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Multiplicity and Stability of Equilibrium States of Three-Dimensional Nonlinear System*

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Abstract The multiplicity and stability of the equilibrium states of a three-dimensional differential system with initial conditions and three cross terms are studied in this paper. The existence and multiplicity of equilibrium states are given under the different qualifications of parameters. Besides, the local stability of the equilibrium state is shown by analyzing the eigenfunction of the Jacobi matrix. The global stability of the equilibrium state is obtained by constructing the Lyapunov function. Furthermore, the numerical simulation intuitively reflected the relationship of variables and verified the correctness of theoretical analysis.

Keywords Equilibrium states, multiplicity, local stability, global stability, numerical simulation.

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1. Introduction

Kinematics, dynamics problem involving force in physics can be analyzed by ordinary differential equation. In addition, ordinary differential equations have been widely used in the fields of chemistry, biology, economics, and demographics. People realized the importance of differential equations [7] and the powerful role of mathematical deduction until the Neptune is discovered.

The development of differential equations has gone through the classical stage, the stage of well-posedness theory, the stage of analytical theory and the stage of qualitative theory. As we all know, Poincaré put forward and studied the qualitative theory of differential equations in the 19th century. And the study of stability theory is initiated by Lyapunov. For centuries, mathematicians have never stopped studying the existence, stability, and well-posedness of solutions of differential equations([8, 9, 11, 12, 14–16]). Fortunately, the content of this aspect is constantly improved. The solution that makes the derivative of the differential system equal to zero is called equilibrium state. Equilibrium state is not an abstract mathematical concept. The examples are more common in life. The highest point and the lowest point of the single pendulum motion in the mechanical system, the coexistence of

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the two groups in the ecosystem which quantity remains unchanged, the unchanged supply and demand in the economic system, the unchanged concentration of the substance in the chemical reaction and so on. The equilibrium state of the differential system is one of the most important states of the system. Moreover, the stability of the system can be directly reflect by equilibrium state. The study of the equilibrium state of the infectious disease model can better guide us to prevent, treat, and control the infection of the disease; the study of the equilibrium state of the predator-prey model contributes to the sustainable development of the environment; the study of the equilibrium state of the fishing model can obtain greater economic benefits under the premise of ecological balance. So the study of equilibrium stability has important practical significance which attracts many mathematics scholars. About the theoretical research on equilibrium state, one can refer to [1–6, 10, 13].

The stability of equilibrium states of linear systems can be shown by solving the eigenvalues of Jacobi matrices. However, the stability of the equilibrium state of nonlinear systems is complicated with the appearance of cross terms, high order terms, and so on. Inspired by the reference [5, 8, 10, 12, 16], we studied the following problem

$$\begin{cases} \frac{dx}{dt} = uN + \alpha z - \beta xy - \gamma xz - (u+\xi)x, \\ \frac{dy}{dt} = \beta xy - \epsilon yz - (\delta + u)y, \\ \frac{dz}{dt} = \epsilon yz + \delta y + \gamma xz + \xi x - (\alpha + u)z, \\ x + y + z = N, \\ x(0) = x_0 > 0,\ y(0) = y_0 > 0, z(0) = z_0 > 0, \end{cases} \tag{1.1}$$

where u, α, β, γ, ξ, ε are positive parameters. Particularly, (1.1) can be reduced to infectious disease model with vaccination when $\gamma = \epsilon = 0$.

We shall apply eigenvalue theory and Lyapunov function [5, 10] to obtain the local stability and global stability of three-dimensional differential dynamic system with initial value conditions and nonlinear terms. With the xy, yz, xz taken into consideration, difficulties such as how to deal with the complex eigenfunction with multiple parameters and how to structure the Lyapunov function have to be overcome.

This paper is organized as follows. In Section 2, some notations are given which are critical to main results. In Section 3, existence and multiplicity of equilibrium state are shown. In Section 4, the local stability of equilibrium state is shown by analyzing the eigenfunction of Jacobi matrix. In Section 5, the global stability of equilibrium is obtained by the Lyapunov function. In Section 6, the numerical simulation is presented to illustrate the correctness and realizability of our theoretical results.

