Journal of Nonlinear Modeling and Analysis
Volume 2, Number 1, March 2020, 57–78

http://jnma.ca; http://jnma.ijournal.cn
DOI:10.12150/jnma.2020.57

Bi-center Problem in a Class of Z_2-equivariant Quintic Vector Fields*

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Abstract In this paper, we study the center problem for Z_2-equivariant quintic vector fields. First of all, for convenience in analysis, the system is simplified by using some transformations. When the system has two nilpotent points at $(0, \pm 1)$ with multiplicity three, the first seven Lyapunov constants at the singular points are calculated by applying the inverse integrating factor method. Then, fifteen center conditions are obtained for the two nilpotent singular points of the system to be centers, and the sufficiency of the first seven center conditions are proved. Finally, the first five Lyapunov constants are calculated at the two nilpotent points $(0, \pm 1)$ with multiplicity five by using the method of normal forms, and the center problem of this system is partially solved.

Keywords Nilpotent singular point, Center–focus problem, Bi-center, Lyapunov constant.

MSC(2010) 34C05, 34C07.

1. Introduction

Consider the following planar differential system,

$$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y), \tag{1.1}$$

where P and Q are polynomials. The second part of Hilbert's 16th problem is to find the upper bound of the limit cycles that system (1.1) can have. One important problem related to the bifurcation of limit cycles is to determine whether a singular point of system (1.1) is a center or not, which is called center problem. The distinction between a center case and a non-center case has great difference on the determination of limit cycles.

As a special class of system (1.1), the Z_n-equivariant vector fields have attractive properties because of their symmetry. It is well known that better results on the number of limit cycles are often obtained from Z_n-equivariant vector fields. In

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*This research was partially supported by the National Natural Science Foundation of China, No. 11601212 (F. Li), Shandong Natural Science Foundation, No. ZR2018MA002 (F. Li) and the Natural Sciences and Engineering Research Council of Canada, No. R2686A02 (P. Yu).

recent years, more and more attention has been paid to the center problem of Z_n-equivariant vector fields. For example, Liu and Li [1] gave a complete study on the bi-center problem of a class of Z_2-equivariant cubic vector fields. Romanovski *et al.* [2] studied the bi-center problem of some Z_2-equivariant quintic systems. Giné [3] investigated the coexistence of centres in two families of planar Z_n-equivariant systems. Theory of rotated equations was discussed by Han *et al.* [4] and applied to study a population model. The Poincaré return map and generalized focal values of analytic planar systems with a nilpotent focus or center were considered in [5] where the classical Hopf bifurcation theory was generalized. Global phase portraits of symmetrical cubic Hamiltonian systems with a nilpotent singular point were discussed in [6]. Recently, Yu *el al.* [7] applied the method of normal forms to improve the results on the number of limit cycles bifurcating from a non-degenerate center of various homogeneous polynomial differential systems. A special type of bifurcation of limit cycles from a nilpotent critical point was studied in [8]. However, for degenerate singular points, because of difficulty, there are very few results obtained even for Z_2-equivalent systems with two nilpotent singular points. In [9], the authors proved that the origin of any Z_2-symmetric system is a nilpotent center if and only if there exists a local analytic first integral. Recently, we studied bifurcation of limit cycles in a class of Z_2-equivalent cubic planar differential systems with two nilpotent singular points, described by

$$\begin{aligned}\frac{dx}{dt} &= A_{10}x + A_{01}y + A_{30}x^3 + A_{21}x^2y + A_{12}xy^2 + A_{03}y^3 = X(x,y),\\ \frac{dy}{dt} &= B_{10}x + B_{01}y + B_{30}x^3 + B_{21}x^2y + B_{12}xy^2 + B_{03}y^3 = Y(x,y).\end{aligned} \tag{1.2}$$

In [10], sufficient and necessary conditions for the critical points of the system (1.2) to be centers were obtained. In addition, the existence of 12 small-amplitude limit cycles bifurcating from the critical points was proved.

In this paper, a class of Z_2-equivariant quintic planar differential systems with two nilpotent singular points, given by

$$\begin{aligned}\frac{dx}{dt} =& A_{10}x + A_{01}y + A_{50}x^5 + A_{41}x^4y + A_{32}x^3y^2 + A_{23}x^2y^3\\ &+ A_{14}xy^4 + A_{05}y^5 = X(x,y),\\ \frac{dy}{dt} =& B_{10}x + B_{01}y + B_{50}x^5 + B_{41}x^4y + B_{32}x^3y^2 + B_{23}x^2y^3\\ &+ B_{14}xy^4 + B_{05}y^5 = Y(x,y),\end{aligned} \tag{1.3}$$

are studied. Necessary conditions for the singular points of system (1.3) to be centers are derived.

