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Existence and Multiplicity of Positive Solutions for Fractional Differential Equation with Parameter*

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Abstract In this paper, by using the fixed point theorem for a cone map, we study the existence and multiplicity of positive solutions for a class of fractional differential equation with parameter.

Keywords Fractional differential equation, Green function, Cone.

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1. Introduction

Fractional calculus has played a significant role in engineering, science, economy, and other fields, during the last few decades. There has been a significant development in ordinary and partial differential equations involving fractional derivatives. There are many important results about the existence of solutions for fractional differential equation, see [1, 2, 5–8, 10, 11] for more details.

In [1], Bai and Lü considered the positive solutions for boundary value problem of fractional order differential equation

$$\begin{cases} D^{\alpha}_{0+}u(t) + f(t,\ u(t)) = 0,\ t \in (0,\ 1), \\ u(0) = 0, u(1) = 0, \end{cases}$$

where $1 < \alpha \leq 2, f : [0, 1] \times [0, +\infty) \to [0, +\infty)$ is continuous.

In this paper, under different growth conditions of f, we obtain the existence and multiplicity of positive solutions for boundary value problem of fractional differential equation with parameter

$$\begin{cases} D^{\alpha}_{0+}u(t) + \lambda f(t,\ u(t)) = 0,\ t \in (0,\ 1), \\ u(0) = 0, u(1) = 0, \end{cases} \tag{1.1}$$

where $1 < \alpha \leq 2, f : [0, 1] \times [0, +\infty) \to [0, +\infty)$ is continuous, $\lambda > 0$ is a parameter.

Remark 1.1. When $\alpha = 2$, problem (1.1) is reduced to the problem of paper [4].

We make the following hypotheses:

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(A) $f(t,u) : [0,1]\times[0,+\infty)\to[0,+\infty)$ is continuous and there exists $g\in C((0,+\infty),(0,+\infty)), q_1, q_2\in C((0,1),(0,+\infty))$ such that $q_1(t)g(y)\le f(t,t^{\alpha-2}y)\le q_2(t)g(y), t\in[0,1], y\in[0,+\infty)$.

For the convenience, we take some notations. Let

$$g_0=\lim_{y\to 0^+}\frac{g(y)}{y},\qquad g_\infty=\lim_{y\to+\infty}\frac{g(y)}{y}.$$

i_0=numbers of zeros in the set $\{g_0, g_\infty\}$; i_∞=numbers of infinities in the set $\{g_0, g_\infty\}$.

$$M(p)=\max_{0\le y\le p}\{g(y)\},\qquad m(p)=\min_{\frac{(\alpha-1)p}{16}\le y\le p}\{g(y)\}.$$

The tool theorem is following:

Theorem 1.1. *Let E be a Banach space, $K\subset E$ is a cone, Ω_1,Ω_2 are bounded open subsets of E, $0\in\Omega_1,\overline{\Omega}_1\subset\Omega_2$, suppose that $A: K\bigcap(\overline{\Omega}_2\setminus\Omega_1)\to K$ is completely continuous and satisfies :*

(i) $\|Ax\|\le\|x\|, x\in K\bigcap\partial\Omega_1$, and $\|Ax\|\ge\|x\|, x\in K\bigcap\partial\Omega_2$; or

(ii) $\|Ax\|\ge\|x\|, x\in K\bigcap\partial\Omega_1$, and $\|Ax\|\le\|x\|, x\in K\bigcap\partial\Omega_2$;

Then A has a fixed point in $K\bigcap(\overline{\Omega}_2\setminus\Omega_1)$.

Definition 1.1. We call $D^\alpha_{0+}f(x)=\frac{1}{\Gamma(n-\alpha)}(\frac{d}{dx})^n\int_0^x\frac{f(t)}{(x-t)^{\alpha-n+1}}dt, \alpha>0, n=[\alpha]+1$ is the Riemann-Liouville fractional derivative of order α.$[\alpha]$ denotes the integer part of number α.

Definition 1.2. We call $I^\alpha_{0+}f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}f(t)dt, x>0, \alpha>0$ is Riemann-Liouville fractional integral of order α.