2. Some notations and Lemmas

To illustrate the main results, we give the following notations and lemmas.
$\Delta = (\alpha + \gamma N + u + \xi)^2 - 4\gamma(uN + \alpha N)$,
$\Delta_1 = b^2 - 4ac$, $a = \beta(\beta + \epsilon - \gamma)$,
$b = -u\epsilon + \beta\alpha - \epsilon\xi + \gamma(\delta + u) - \beta\delta - u\beta - N\epsilon\beta$,
$c = uN\epsilon - (\delta + u)(\alpha + u) + u\delta + u^2$,

$a_1 = \gamma z^* + u + \xi$, $a_2 = -\beta x^* + \epsilon z^* + \delta + u$,
$a_3 = -\gamma x^* + \alpha + u$, $e_1 = \beta y^* + \gamma z^* + u + \xi$,
$e_2 = -\beta x^* + \epsilon z^* + \delta + u$, $e_3 = -\epsilon y^* - \gamma x^* + \alpha + u$,
$h_1 = e_1e_2 + e_1e_3 + e_2e_3 + \epsilon y^*(\epsilon\gamma z^* + \delta) + \beta^2 y^* x^* + (\gamma z^* + \xi)(\gamma x^* - \alpha)$,
$h_2 = e_1e_2e_3 + \epsilon y^*(\epsilon z^* + \delta)e_1 + \beta^2 y^* x^* e_3 + \beta y^*(\epsilon z^* + \delta)(\gamma x^* - \alpha) - (\gamma z^* + \xi)\beta\epsilon y^* x^* + e_2(\gamma z^* + \xi)(\gamma x^* - \alpha)$,
(A_0) $\triangle = 0$,
(A_1) $\triangle > 0$,
(A_2) $a = 0$, $bc < 0$,
(A_3) $ac > 0$, $\triangle_1 > 0$, $-\frac{b}{2a} > 0$,
(A_4) $ac < 0$, $b > 0$, $\triangle_1 > 0$,
(A_5) $a \neq 0$, $b = 0$, $ac < 0$,
(T_1) $a_1 + a_2 + a_3 > 0$, $a_1a_2 + a_1a_3 + a_2a_3 + (\gamma z^* + \xi)(\gamma x^* - \alpha) > 0$, $a_1a_2a_3 + a_2(\gamma z^* + \xi)(\gamma x^* - \alpha) > 0$,
(T_2) $e_1 + e_2 + e_3 > 0$, $h_1 > 0$, $h_2 > 0$.

Lemma 2.1 ([10]). *Let*

$$\frac{dx}{dt} = Ax, \ \ A = (a_{ij})_{n\times n}, \tag{2.1}$$

where $x = (x_1, x_2, \cdots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$.

Then the zero solution of (2.1) *is locally asymptotically stable when all the eigenvalues* λ *of the matrix* A *have a negative real part; the zero solution of* (2.1) *is stable when all eigenvalues of the matrix* A *have non-positive real parts, and the eigenvalues with zero real parts are the single roots of the eigenpolynomial; the zero solution of the system* (2.1) *is not stable, when the matrix* A *has a positive eigenvalue, or the eigenvalue with zero real is the multiple roots of the eigenpolynomial.*

Lemma 2.2 ([10]). *Assume that* $x = (x_1, x_2, \cdots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$ *is the solution of the following* n*-dimensional autonomous differential system*

$$\frac{dx}{dt} = f(x), \ \ f(0) = 0. \tag{2.2}$$

Then $\dot{V}(x(t)) = \frac{dV(x(t))}{dt} = \sum\limits_{k=1}^{n} \frac{\partial V}{\partial x_k}\frac{dx_k}{dt} = \sum\limits_{k=1}^{n} \frac{\partial V}{\partial x_k} f_k(x)$ *is the total derivative of* $V(x(t))$ *along the trajectory of the system* (2.2).

The zero solution of the system (2.2) *is stable if there is a neighborhood* δ *and a positive (negative) function* $V(x)$, *satisfying* $\dot{V}(x)$ *is semi-negative(semi-positive).*