The rest of the paper is organized as follows. In the next section, we simplify system (1.3) for convenience in analysis. In Section 3, the first seven Lyapunov constants at an order-3 nilpotent singular point are computed by using the inverse integrating factor method or the method of normal forms. Bi-center conditions in Z_2-equivariant vector fields are discussed, and fifteen bi-center conditions are obtained for system (1.3). Further, the first five Lyapunov constants at an order-5 nilpotent singular point are computed by using the method of normal forms, yielding one more bi-center condition.

2. Simplification of system (1.3)

Suppose $(0, \pm 1)$ are isolated singular points of system (1.3). Then,

$$A_{01} = -A_{05}, \quad B_{01} = -B_{05}, \tag{2.1}$$

and the Jacobin matrix of system (1.3) evaluated at $(0, \pm 1)$ is given by

$$J_0 = \begin{pmatrix} \frac{\partial X}{\partial x} & \frac{\partial X}{\partial y} \\ \frac{\partial Y}{\partial x} & \frac{\partial Y}{\partial y} \end{pmatrix}_{(0,\ \pm 1)} = \begin{pmatrix} A_{10} + A_{14} & 4A_{05} \\ B_{10} + B_{14} & 4B_{05} \end{pmatrix}. \tag{2.2}$$

We have the following result.

Lemma 2.1. *Suppose* $(0, \pm 1)$ *are isolated nilpotent singular points of* (1.3)*. Then*

$$A_{05} \neq 0.$$

Proof. Suppose $A_{05} = 0$. Then J_0 is a triangular matrix with two eigenvalues,

$$\lambda_1 = A_{10} + A_{14}, \quad \lambda_2 = 4B_{05}.$$

Since $(0, \pm 1)$ are isolated nilpotent singular points, we have

$$\lambda_1 = \lambda_2 = 0,$$

under which together with (2.1), system (1.3) is reduced to

$$\begin{aligned} \frac{dx}{dt} &= x(A_{10} + A_{50}x^4 + A_{41}x^3y + A_{32}x^2y^2 + A_{23}xy^3 + A_{14}y^4), \\ \frac{dy}{dt} &= x(B_{10} + B_{50}x^4 + B_{41}x^3y + B_{32}x^2y^2 + B_{23}xy^3 + B_{14}y^4), \end{aligned} \tag{2.3}$$

which has a common factor x in the two equations, implying that $(0, \pm 1)$ are not isolated, and so Lemma 2.1 is proved. $\square$

Lemma 2.2. *Suppose* $(0, \pm 1)$ *are isolated nilpotent singular points of system* (1.3)*. Then without loss of generality, it can be assumed that*

$$\begin{aligned} &A_{10} = -A_{14}, \quad A_{05} = \frac{1}{4}, \\ &B_{10} = -B_{14}, \quad B_{05} = 0. \end{aligned} \tag{2.4}$$

Proof. Suppose $(0, \pm 1)$ are isolated nilpotent singular points of system (1.3). So $A_{05} \neq 0$ by Lemma 2.1. Consider the following non-degenerate transformation,

$$x = 4A_{05}\xi, \quad y = 4B_{05}\xi + \eta, \tag{2.5}$$

which has fixed points $(0, 0)$ and $(0, \pm 1)$. By applying the transformation (2.5), it is easy to obtain the Jacobin matrix evaluated at $(0, \pm 1)$, given by

$$J_1 = \begin{pmatrix} \mathrm{Tr}(J_0) & 1 \\ -\mathrm{Det}(J_0) & 0 \end{pmatrix}. \tag{2.6}$$

Since $(0,\pm 1)$ are nilpotent singular points of system (1.3), we have

$$\mathrm{Tr}(J_0)=\mathrm{Det}(J_0)=0,$$

and then (2.6) becomes

$$J_1^*=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}.$$

Namely, we can always suppose

$$J_0=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}.$$

Otherwise, J_0 can be transformed into J_1^* by the transformation (2.5), so Lemma 2.2 is proved. □