2. Preliminaries

Lemma 2.1 (Lemma 2.3, [1]). *The solutions of problem*

$$\begin{cases} D^\alpha_{0+}u(t)+\lambda f(t,\ u(t))=0,\ t\in(0,\ 1),\\ u(0)=0, u(1)=0\end{cases}$$

is equivalent to the solutions of the integral equation

$$u(t)=\lambda\int_0^1 G(t,s)f(s,u(s))ds, \tag{2.1}$$

where

$$G(t,s)=\begin{cases}\frac{[t(1-s)]^{\alpha-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)},\ 0\le s\le t\le 1;\\ \frac{[t(1-s)]^{\alpha-1}}{\Gamma(\alpha)},\qquad 0\le t\le s\le 1.\end{cases}$$

Lemma 2.2 (Proposition 1, [5]). *The Green function $G(t,s)$ has the following properties:*

(i) $G(t,s)\in C([0,1]\times[0,1])$ and $G(t,s)>0, \forall t,s\in(0,1)$;

(ii) There exists s positive function $\gamma \in C(0,1)$ such that

$$\min_{\frac{1}{4}\leq t\leq\frac{3}{4}} G(t,s) \geq \gamma(s) \max_{0\leq t\leq 1} G(t,s) \geq \gamma(s)G(s,s), s \in (0,1).$$

Lemma 2.3 (Lemma 2.3, [5])**.** *The Green function $G(t,s)$ satisfies*

$$\frac{\alpha-1}{\Gamma(\alpha)}t^{\alpha-1}(1-t)(1-s)^{\alpha-1} \leq G(t,s) \leq \frac{1}{\Gamma(\alpha)}t^{\alpha-1}(1-t)(1-s)^{\alpha-2}, t,s \in (0,1).$$

Let $G^*(t,s) := t^{2-\alpha}G(t,s)$, then

$$\frac{\alpha-1}{\Gamma(\alpha)}t(1-t)s(1-s)^{\alpha-1} \leq G^*(t,s) \leq \frac{1}{\Gamma(\alpha)}s(1-s)^{\alpha-1}, t,s \in (0,1).$$

Define

$$K := \{y \in C[0,1] \mid y(t) \geq 0, y(t) \geq (\alpha-1)t(1-t)\|y\|\},$$

where $\|y\| = \max_{0\leq t\leq 1} |y(t)|$.

Lemma 2.4. *Let $(T_\lambda y)(t) := \lambda\int_0^1 G^*(t,s)f(s,s^{\alpha-2}y(s))ds$, then $T_\lambda : K \to K$ is completely continuous.*

Proof. The continuity of T_λ is obvious. Following we prove $T_\lambda : K \to K$. From Lemma 2.2 and condition (A), we can get that $T_\lambda y(t) \geq 0, t \in [0,1]$. For $\forall y \in K$, from Lemma 2.3, we have

$$\begin{aligned} T_\lambda y(t) &= \lambda \int_0^1 G^*(t,s)f(s,s^{\alpha-2}y(s))ds \\ &\geq \frac{\lambda(\alpha-1)}{\Gamma(\alpha)}t(1-t)\int_0^1 s(1-s)^{\alpha-1}f(s,s^{\alpha-2}y(s))ds. \end{aligned}$$

On the other hand

$$\|T_\lambda y\| = \max_{0\leq t\leq 1}|T_\lambda y(t)| \leq \frac{\lambda}{\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-1}f(s,s^{\alpha-2}y(s))ds.$$

Hence

$$T_\lambda y(t) \geq (\alpha-1)t(1-t)\|T_\lambda y\|.$$

So $T_\lambda : K \to K$.

Next we show that T_λ is uniformly bounded and equicontinuous.

Let $\forall D \subset K$ be bounded, so there exists a constant $L > 0$ such that $\|y\| \leq L$ for $\forall y \in D$.

Let $M = \max_{0\leq y\leq L}|g(y)| + 1$, then

$$\begin{aligned} |T_\lambda y(t)| &\leq \lambda \int_0^1 |G^*(t,s)f(s,s^{\alpha-2}y(s))|ds \\ &\leq \lambda \int_0^1 |G^*(t,s)q_2(s)g(y(s))|ds \\ &\leq M\lambda \int_0^1 \frac{1}{\Gamma(\alpha)}s(1-s)^{\alpha-1}q_2(s)ds. \end{aligned}$$