3. Existence of equilibrium state

Theorem 3.1. *The equilibrium state of system* (1.1) *is* $E_0 = (x^*, 0, N - x^*)$ *when* (A_0) *is satisfied; equilibrium states of system* (1.1) *are* $E_1 = (x_1^*, 0, N - x_1^*)$, $E_2 = (x_2^*, 0, N - x_2^*)$, *when* (A_1) *is satisfied; the equilibrium state of system* (1.1) *is* $E_3 = (\frac{-c}{b}, y_3^*, z_3^*)$, *when* (A_2) *is satisfied; equilibrium states of system* (1.1) *are* $E_4 = (\frac{-b+\sqrt{\Delta_1}}{2a}, y_4^*, z_4^*)$, $E_5 = (\frac{-b-\sqrt{\Delta_1}}{2a}, y_5^*, z_5^*)$, *when* (A_3) *is satisfied; the equilibrium state of system* (1.1) *is* $E_6 = (\frac{-b+\sqrt{\Delta_1}}{2a}, y_6^*, z_6^*)$, *when* (A_4) *is satisfied; the equilibrium state of system* (1.1) *is* $E_7 = (\sqrt{\frac{-c}{a}}, y_7^*, z_7^*)$, *when* (A_5) *is satisfied.*

Proof. By the definition of equilibrium state, we have

$$\begin{cases} \frac{dx}{dt} = 0, \\ \frac{dy}{dt} = 0, \\ \frac{dz}{dt} = 0. \end{cases} \tag{3.1}$$

By (3.1), we obtain $(\beta x - \epsilon z - \delta - u)y = 0$. In what follows, we analyze the equilibrium state when $y = 0$ and $y \neq 0$.
1) $y = 0$

Considering $N = x + z$ and (3.1), we obtain

$$\begin{cases} (\alpha + u - \alpha + \xi - u - \xi)x = 0, \\ (u + \xi + \alpha - \xi - \alpha - u)z = 0, \end{cases} \tag{3.2}$$

and $uN + \alpha(N - x) - \gamma x(N - x) - (u + \xi)x = 0$ i.e.,

$$f(x) = \gamma x^2 - (\alpha + \gamma N + u + \xi)x + uN + \alpha N = 0. \tag{3.3}$$

Clearly, (3.3) is the quadratic function of x. By the analysis of images and Veda's theorem, we have the following conclusions:
(i)$\Delta = 0$, (3.3) has the unique root $x^* = \frac{\alpha+u+\xi+\gamma N}{2\gamma} > 0$.
(ii)$\Delta > 0$, (3.3) has two roots $x_1^*, x_2^* = \frac{(\alpha+\gamma N+u+\xi)\pm\sqrt{\Delta}}{2\gamma}$. By $f(0) = uN + \alpha N > 0$, $x_1^*.x_2^* > 0$, we obtain $x_1^* > 0$, $x_2^* > 0$.
(iii)$\Delta < 0$, (3.3) has no root.

In summary: the equilibrium state of system (1.1) is $E_0 = (x^*, 0, N - x^*)$ when (A_0) is satisfied; the equilibrium states of system (1.1) are $E_1 = (x_1^*, 0, N - x_1^*)$, $E_2 = (x_2^*, 0, N - x_2^*)$, when (A_1) is satisfied.
2)$y \neq 0$

Combining $\beta x - \epsilon z - \delta - u = 0$ with (3.1), we obtain

$$z = \frac{\beta x - \delta - u}{\epsilon}, \tag{3.4}$$

$$y = \frac{(\alpha + u)(\beta x - \delta - u) - \epsilon\xi x - \gamma x(\beta x - \delta - u)}{\epsilon(\beta x - u)}. \tag{3.5}$$

By (3.4), (3.5) and $\frac{dx}{dt} = 0$, one has

$$uN + \alpha\frac{\beta x - \delta - u}{\epsilon} - \beta x\frac{(\alpha+u)(\beta x - \delta - u) - \epsilon\xi x - \gamma x(\beta x - \delta - u)}{\epsilon(\beta x - u)} - \gamma x\frac{\beta x - \delta - u}{\epsilon} - (u+\xi)x = 0,$$

i.e.,

$$\begin{aligned} &- u^2 N\epsilon + uN\epsilon\beta x + \alpha(\beta x - \delta - u)(\beta x - u) + ((\delta + u)\gamma x - \gamma\beta x^2)(\beta x - u) \\ &- \beta x[-(\alpha+u)(\delta+u) + (\alpha+u)\beta x - \epsilon\xi x + (\delta+u)\gamma x - \gamma\beta x^2] - \epsilon(\beta x - u)(u + \xi)x = 0. \end{aligned}$$

It is more difficult to solve the above formula directly. By the initial conditions of the system (1.1), we obtain

$$\frac{\beta x - \delta - u}{\epsilon} + x + \frac{(\alpha + u)(\beta x - \delta - u) - \epsilon\xi x - \gamma x(\beta x - \delta - u)}{\epsilon(\beta x - u)} = N. \tag{3.6}$$