By Lemma 2.2, we have that

Lemma 2.3. *If* $(0,\pm 1)$ *are isolated nilpotent singular points of system* (1.3)*, then the system can be simplified as*

$$\begin{aligned}\frac{dx}{dt}&=-\frac{1}{4}y+\frac{1}{4}y^5-A_{14}x+A_{50}x^5+A_{41}x^4y+A_{32}x^3y^2+A_{23}x^2y^3+A_{14}xy^4,\\ \frac{dy}{dt}&=-B_{14}x+B_{50}x^5+B_{41}x^4y+B_{32}x^3y^2+B_{23}x^2y^3+B_{14}xy^4.\end{aligned}\tag{2.7}$$

Now we discuss the multiplicity of nilpotent singular points $(0,\pm 1)$ of system (2.7). System (2.7) can be transformed to

$$\begin{aligned}\frac{d\xi}{dt}&=\frac{1}{4}\eta(1+\eta)(2+\eta)(\eta^2+2\eta+2)+A_{50}\xi^5+A_{41}\xi^4(1+\eta)\\ &\quad+A_{32}\xi^3\eta(1+\eta)^2+A_{23}\xi^2\eta(1+\eta)^3+A_{14}\xi\eta(2+\eta)(\eta^2+2\eta+2)\\ &=\Phi(\xi,\eta),\\ \frac{d\eta}{dt}&=B_{50}\xi^5+B_{41}\xi^4(1+\eta)+B_{32}\xi^3\eta(1+\eta)^2\\ &\quad+B_{23}\xi^2\eta(1+\eta)^3+B_{14}\xi\eta(2+\eta)(2+\eta(2+\eta))\\ &=\Psi(\xi,\eta),\end{aligned}\tag{2.8}$$

by

$$\xi=\pm x,\quad \eta=\pm y-1.$$

Suppose the only solution of the implicit function equation $\Phi(\xi,\eta)=0$ near $(0,0)$ is

$$\eta=f(\xi)=\sum_{k=2}^{\infty}c_k\xi^k.$$

Denote

$$\begin{aligned}\Psi(\xi,f(\xi))&=\sum_{k=2}^{\infty}\alpha_k\xi^k,\\ \left[\frac{\partial\Phi}{\partial\xi}+\frac{\partial\Psi}{\partial\eta}\right]_{(\xi,f(\xi)}&=\sum_{l=1}^{\infty}\beta_l\xi^l.\end{aligned}\tag{2.9}$$

By using the intersection number of algebraic curves, the definition of multiplicity of a singular point of system (1.1) is given in [11].

Definition 2.1. [11] Suppose that the origin is an isolated singular point of system (2.8) and the conditions given in (2.9) holds. If $\alpha_2 = \alpha_3 = \cdots = \alpha_{k-1} = 0,\ \alpha_k \neq 0$, the origin is called a k-multiple singular point of system (2.8), and the k is called the multiplicity of the origin.

It is not difficult to obtain the coefficients α_k and β_l in (2.9), as given below:

$$\begin{aligned}
\alpha_2 &= B_{23},\\
\alpha_3 &= B_{32} - 4A_{23}B_{14},\\
\alpha_4 &= 16A_{14}A_{23}B_{14} - 4A_{32}B_{14} - 3A_{23}B_{23} + B_{41},\\
\alpha_5 &= -64A_{14}^2A_{23}B_{14} + 8A_{23}^2B_{14} + 16A_{14}A_{32}B_{14} - 4A_{41}B_{14} + 12A_{14}A_{23}B_{23}\\
&\quad - 3A_{32}B_{23} - 2A_{23}B_{32} + B_{50},\\
\alpha_6 &= \frac{1}{2}(512A_{14}^3A_{23}B_{14} - 96A_{14}A_{23}^2B_{14} - 128A_{14}^2A_{32}B_{14} + 24A_{23}A_{32}B_{14}\\
&\quad + 32A_{14}A_{41}B_{14} - 8A_{50}B_{14} - 96A_{14}^2A_{23}B_{23} + 9A_{23}^2B_{23} + 24A_{14}A_{32}B_{23}\\
&\quad - 6A_{41}B_{23} + 16A_{14}A_{23}B_{32} - 4A_{32}b32 - 2A_{23}B_{41}),\\
\beta_1 &= 2(A_{23} + 2b_{14}),\\
\beta_2 &= -4A_{14}A_{23} + 3A_{32} + 3B_{23},\\
\beta_3 &= -2(-8A_{14}^2A_{23} + 3A_{23}^2 + 2A_{14}A_{32} - 2A_{41} + 6A_{23}B_{14} - b32),\\
\beta_4 &= -64A_{14}^3A_{23} + 32A_{14}A_{23}^2 + 16A_{14}^2A_{32} - 12A_{23}A_{32} - 4A_{14}A_{41} + 5A_{50}\\
&\quad + 48A_{14}A_{23}B_{14} - 12A_{32}B_{14} - 6A_{23}B_{23} + B_{41}.
\end{aligned} \tag{2.10}$$

According to Theorems 7.2 and 7.3 in [12], the types of the origin of system (2.7) can be classified as follows.