It follows from (A) that $T_\lambda(D)$ is uniformly bounded. For $\forall\varepsilon > 0, \forall y \in D, t_1, t_2 \in [0,1], t_1 < t_2$, since $G^*(t,s)$ is uniformly continuous on $(t,s) \in [0,1]\times[0,1]$, then there exists $\eta > 0$, when $|t_1 - t_2| < \eta$, we have

$$|G^*(t_1,s) - G^*(t_2,s)| < \frac{\varepsilon}{M\lambda\int_0^1 q_2(s)ds}.$$

Then from condition (A), one has

$$\begin{aligned}
&|T_\lambda y(t_1) - T_\lambda y(t_2)| \\
&= \lambda|\int_0^1 G^*(t_1,s)f(s,s^{\alpha-2}y(s))ds - \lambda\int_0^1 G^*(t_2,s)f(s,s^{\alpha-2}y(s))ds| \\
&\le \lambda\int_0^1 |G^*(t_1,s) - G^*(t_2,s)|f(s,s^{\alpha-2}y(s))ds \\
&\le \lambda\int_0^1 |G^*(t_1,s) - G^*(t_2,s)|q_2(s)g(y(s))ds \\
&< M\lambda\cdot\frac{\varepsilon}{M\lambda\int_0^1 q_2(s)ds}\cdot\int_0^1 q_2(s)ds = \varepsilon.
\end{aligned}$$

By means of the Arzela- Ascoli theorem, $T_\lambda : K \to K$ is completely continuous. □

3. Main results and proofs

Theorem 3.1. *Assume (A) hold,*

(a) If $i_0 = 1$ or 2, then problem (1.1) has i_0 positive solutions for $\lambda > \lambda_0 = \frac{1}{\int_{\frac14}^{\frac34} G^*(\frac12,s)q_1(s)ds\cdot m(1)}$;

(b) If $i_\infty = 1$ or 2, then problem (1.1) has i_∞ positive solutions for $0 < \lambda < \lambda_0 = \{\frac{1}{\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-1}q_2(s)ds\cdot M(1)\}^{-1}$.

Proof. (a) Choose a number $p = 1$, let $\Omega_p = \{y \in C[0,1] \mid \|y\| < p\}$, we have $\frac{(\alpha-1)p}{16} \le y(t) \le p$ for $y \in K\bigcap\partial\Omega_p$ and $t \in [\frac14, \frac34]$.

Hence

$$\begin{aligned}
\|T_\lambda y\| \ge T_\lambda y(\frac12) &= \lambda\int_0^1 G^*(\frac12,s)f(s,s^{\alpha-2}y(s))ds \\
&\ge \lambda\int_{\frac14}^{\frac34} G^*(\frac12,s)q_1(s)g(y(s))ds \\
&\ge \lambda\int_{\frac14}^{\frac34} G^*(\frac12,s)q_1 ds\cdot m(p).
\end{aligned}$$

Then $\|T_\lambda y\| > \|y\|, y \in K\bigcap\partial\Omega_p$ for $\lambda > \lambda_0 = \frac{1}{\int_{\frac14}^{\frac34} G^*(\frac12,s)q_1(s)ds\cdot m(1)}$.

If $g_0 = 0$, then there exists $r \in (0,p)$ such that $g(y) \le \varepsilon y, 0 \le y \le r$, where $\varepsilon > 0$ satisfies

$$\frac{\lambda\varepsilon}{\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-1}q_2(s)ds \le 1.$$

Let $\Omega_r = \{y \in C[0,1] \mid \|y\| < r\}$, from Lemma 2.2 and (A), we have

$$\begin{aligned} T_\lambda y(t) &= \lambda \int_0^1 G^*(t,s) f(s, s^{\alpha-2} y(s)) ds \\ &\le \frac{\lambda}{\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-1} q_2(s) g(y(s)) ds \\ &\le \frac{\lambda\varepsilon}{\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-1} q_2(s) ds \cdot \|y\| \le \|y\|. \end{aligned}$$

So $\|T_\lambda y\| \le \|y\|, y \in K \bigcap \partial\Omega_r$. Therefore, by the (i) of Theorem 1.1, it follows that T_λ has a fixed point $y_1 \in K \bigcap (\overline{\Omega}_p \setminus \Omega_r)$ and $r \le \|y_1\| < p$.

If $g_\infty = 0$, then there exists $M > 0$ such that $g(y) \le \varepsilon y, y > M$.