By (3.4), (3.5) and $x + y + z = N$, (3.6) can be rewritten as

$$ax^2 + bx + c = 0. \tag{3.7}$$

The root of (3.7) is analyzed as follows:
(1): $a = 0$, $bc > 0$, (3.7) has no positive root;
(2): $a = 0$, $bc < 0$, (3.7) has unique root $\bar{x}^* = -\frac{b}{c}$;
(3): $a > 0$, $\Delta_1 > 0$, $-\frac{b}{2a} > 0$, (3.7) has two roots $x_1^* + x_2^* = -\frac{b}{a} > 0$, $x_1^*.x_2^* = \frac{c}{a}$,
if $c > 0$, (3.7) has two roots $\tilde{x}_1^*, \tilde{x}_2^* = \frac{-b\pm\sqrt{\Delta_1}}{2a}$,
if $c < 0$, (3.7) has positive root $\bar{x}_1^* = \frac{-b+\sqrt{\Delta_1}}{2a}$, negative root $\bar{x}_2^* = \frac{-b-\sqrt{\Delta_1}}{2a}$;
(4): $a > 0$, $\Delta_1 > 0$, $-\frac{b}{2a} < 0$, $c > 0$, (3.7) has no positive root;
(5): $a > 0$, $\Delta_1 > 0$, $-\frac{b}{2a} < 0$, $c < 0$, (3.7) has positive root $x_1 = \frac{-b+\sqrt{\Delta_1}}{2a}$, negative root $\frac{-b-\sqrt{\Delta_1}}{2a}$;
(6): $a > 0$, $b = 0$, $c < 0$, (3.7) has positive root $x_0^* = \sqrt{\frac{-c}{a}}$;
(7): $a > 0$, $\Delta_1 < 0$, (3.7) has no positive root;
(8): $a < 0$, $\Delta_1 < 0$, (3.7) has no positive root;
(9): $a < 0$, $\Delta_1 > 0$, $-\frac{b}{2a} > 0$, $c > 0$, (3.7) has positive root $x_3^* = \frac{-b+\sqrt{\Delta_1}}{2a}$;
(10): $a < 0$, $\Delta_1 > 0$, $-\frac{b}{2a} > 0$, $c < 0$, (3.7) has positive root $x_4^*, x_5^* = \frac{-b\pm\sqrt{\Delta_1}}{2a}$;
(11): $a < 0$, $\Delta_1 > 0$, $-\frac{b}{2a} < 0$, $c < 0$, (3.7) has no positive root;
(12): $a < 0$, $\Delta_1 > 0$, $-\frac{b}{2a} < 0$, $c > 0$, (3.7) has positive root $x_6^* = \frac{-b-\sqrt{\Delta_1}}{2a}$;
(13): $a < 0$, $c > 0$, $b = 0$, (3.7) has positive root $x_7^* = \sqrt{\frac{-c}{a}}$.

In summary: the equilibrium state of system (1.1) is $E_3 = (\frac{-c}{b}, y_3^*, z_3^*)$, when (A_2) is satisfied; the equilibrium states of system (1.1) are $E_4 = (\frac{-b+\sqrt{\Delta_1}}{2a}, y_4^*, z_4^*)$, $E_5 = (\frac{-b-\sqrt{\Delta_1}}{2a}, y_5^*, z_5^*)$, when (A_3) is satisfied; the equilibrium state of system (1.1) is $E_6 = (\frac{-b+\sqrt{\Delta_1}}{2a}, y_6^*, z_6^*)$, when (A_4) is satisfied; the equilibrium state of system (1.1) is $E_7 = (\sqrt{\frac{-c}{a}}, y_7^*, z_7^*)$, when (A_5) is satisfied. □

4. The local stability of positive equilibrium state

4.1. The local stability of positive equilibrium state when $y = 0$

Theorem 4.1. *If (A_0)(or(A_1)),$(T_1)$are satisfied, the equilibrium state of the model (1.1)is locally asymptotically stable.*

