) is also a saddle-node (see Figure 3 (b) and (c)).

Next consider the singularities of system (A_{21}) at the infinity. By making Poincaré transformation $(x, y) \rightarrow (1/z, u/z)$ and the time transformation $d\tau = dt/z$, system (A_{21}) is changed to

$$
\begin{cases}
\frac{dz}{d\tau} = -u^2 z - z^2, \\[2mm]
\frac{du}{d\tau} = z - uz + au^2 - u^3.
\end{cases}
\tag{3.4}
$$

System (3.4) has two singularities at $A(0, 0)$ and at $B(0, a)$ respectively on the u-axis. It is easy to verify that B is a stable node. So we only need to discuss the singularity of system (3.4) at A. Let $A(z, u) = -uz + au^2 - u^3$ and $B(z, u) = -u^2 z - z^2$. By $z + A(z, u) = 0$, we get $z = f(u) = -au^2 + (1 - a)u^3 + \cdots$, $F(u) = B(f(u), u) = (a - a^2)u^4 + o(u^5)$ and

$$
G(u) = \left(\frac{\partial}{\partial z}(-u^2 z - z^2) + \frac{\partial}{\partial u}(z - uz + au^2 - u^3) \right) \Big|_{z=f(u)} = 2au + o(u^2).
$$

From Lemma 2.4, we know that A is a saddle-node. In order to study the singularity of system (A_{21}) at the infinity on the y-axis direction, we make the Poincaré transformation $(x, y) \rightarrow (v/z, 1/z)$ and the time rescale $d\tau = dt/z$, it follows that

$$
\begin{cases}
\frac{dz}{d\tau} = -az - z^2 v, \\[2mm]
\frac{dv}{d\tau} = 1 + zv - av - zv^2.
\end{cases}
\tag{3.5}
$$

It is easy to find that $(0,0)$ is not a singularity of system (3.5), i.e., system (A_{21}) has no singularity at infinity in the y-axis direction.

From Lemma 2.5, it follows that (A_{21}) has no limit cycle because the index of the unique finite singularity is zero. Finally, combining all of the above information, we can obtain the global phase portrait of (A_{21}), see Figure 1.

(3) The phase portrait of (A_{22}).

We will firstly apply Lemma 2.3 to investigate the singularity of system (A_{22}) at the origin. It follows from $y^2 + x = 0$ that $x = f(y) = -y^2$ and $\psi(y) = y^2$. Therefore, the origin of system (A_{22}) is a saddle-node as being shown in Figure 3 (d).

Next consider the singularities of the system (A_{22}) at the infinity. After making the Poincaré transformation $(x, y) \to (1/z, u/z)$ and the time transformation $d\tau = dt/z$, system (A_{22}) becomes

$$\begin{cases} \frac{dz}{d\tau} = -u^2 z - z^2, \\ \frac{du}{d\tau} = u^2 - u^3 - uz. \end{cases} \tag{3.6}$$

System (3.6) has two singularities at $A(0,0)$ and at $B(0,1)$ respectively on the u-axis. It is easy to check that B is a stable node. Since A is a high order degenerate singularity, we will blow-up it by using the transformation (2.3) with $\alpha = \beta = 1$. To simplify the symbols, we replace u, z, τ in the system (3.6) with x, y, t respectively, and get

$$\begin{cases} \frac{dx}{dt} = x^2 - x^3 - xy, \\ \frac{dy}{dt} = -x^2 y - y^2. \end{cases} \tag{3.7}$$

Then on the coordinate cards $\{\bar{x} = \pm 1\}$, taking $(x, y) = (\pm\mu, \mu\bar{y})$ and time transformation $\mu dt = d\tau$, system (3.7) becomes

$$\begin{cases} \frac{d\mu}{d\tau} = \pm\mu - \mu^2 - \mu\bar{y}, \\ \frac{d\bar{y}}{d\tau} = \mp\bar{y}. \end{cases} \tag{3.8}$$