Let

$$R = \max\Big\{ \frac{\lambda \max\limits_{0\le y\le M}\{g(y(s))\} \int_0^1 s(1-s)^{\alpha-1} q_2(s) ds}{\Gamma(\alpha) - \lambda\varepsilon \int_0^1 s(1-s)^{\alpha-1} q_2(s) ds}, p+M \Big\},$$

$$\Omega_R = \{y \in C[0,1] \mid \|y\| < R\}.$$

From Lemma 2.3 and condition (A), we have

$$\begin{aligned} T_\lambda y(t) &= \lambda \int_0^1 G^*(t,s) f(s, s^{\alpha-2} y(s)) ds \\ &\le \frac{\lambda}{\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-1} q_2(s) g(y(s)) ds \\ &\le \frac{\lambda}{\Gamma(\alpha)} \int_{0\le y\le M} s(1-s)^{\alpha-1} q_2(s) g(y(s)) ds \\ &+ \frac{\lambda}{\Gamma(\alpha)} \int_{M<y\le R} s(1-s)^{\alpha-1} q_2(s) g(y(s)) ds \\ &\le \frac{\lambda}{\Gamma(\alpha)} (\max_{0\le y\le M}\{g(y(s))\} + \varepsilon\|y\|) \int_0^1 s(1-s)^{\alpha-1} q_2(s) ds \le R = \|y\|. \end{aligned}$$

So $\|T_\lambda y\| \le \|y\|, y \in K \bigcap \partial\Omega_R$. Therefore, by the part (ii) of Theorem 1.1, it follows that T_λ has a fixed point $y_2 \in K \bigcap (\overline{\Omega}_R \setminus \Omega_p)$ and $p < \|y_2\| \le R$.

Hence T_λ has two fixed points y_1, y_2 and $r \le \|y_1\| < p < \|y_2\| \le R$. It follows from above that if $g_0 = g_\infty = 0$, then T_λ has two positive solutions for $\lambda > \lambda_0$ and satisfies

$$y_i(t) = \lambda \int_0^1 G^*(t,s) f(s, s^{\alpha-2} y_i(s)) ds, t \in [0,1], i = 1,2, \text{ and } \|y_i\| \le R.$$

It is obvious that $u_i(t) = t^{\alpha-2} y_i(t), i = 1,2$ are two positive solutions of problem (2.1) for $t \in [0,1]$, i.e

$$u_i(t) = \lambda \int_0^1 G(t,s) f(s, u_i(s)) ds, t \in [0,1], i = 1,2.$$

Next, we will prove $u_i(0) = 0, i = 1,2$. From $y_i \in C[0,1]$ and condition (A), we have

$$\lim_{t\to 0^+} u_i(t) = \lambda \lim_{t\to 0^+} \int_0^1 G(t,s) f(s, u_i(s)) ds$$

$$= \lambda \lim_{t\to 0^+} \int_0^1 G(t,s) f(s, s^{\alpha-2} y_i(s)) ds$$
$$\leq \lambda \lim_{t\to 0^+} \int_0^1 G(t,s) q_2(s) g(y_i(s)) ds$$
$$\leq \lambda \lim_{t\to 0^+} \int_0^1 G(t,s) q_2(s) ds \cdot \max_{\|y_i\|\leq 1} g(y_i) = 0, \quad i = 1, 2.$$

Thus, $u_i(0) = 0, i = 1, 2$. Then $u_i(t) = t^{\alpha-2} y_i(t), i = 1, 2$ are two positive solutions of (2.1) for $t \in [0, 1]$, from Lemma 2.1, problem (1.1) has two positive solutions $u_1(t) = t^{\alpha-2} y_1(t), u_2(t) = t^{\alpha-2} y_2(t)$.

(b) Choose a number $p = 1$, let $\Omega_p = \{y \in C[0, 1] \mid \|y\| < p\}$, then for $y \in K \bigcap \partial\Omega_p$, we have

$$T_\lambda y(t) \leq \frac{\lambda}{\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-1} f(s, s^{\alpha-2} y(s)) ds$$
$$\leq \frac{\lambda}{\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-1} q_2(s) g(y(s)) ds$$
$$\leq \frac{\lambda}{\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-1} q_2(s) ds \cdot M(p).$$

So $\|T_\lambda y\| < \|y\|$ for $0 < \lambda < \lambda_0 = \{\frac{1}{\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-1} q_2(s) ds \cdot M(1)\}^{-1}$, $y \in K \bigcap \partial\Omega_p$.