Proof. By transformation $\tilde{x} = x - x^*$, $\tilde{z} = z - z^*$, $\tilde{y} = y$, (1.1) can be written as

$$\begin{cases} \frac{d\tilde{x}}{dt} = \alpha\tilde{z} - \beta\tilde{x}\tilde{y} - \gamma\tilde{x}\tilde{z} - (u+\xi)\tilde{x} + uN + \alpha z^* - \gamma z^* x^* - (u+\xi)x^* - \beta x^*\tilde{y} - \gamma\tilde{x}z^* - \gamma x^*\tilde{z}, \\ \frac{d\tilde{y}}{dt} = \beta\tilde{x}\tilde{y} - \epsilon\tilde{y}\tilde{z} - (\delta+u)\tilde{y} + \beta x^*\tilde{y} - \epsilon\tilde{y}z^*, \\ \frac{d\tilde{z}}{dt} = \epsilon\tilde{y}\tilde{z} + \epsilon\tilde{y}z^* + \delta\tilde{y} + \gamma\tilde{x}\tilde{z} + \gamma x^*\tilde{z} + \gamma x^* z^* + \gamma\tilde{x}z^* + \xi\tilde{x} + \xi x^* - (\alpha+u)\tilde{z} - z^*(\alpha+u). \end{cases} \tag{4.1}$$

The Jacobi matrix linearization system of (4.1) is

$$A = \begin{pmatrix} -\gamma z^* - u - \xi & -\beta x^* & \alpha - \gamma x^* \\ 0 & \beta x^* - \epsilon z^* - \delta - u & 0 \\ \gamma z^* + \xi & \epsilon z^* + \delta & \gamma x^* - \alpha - u \end{pmatrix}.$$

The eigenvalue polynomial is

$$\begin{aligned} g(\lambda) =& \lambda^3 + (a_1 + a_2 + a_3)\lambda^2 + [a_1a_2 + a_1a_3 + a_2a_3 + (\gamma z^* + \xi)(\gamma x^* - \alpha)]\lambda \\ &+ a_1a_2a_3 + a_2(\gamma z^* + \xi)(\gamma x^* - \alpha). \end{aligned} \tag{4.2}$$

Assume that $\lambda_1 = C_1$, $\lambda_2 = C_2$, $\lambda_3 = C_3$, then

$$\lambda^3 - (C_1 + C_2 + C_3)\lambda^2 + (C_1C_2 + C_1C_3 + C_2C_3)\lambda - C_1C_2C_3 = 0$$

hold. According to the coefficient of the same term is equal, we have

$$-(C_1 + C_2 + C_3) = a_1 + a_2 + a_3, \ - C_1C_2C_3 = a_1a_2a_3 + a_2(\gamma z^* + \xi)(\gamma x^* - \alpha).$$

Since a_1, a_2, a_3, γ, z^*, x^*, α, ξ are real number, so C_1, C_1, C_3 are real number. By calculation, $\lambda < 0$ when T_1 is satisfied. By the lemma2.1, we obtain the local stability of equilibrium state of system (1.1) when $T_1 \cap (A(i))$, $i = 0, 1$. □

4.2. The local stability of positive equilibrium state when $y \neq 0$

Theorem 4.2. *If* $\bigcup_{i=2}^{5}(A_i))$*,*(T_2)*are satisfied, the equilibrium state of the model* (1.1)*is locally asymptotically stable.*

Proof. By transformation $\tilde{x} = x - x^*$, $\tilde{z} = z - z^*$, $\tilde{y} = y - y^*$, (1.1) can be written as

$$\begin{cases} \frac{d\tilde{x}}{dt} = \alpha(\tilde{z} + z^*) - \beta(\tilde{x} + x^*)(\tilde{y} + y^*) - \gamma(\tilde{x} + x^*)(\tilde{z} + z^*) - (u + \xi)(\tilde{x} + x^*), \\ \frac{d\tilde{y}}{dt} = \beta(\tilde{x} + x^*)(\tilde{y} + y^*) - \epsilon(\tilde{y} + y^*)(\tilde{z} + z^*) - (\delta + u)(\tilde{y} + y^*), \\ \frac{d\tilde{z}}{dt} = \epsilon(\tilde{y}+y^*)(\tilde{z}+z^*)+\delta(\tilde{y}+y^*)+\gamma(\tilde{x}+x^*)(\tilde{z}+z^*) + \xi(\tilde{x} + x^*) - (\alpha + u)(\tilde{z} + z^*) \end{cases} \tag{4.3}$$

The Jacobi matrix linearization system of (4.3) is

$$A = \begin{pmatrix} -e_1 & -\beta x^* & \alpha - \gamma x^* \\ \beta y^* & -e_2 & -\epsilon y^* \\ \gamma z^* + \xi & \epsilon z^* + \delta & -e_3 \end{pmatrix}.$$