Clearly, the origin of systems (3.8) is a saddle. On the coordinate cards $\{\bar{y} = \pm 1\}$, we take the transformation $(x, y) = (\pm\mu\bar{x}, \mu)$ and the time transformation $\mu dt = d\tau$ to change system (3.7) into

$$\begin{cases} \frac{d\mu}{d\tau} = \mp\mu - \mu^2\bar{x}^2, \\ \frac{d\bar{x}}{d\tau} = \bar{x}^2. \end{cases} \tag{3.9}$$

It is not hard to see from Lemma 2.3 that the origin of systems (3.9) is a saddle-node. From the properties of the singularities on both coordinate cards $\{\bar{x} = \pm 1\}$ and coordinate cards $\{\bar{y} = \pm 1\}$, we can obtain the phase portrait of system (3.6) near the singularity $A(0,0)$, see Figure 4, where (a) is a graph on the four coordinate cards, (b) is a graph of the singularity $A(0,0)$ of system

(3.7) after blowing up, and, (c) is a graph of the singularity $A(0,0)$ of system (3.7).
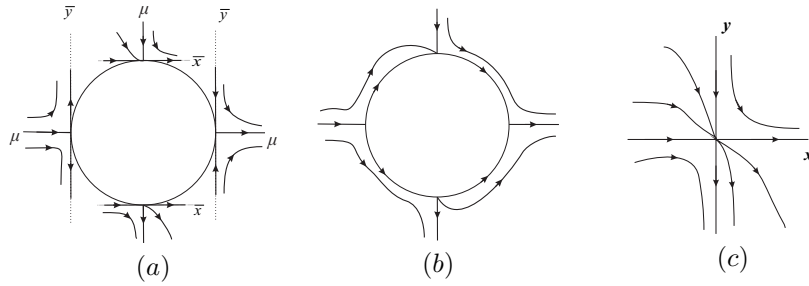

(a) (b) (c)

Figure 4.

Let's study the singularity at the infinity on the y-axis direction. By the transformation $(x,y) \to (v/z, 1/z)$ and $d\tau = dt/z$, system (A_{22}) is changed into

$$\begin{cases} \frac{dz}{d\tau} = -z, \\ \frac{dv}{d\tau} = 1 + zv - v. \end{cases} \qquad (3.10)$$

We find that $(0,0)$ is not a singularity of system (3.10).

On the other hand, system (A_{22}) has no limit cycle because it has an invariant straight line $y = 0$. By combining all of the above information, we can obtain the global phase portrait of (A_{22}). Please see Figure 1.

(4) The phase portrait of (A_{23}).

Similarly, we first utilize Lemma 2.3 to study the singularity of system (A_{23}) at the origin. Through $y^2 + x = 0$, we get $x = f(y) = -y^2$ and $\psi(y) = -y^2$. From Lemma 2.3, we know that the origin is a saddle-node as was shown in Figure 3 (e).

Next consider the singularities of system (A_{23}) at the infinity. After making the Poincaré transformation $(x, y) \to (1/z, u/z)$ and the time transformation $d\tau = dt/z$, system (A_{23}) becomes

$$\begin{cases} \frac{dz}{d\tau} = -u^2 z - z^2, \\ \frac{du}{d\tau} = z - uz - u^3. \end{cases} \qquad (3.11)$$

We notice that on the u-axis, system (3.11) has a degenerate singularity at $A(0,0)$ which can be studied by applying Lemma 2.4. In fact, by setting $A(z,u) = -uz - u^3$, $B(z,u) = -u^2 z - z^2$. By $z + A(z,u) = 0$, we get $z = f(u) = u^3 + u^4 + \cdots$, $F(u) = B(f(u), u) = -u^5 + o(u^6)$ and

$$G(u) = \left(\frac{\partial}{\partial z}(-u^2 z - z^2) + \frac{\partial}{\partial u}(z - uz - u^3) \right) \Big|_{z=f(u)} = -4u^2 + o(u^3).$$

which yields that $A(0,0)$ is a stable node. On the other hand, by direct calculation we find that system (A_{23}) has no singularity at infinity on the y-axis direction.

Finally, taking into account Lemma 2.5 which implies that (A_{23}) has no limit cycle because the index of the unique finite singularity is zero, and, combining all of the above information, we can obtain the global phase portrait of system (A_{23}) which was shown in Figure 1.