If $g_0 = \infty$, then there exists $r \in (0, p)$, such that $g(y) \geq My, 0 \leq y \leq r$, where

$$\frac{\lambda M(\alpha-1)^2}{64\Gamma(\alpha)} \int_{\frac{1}{4}}^{\frac{3}{4}} s(1-s)^{\alpha-1} q_1(s) ds \geq 1.$$

Let $\Omega_r = \{y \in C[0, 1] \mid \|y\| < r\}$, from Lemma 2.3 and (A), we have

$$\|T_\lambda y\| \geq T_\lambda y(\frac{1}{2}) = \lambda \int_0^1 G^*(\frac{1}{2}, s) f(s, s^{\alpha-2} y(s)) ds$$
$$\geq \frac{\lambda(\alpha-1)}{4\Gamma(\alpha)} \int_{\frac{1}{4}}^{\frac{3}{4}} s(1-s)^{\alpha-1} q_1(s) g(y(s)) ds$$
$$\geq \frac{\lambda M(\alpha-1)}{4\Gamma(\alpha)} \int_{\frac{1}{4}}^{\frac{3}{4}} s(1-s)^{\alpha-1} q_1(s) y(s) ds$$
$$\geq \frac{\lambda M(\alpha-1)^2}{64\Gamma(\alpha)} \int_{\frac{1}{4}}^{\frac{3}{4}} s(1-s)^{\alpha-1} q_1(s) ds \cdot \|y\| \geq \|y\|.$$

Therefore, by the (ii) of Theorem 1.1, it follows that T_λ has a fixed point $y_1 \in K \bigcap (\overline{\Omega}_p \setminus \Omega_r)$ and $r \leq \|y_1\| < p$.

If $g_\infty = \infty$, then there exists $M^* > 0$ such that $g(y) \leq \rho y, y > M^*$. Let

$$R = p + \frac{16}{\alpha-1} M^*, \quad \Omega_R = \{y \in C[0, 1] \mid \|y\| < R\}.$$

So $y(t) \geq \frac{\alpha-1}{16}\|y\| > M^*$ for $y \in K\bigcap \partial\Omega_R, t \in [\frac{1}{4}, \frac{3}{4}]$, and

$$\begin{aligned}\|T_\lambda y\| \geq T_\lambda y(\frac{1}{2}) &\geq \lambda \int_{\frac{1}{4}}^{\frac{3}{4}} G^*(\frac{1}{2}, s)q_1(s)\rho y(s)ds \\ &\geq \frac{\rho\lambda(\alpha-1)}{16}\int_{\frac{1}{4}}^{\frac{3}{4}} G^*(\frac{1}{2}, s)q_1(s)ds \cdot \|y\| \\ &\geq \frac{\rho\lambda(\alpha-1)^2}{64\Gamma(\alpha)}\int_{\frac{1}{4}}^{\frac{3}{4}} s(1-s)^{\alpha-1}q_1(s)ds \cdot \|y\| \geq \|y\|.\end{aligned}$$

Therefore, by the (i) of Theorem 1.1, it follows that $\|T_\lambda y\| \geq \|y\|, y \in K\bigcap \partial\Omega_R$. Thus T_λ has a fixed point $y_2 \in K\bigcap(\overline{\Omega}_R \setminus \Omega_p)$, and $p < \|y_2\| \leq R$. Consequently, T_λ has two fixed points y_1, y_2 and $r \leq \|y_1\| < p < \|y_2\| \leq R$. It follows from above that if $g_0 = g_\infty = \infty$, then T_λ has two positive solutions for $0 < \lambda < \lambda_0$.

It is similar with (a) that $u_i(t) = t^{\alpha-2}y_i(t), i = 1, 2$ are two positive solutions of problem (1.1). □

Theorem 3.2. *Assume (A) hold and $i_0 = i_\infty = 0$, if*

$$\frac{64\Gamma(\alpha)}{(\alpha-1)^2\int_{\frac{1}{4}}^{\frac{3}{4}} s(1-s)^{\alpha-1}q_1(s)ds \cdot \max\{g_\infty, g_0\}} < \lambda < \frac{\Gamma(\alpha)}{\int_0^1 s(1-s)^{\alpha-1}q_2(s)ds \cdot \min\{g_\infty, g_0\}},$$

then (1.1) has a positive solution.