The eigenvalue polynomial is

$$\begin{aligned} g_1(\lambda) =& \lambda^3 + (e_1 + e_2 + e_3)\lambda^2 + [e_1e_2 + e_1e_3 + e_2e_3 + \epsilon y^*(\epsilon z^* + \delta) + \beta^2 y^* x^* + (\gamma z^* \\ &+ \xi)(\gamma x^* - \alpha)]\lambda + e_1e_2e_3 + \epsilon y^*(\epsilon z^* + \delta)e_1 + \beta^2 y^* x^* e_3 + \beta y^*(\epsilon z^* + \delta)(\gamma x^* \\ &- \alpha) - (\gamma z^* + \xi)\beta\epsilon y^* x^* + e_2(\gamma z^* + \xi)(\gamma x^* - \alpha). \end{aligned}$$

$g_1(\lambda)$ can be written as

$$g_1(\lambda) = \lambda^3 + (e_1 + e_2 + e_3)\lambda^2 + h_1\lambda + h_2.$$

Assume that $\tilde{\lambda}_1$, $\tilde{\lambda}_2$, $\tilde{\lambda}_3$ are the root of $g_1(\lambda) = 0$. We obtain

$$\lambda^3 - (\tilde{\lambda}_1 + \tilde{\lambda}_2 + \tilde{\lambda}_3)\lambda^2 + (\tilde{\lambda}_1\tilde{\lambda}_2 + \tilde{\lambda}_1\tilde{\lambda}_3 + \tilde{\lambda}_2\tilde{\lambda}_3)\lambda - \tilde{\lambda}_1\tilde{\lambda}_2\tilde{\lambda}_3 = 0.$$

According to the coefficient of the same term is equal, we have

$$\begin{cases} -(\tilde{\lambda}_1 + \tilde{\lambda}_2 + \tilde{\lambda}_3) = e_1 + e_2 + e_3, \\ \tilde{\lambda}_1\tilde{\lambda}_2 + \tilde{\lambda}_1\tilde{\lambda}_3 + \tilde{\lambda}_2\tilde{\lambda}_3 = h_1, \\ -\tilde{\lambda}_1\tilde{\lambda}_2\tilde{\lambda}_3 = h_2. \end{cases}$$

So $\tilde{\lambda}_1$, $\tilde{\lambda}_2$, $\tilde{\lambda}_3$ are negative real number under the condition T_2. By the lemma2.1, we obtain the local stability of equilibrium state of system (1.1), when $T_2 \cap (A(i))$, $i = 2, 3, 4, 5$. □

5. The global stability of positive equilibrium state

5.1. The global stability of positive equilibrium state when $y = 0$

Theorem 5.1. *The locally stable equilibrium state is globally asymptotically stable when* $\beta N - (\delta + u) < 0$*,* $u + \alpha - \xi - \gamma N < 0$*.*

Proof. Since $x^* = \frac{N}{2} + \frac{\alpha+u+\xi}{2\gamma}$, $x^* + z^* = N$, we have $\frac{N}{2} < x^* < N$, $0 < z^* < N$. Obviously, $D = \{(x, y)|x \geq 0, y \geq 0, x + y \leq N\}$ is the positive invariant field of (1.1). The trajectory of (1.1) starting from $t = 0$ will not run outside of D as t increasing. We obtain $\frac{dy}{dt} \leq (\beta N - (\delta+u))y$ by using $x < N$, $\frac{dy}{dt} = (\beta x - \epsilon z - (\delta+u))y$. So $\lim\limits_{t\to\infty} y(t) = 0$, when $\beta N - (\delta + u) < 0$.

The following we apply vertical isocline to analyze changes of trajectories in D. The vertical isocline of (1.1) can defined by

$$L : x(\beta y + \gamma z^* + u + \xi) = uN + \alpha z^*.$$

Giving $z = z^*$, we obtain that L divides D into D_1 and D_2. Moreover, $\frac{dx}{dt} = 0$, on the curve L, $\frac{dx}{dt} < 0$, when $(x, y) \in D_1$, $\frac{dx}{dt} > 0$, when $(x, y) \in D_2$. Since $\lim\limits_{t\to\infty} y(t) = 0$, we have the trajectory from D_2 must enter D_1 after a certain time t_1 under the condition $u + \alpha - \xi - \gamma N < 0$. Since D is the positive invariant domain of (1.1) and the trajectory of L is vertically passed from below, the trajectories in D must go in D_1 and stay in D_1 forever. $\lim\limits_{t\to\infty} x(t) = x^*$, $\lim\limits_{t\to\infty} z(t) = N - x^*$ in D_1. So the locally stable equilibrium state $E(x^*, y^*)$ is global stable([5]), when $\beta N - (\delta + u) < 0$ in D . □