(5) The phase portrait of (A_{24}).

Let's first apply Lemma 2.4 to study the singularity of system (A_{24}) at the origin. Similar to the analysis of system (A_1), it is not hard to get that $F(y) = y^2, G(y) = 2y$. From Lemma 2.4, the singularity $(0,0)$ is a cusp as was shown in Figure 3 (f). Next we take the Poincaré transformation $(x, y) \to (1/z, u/z)$ and the time transformation $d\tau = dt/z$, then system (A_{24}) becomes

$$\begin{cases} \frac{dz}{d\tau} = -u^2 z, \\ \frac{du}{d\tau} = z + u^2 - u^3. \end{cases} \tag{3.12}$$

System (3.12) has two singularities at $A(0,0)$ and at $B(0,1)$ respectively on the u-axis. It is easy to check that B is a stable node. Then, we discuss the singularity at A with Lemma 2.4. Let $A(z,u) = u^2 - u^3, B(z,u) = -u^2 z$. By $z + A(z,u) = 0$, we get that $z = f(u) = -u^2 + u^3$, $F(u) = B(f(u), u) = u^4 + o(u^5)$ and

$$G(u) = \left(\frac{\partial}{\partial z}(-u^2 z) + \frac{\partial}{\partial u}(z + u^2 - u^3) \right) \Big|_{z=f(u)} = 2u + o(u^2).$$

From Lemma 2.4, we know that the singularity at $A(0,0)$ is a saddle-node. On the other hand, by a direct computation we find that system (A_{24}) has no singularity at infinity on the y-axis direction. Thus, from the above results and Lemma 2.5 which implies that system (A_{24}) has no limit cycle because the index of the unique finite singularity is zero, we can obtain the global phase portrait of (A_{24}), which was shown in Figure 1.

(6) The phase portrait of (A_{25}).

We first make the linear transformation $x = \bar{y}$, $y = \bar{x} + \bar{y}$ and transform system (A_{25}) into the form

$$\begin{cases} \dot{\bar{x}} = \bar{x}^2 + 2\bar{x}\bar{y} + \bar{y}^2, \\ \dot{\bar{y}} = \bar{y}. \end{cases} \tag{3.13}$$

Then we apply Lemma 2.3 to study the singularity of system (3.13) at the origin. It is easy to obtain that $\psi(\bar{x}) = \bar{x}^2$, so from Lemma 2.3, the origin of system (A_{25}) is a saddle-node as was shown in Figure 3 (g).

Next consider the singularities of system (A_{25}) at the infinity. After making the Poincaré transformation $(x, y) \to (1/z, u/z)$ and the time rescale $d\tau = dt/z$, system (A_{25}) becomes

$$\begin{cases} \frac{dz}{d\tau} = -z^2, \\ \frac{du}{d\tau} = z + u^2 - uz. \end{cases} \tag{3.14}$$

System (3.14) has a singularity at $A(0,0)$ on the u-axis. In order to apply Lemma 2.4, we let $A(z,u) = u^2 - uz$ and $B(z,u) = -z^2$. By $z + A(z,u) = 0$,

we get that $z = f(u) = -u^2 + \cdots, \quad F(u) = B(f(u), u) = -u^4 + o(u^5)$ and

$$G(u) = \left(\frac{\partial}{\partial z}(-z^2) + \frac{\partial}{\partial u}(z + u^2 - uz) \right) \Big|_{z=f(u)} = 2u + o(u^2).$$

Therefore the singularity $A(0,0)$ is a saddle-node. Next, by taking the change $(x, y) \rightarrow (v/z, 1/z)$ and $d\tau = dt/z$, system (A_{25}) becomes

$$\begin{cases} \frac{dz}{d\tau} = -z - z^2 v, \\ \frac{dv}{d\tau} = zv - v - zv^2. \end{cases} \quad (3.15)$$

It is easy to see that the singularity at $(0,0)$ is a stable node of system (3.15). Finally, taking into account the above results and the fact that (A_{25}) has an invariant straight line $x = 0$ which yields that this system has no limit cycle, we can easily get the global phase portrait of (A_{25}). Please see Figure 1.

$\square$

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