Proof. If $g_\infty > g_0$, then

$$\frac{64\Gamma(\alpha)}{(\alpha-1)^2\int_{\frac{1}{4}}^{\frac{3}{4}} s(1-s)^{\alpha-1}q_1(s)ds \cdot g_\infty} < \lambda < \frac{\Gamma(\alpha)}{\int_0^1 s(1-s)^{\alpha-1}q_2(s)ds \cdot g_0}.$$

It is easy to see that there exists $0 < \varepsilon < g_\infty$ such that

$$\frac{64\Gamma(\alpha)}{(\alpha-1)^2\int_{\frac{1}{4}}^{\frac{3}{4}} s(1-s)^{\alpha-1}q_1(s)ds \cdot (g_\infty-\varepsilon)} < \lambda < \frac{\Gamma(\alpha)}{\int_0^1 s(1-s)^{\alpha-1}q_2(s)ds \cdot (g_0+\varepsilon)}.$$

Now turning to g_0 and g_∞, there is $r_1 > 0$ such that $g(y) \leq (g_0+\varepsilon)y, 0 \leq y \leq r_1$ for $y \in K\bigcap\partial\Omega_{r_1}$. From Lemma 2.3, we have

$$\begin{aligned}\|T_\lambda y\| &\leq \frac{\lambda}{\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-1}q_2(s)(g_0+\varepsilon)y(s)ds \\ &\leq \frac{\lambda}{\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-1}q_2(s)ds \cdot (g_0+\varepsilon)\|y\| < \|y\|.\end{aligned}$$

On the other hand, there exists $H > r_1$, such that $g(y) \geq (g_\infty - \varepsilon)y, y \geq H$. Let $r_2 = \max\{2r_1, \frac{16H}{\alpha-1}\}$, then

$$y(t) \geq \frac{\alpha-1}{16}\|y\| \geq H, \quad y \in K\bigcap\partial\Omega_{r_2}, \quad t \in [\frac{1}{4}, \frac{3}{4}].$$

Thus $g(y) \geq (g_\infty - \varepsilon)y(t)$ for $y \in K \bigcap \partial\Omega_{r_2}, t \in [\frac{1}{4}, \frac{3}{4}]$.

So

$$\|T_\lambda y\| \geq T_\lambda y(\frac{1}{2}) \geq \lambda \int_{\frac{1}{4}}^{\frac{3}{4}} G^*(\frac{1}{2}, s) q_1(s) g(y(s)) ds$$
$$\geq \frac{\lambda(\alpha-1)^2}{64\Gamma(\alpha)} \int_{\frac{1}{4}}^{\frac{3}{4}} s(1-s)^{\alpha-1} q_1(s) ds \cdot (g_\infty - \varepsilon)\|y\| > \|y\|.$$

Consequently, T_λ has a fixed point $y \in K \bigcap (\overline{\Omega}_{r_2} \setminus \Omega_{r_1})$. Hence $u(t) = t^{\alpha-2}y(t)$ is the solution of (1.1).

If $g_\infty < g_0$, then $\frac{64\Gamma(\alpha)}{(\alpha-1)^2 \int_{\frac{1}{4}}^{\frac{3}{4}} s(1-s)^{\alpha-1} q_1(s) ds \cdot g_0} < \lambda < \frac{\Gamma(\alpha)}{\int_0^1 s(1-s)^{\alpha-1} q_2(s) ds \cdot g_\infty}$. There exists $0 < \varepsilon < g_0$ such that

$$\frac{64\Gamma(\alpha)}{(\alpha-1)^2 \int_{\frac{1}{4}}^{\frac{3}{4}} s(1-s)^{\alpha-1} q_1(s) ds \cdot (g_0 - \varepsilon)} < \lambda < \frac{\Gamma(\alpha)}{\int_0^1 s(1-s)^{\alpha-1} q_2(s) ds \cdot (g_\infty + \varepsilon)}.$$