5.2. The global stability of positive equilibrium state when $y \neq 0$

Theorem 5.2. *The function* $V(x(t),y(t),z(t)) = \frac{1}{2}(\frac{x-x^*}{x^*})^2 + \frac{2a_0}{\beta(x^*)^2}(y - y^* - y^* ln(\frac{y}{y^*})) + \frac{1}{2}(\frac{z^*}{x^*})^2(\frac{z-z^*}{z^*})^2$ *is positive definite function and* $\frac{dV}{dt} < 0$*, where* $0 < a_0 < \min\{u + \xi, \alpha + u\}$*.*

Proof. Let $Z = y - y^* - y^* ln(\frac{y}{y^*})$. Since $y^* > 0$, the positive and negative of $\frac{Z}{y^*}$ are consistent with Z. A straightforward computation shows that

$$\frac{Z}{y^*} = \frac{y}{y^*} - 1 - ln(\frac{y}{y^*}).$$

Define $\frac{y}{y^*} = m$, $\frac{Z}{y^*} = n$, we have $n = m - ln(m) - 1$. By the derivative operation, $\frac{dn}{dm} = 1 - \frac{1}{m}$ holds. Since $\frac{dn}{dm} > 0$ when $m > 1$, we have $n_{min} = 0$. Similarly, $\frac{dn}{dm} < 0$ when $0 < m < 1$, we have $n_{min} = 0$. So $n > 0$, $Z > 0$. Clearly, $\frac{1}{2}(\frac{x-x^*}{x^*})^2 > 0$, $\frac{1}{2}(\frac{z^*}{x^*})^2(\frac{z-z^*}{z^*})^2 > 0$. So V is positive definite function. Besides, we have

$$\begin{aligned}
\frac{dV}{dt} =& \frac{x-G}{G^2}(uN + \alpha z - \beta xy - \gamma xz - (u+\xi)x) + \frac{2a_0}{\beta G^2}(y-y^*)(\beta x - \epsilon z - \delta - u) \\
&+ \frac{(z-z^*)}{G^2}(\epsilon yz + \delta y + \gamma xz + \xi x - (\alpha+u)z) \\
=& \frac{-(u+\xi)(x-G)^2}{G^2} + \frac{2a_0}{G^2}(y-y^*)(x-G) + \frac{-(\alpha+u)(z-z^*)^2}{G^2} \\
=& \frac{-(u+\xi)(x-G)^2}{G^2} + \frac{2a_0}{G^2}[N - x - z - (N - G - z^*)](x-G) + \frac{-(\alpha+u)(z-z^*)^2}{G^2} \\
=& \frac{-(u+\xi)(x-G)^2}{G^2} - \frac{2a_0}{G^2}(z-z^*)(x-G) - \frac{2a_0}{G^2}(x-G)^2 - \frac{(\alpha+u)(z-z^*)^2}{G^2} \\
\leq& \frac{-a_0[(x-G)^2 + (z-z^*)^2] - 2a_0(z-z^*)(x-G)}{G^2} - \frac{2a_0(x-G)^2}{G^2} \\
=& \frac{-a_0(x - G + z - z^*)^2}{G^2} - \frac{2a_0(x-G)^2}{G^2} < 0
\end{aligned}$$

where $x^* = G$. □

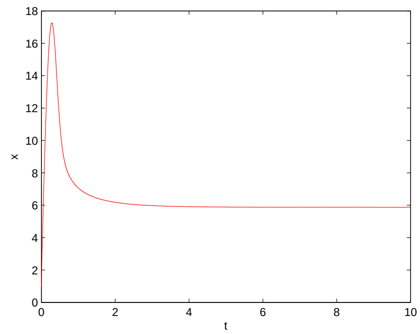
By theorem 5.2 and lemma 2.2, the equilibrium state of system (1.1) is global stable.

6. Numerical simulation

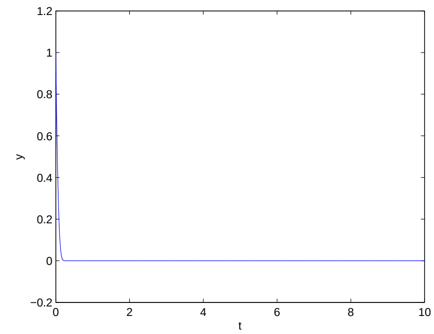
In this part, we apply MATLAB software to draw corresponding pictures to illustrate the correctness and reliability of theoretical results.

We choose $u = 1$, $\beta = 0.0002$, $\gamma = 0.5$, $\alpha = 2$, $\xi = 1$, $\epsilon = 10$, $\delta = 2$, $N = 100$, $\Delta = 2316$, $y^* = 0$, $x^* = 5.8752$, $z^* = 94.1248$, $a_1 + a_2 + a_3 = 993.3720$, $a_1a_2 + a_1a_3 + a_2a_3 + (\gamma z^* + \xi)(\gamma x^* - \alpha) > 0 = 4.6434e + 004 > 0$, $a_1a_2a_3 + a_2(\gamma z^* + \xi)(\gamma x^* - \alpha) = 4.5442e + 004$.

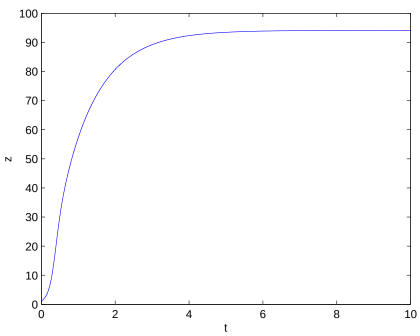
Observe the image, we can see x is stable at equilibrium $x^* = 5.8752$(Figure(a)); y is stability at equilibrium $y^* = 0$(Figure (b)); z is stability at equilibrium $z^* = 94.1248$(Figure (c)). This is consistent with the existence of the equilibrium state of the theorem 3.1 , while satisfying the T_1 condition in the local stability analysis. Figure (d) shows the relationship diagram of x, y, z in three-dimensional space, which intuitively understands the overall change of the system solution curve. Figure (e), Figure (f), and Figure (g) respectively give the relationship between x, y, x, z and y, z, moeover, which shows the interaction between variables. The images verify the existence of equilibrium states and the correctness of stability theory.



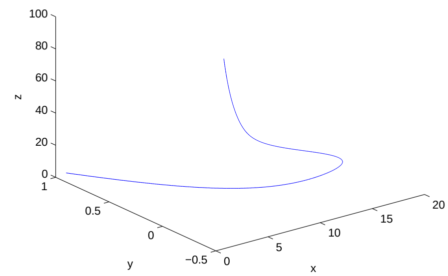
(a) phase diagram of x



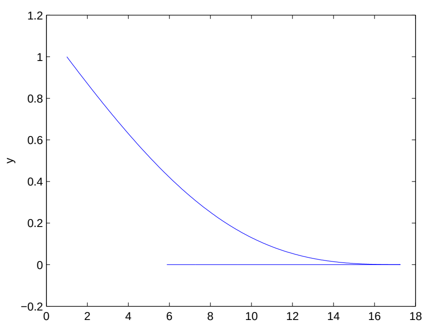
(b) phase diagram of y



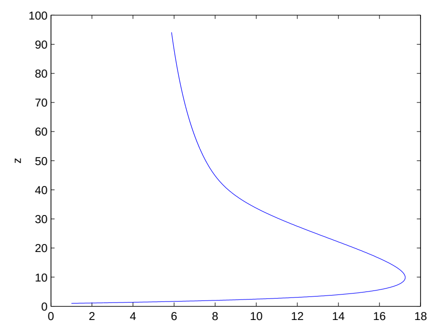
(c) phase diagram of z



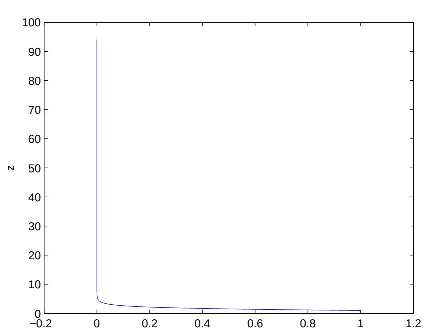
(d) relation diagram of x,y,z



(e) relation diagram of x,y



(f) relation diagram of x,z



(g) relation diagram of y,z

Figure 1. A simulation result of example

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