Now turning to g_0 and g_∞, there exists $r_1 > 0$ such that $g(y) \geq (g_0 - \varepsilon)y, 0 \leq y \leq r_1$, so $g(y(t)) \geq (g_0 - \varepsilon)y(t)$ for $y \in K \bigcap \partial\Omega_{r_1}, t \in [\frac{1}{4}, \frac{3}{4}]$.
Hence

$$\|T_\lambda y\| \geq T_\lambda y(\frac{1}{2}) \geq \lambda \int_{\frac{1}{4}}^{\frac{3}{4}} G^*(\frac{1}{2}, s) q_1(s) g(y(s)) ds$$
$$\geq \lambda \int_{\frac{1}{4}}^{\frac{3}{4}} G^*(\frac{1}{2}, s) q_1(s) \cdot (g_0 - \varepsilon) y(s) ds$$
$$\geq \frac{\lambda(\alpha-1)^2}{64\Gamma(\alpha)} \int_{\frac{1}{4}}^{\frac{3}{4}} s(1-s)^{\alpha-1} q_1(s) ds \cdot (g_0 - \varepsilon)\|y\| > \|y\|.$$

On the other hand, there exists $H > r_1$ such that $g(y) \leq (g_\infty + \varepsilon)y, y \geq H$. Let $r_2 = \max\{2r_1, \frac{16H}{\alpha-1}\}$, then

$$y(t) \geq \frac{\alpha-1}{16}\|y\| \geq H \quad \text{for } y \in K \bigcap \partial\Omega_{r_2}, t \in [0,1].$$

Thus $g(y(t)) \leq (g_\infty + \varepsilon)y(t)$ for $y \in K \bigcap \partial\Omega_{r_2}, t \in [0,1]$. From Lemma 2.3, we have

$$\|T_\lambda y(t)\| \leq \frac{\lambda}{\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-1} q_2(s) g(y(s)) ds$$
$$\leq \frac{\lambda}{\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-1} q_2(s) ds \cdot (g_\infty + \varepsilon)\|y\| < \|y\|.$$

Consequently, T_λ has a fixed point $y \in K \bigcap (\overline{\Omega}_{r_2} \setminus \Omega_{r_1})$. Hence $u(t) = t^{\alpha-2}y(t)$ is the solution of (1.1) □

Example 3.1. Consider the boundary value problem

$$\begin{cases} D_{0+}^\alpha u(t) + \lambda(u^a(t) + u^b(t)) = 0, \ 0 < a < 1 < b < \frac{1}{2-\alpha}, 1 < \alpha < 2, \\ u(0) = 0, u(1) = 0. \end{cases} \tag{3.1}$$

Then problem (3.1) has two positive solutions u_1 and u_2 for each $0 < \lambda < \lambda_0$, where λ_0 is some positive constant.

Proof. We will apply Theorem 1.1(ii) to this end, we take $f(t,u) = u^a + u^b$, then $f(t, t^{\alpha-2}y) = t^{a(\alpha-2)}y^a + t^{b(\alpha-2)}y^b$. Let $q_1(t) = t^{a(\alpha-2)}, q_2(t) = t^{b(\alpha-2)}$and $g(y) = y^a + y^b$, then

$$q_1(t)g(y) \le f(t, t^{\alpha-2}y) \le q_2(t)g(y)$$

and $q_1, q_2 \in C(0,1), g \in C((0,+\infty),(0,+\infty))$, thus condition (A) is satisfied. Note $g_0 = \infty, g_\infty = \infty$, so $i_\infty = 2$. Since

$$\begin{aligned}
\frac{1}{\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-1}q_2(s)ds &= \frac{1}{\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-1}s^{b(\alpha-2)}ds \\
&= \frac{1}{\Gamma(\alpha)}\int_0^1 s^{(2+b(\alpha-2))-1}(1-s)^{\alpha-1}ds \\
&= \frac{1}{\Gamma(\alpha)}\frac{\Gamma(2+b(\alpha-2))\Gamma(\alpha)}{\Gamma(2+b(\alpha-2)+\alpha)} = \frac{\Gamma(2+b(\alpha-2))}{\Gamma(2+b(\alpha-2)+\alpha)},
\end{aligned}$$

and

$$M(1) = \max_{0\le y\le 1}\{g(y(s))\} = \max_{0\le y\le 1}\{y^a + y^b)\} = 2.$$

So

$$\lambda_0 = \{\frac{1}{\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-1}q_2(s)ds \cdot M(1)\}^{-1} = [\frac{2\Gamma(2+b(\alpha-2))}{\Gamma(2+b(\alpha-2)+\alpha)}]^{-1}.$$

Then from $0 < \lambda < \lambda_0$, we have $\|T_\lambda y\| < \|y\|$. The result is now from Theorem 1.1(ii). In particular, if $\alpha \to 2$, then $\lambda_0 \to 3$. □

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