For $k = 2m,\ \alpha_k \neq 0$,

$$\begin{cases} \beta_n = 0, \text{ degenerate point},\\ \beta_n \neq 0 \ \begin{cases} n \geq m, \text{ degenerate point},\\ n < m, \text{ saddle-node point}.\end{cases}\end{cases}$$

For $k = 2m+1,\ \lambda = \beta_n^2 + 4(m+1)\alpha_{2m+1}$,

$$\begin{cases} \alpha_{2m+1} > 0, \text{ saddle},\\ \alpha_{2m+1} < 0, \begin{cases} \beta_n = 0, \text{ center or focus},\\ \beta_n \neq 0 \ \begin{cases} n > m, & \\ \text{or } n = m,\ \lambda < 0, & \text{center or focus},\\ n < m, & \\ \text{or } n = m,\ \lambda \geq 0, & \begin{cases} n \text{ even, node},\\ n \text{ odd, degenerate point}.\end{cases}\end{cases}\end{cases}\end{cases}$$

Proposition 2.1. *The nilpotent singular points* $(0, \pm 1)$ *of* (2.7) *are degenerate singular points when* $\alpha_2 = B_{23} \neq 0$.

Phase portraits of some systems were given in [13] with the help of Maple programme P4. We also use the Maple programme P4 to show some simulations.

Example 2.1. When $A_{50} = A_{41} = A_{32} = A_{23} = A_{14} = B_{50} = B_{41} = B_{32} = B_{23} = B_{14} = 1$, obviously we have $\alpha_2 = B_{23} = 1 \neq 0$. So according to the classification, $(0, \pm 1)$ are degenerate singular points, as shown in Figure 1.

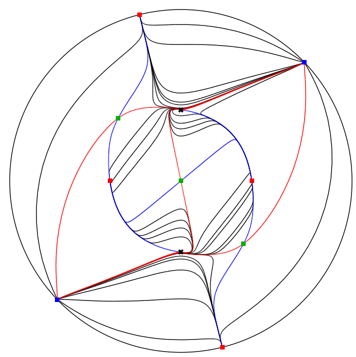

Figure 1. Phase portrait showing $(0, \pm 1)$ to be degenerate singular points for $A_{50} = A_{41} = A_{32} = A_{23} = A_{14} = B_{50} = B_{41} = B_{32} = B_{23} = B_{14} = 1$.

Proposition 2.2. *The nilpotent singular points* $(0, \pm 1)$ *of* (2.7) *are saddles for* $B_{23} = 0, \alpha_3 > 0$*; but are centers or foci when the following conditions hold:*

$$B_{23} = 0, \quad \alpha_3 < 0, \quad \beta_1 \neq 0, \quad \lambda_1 = \beta_1^2 + 8\alpha_3 < 0.$$

Proposition 2.3. *The nilpotent singular points* $(0, \pm 1)$ *of* (2.7) *are saddle-node points when* $B_{23} = 0, \alpha_3 = 0, \alpha_4 \beta_1 \neq 0$*; and are degenerate singular points when* $B_{23} = 0,\ \alpha_3 = \beta_1 = 0,\ \alpha_4 \neq 0$.

Example 2.2. Taking $A_{50} = A_{41} = A_{32} = A_{14} = B_{50} = B_{41} = B_{32} = B_{14} = 1, B_{23} = 0,\ A_{23} = -2, B_{32} = -10$, we have $\alpha_2 = B_{23} = 0,\ \alpha_3 = 1 > 0$, so $(0, \pm 1)$ are saddle-node points, see Figure 2.

When $A_{50} = A_{41} = A_{32} = A_{14} = B_{50} = B_{41} = B_{14} = 1,\ B_{23} = 0,\ A_{23} = -2,\ B_{32} = -20$, we have $\alpha_2 = B_{23} = 0,\ \alpha_3 = -1 > 0,\ \lambda = 1 > 0$, hence $(0, \pm 1)$ are degenerate singular points, as depicted in Figure 3.

When $B_{23} = 0, \alpha_3 = \alpha_4 = 0$, we have the following result.

Proposition 2.4. *The nilpotent singular points* $(0, \pm 1)$ *of* (2.7) *can be classified*

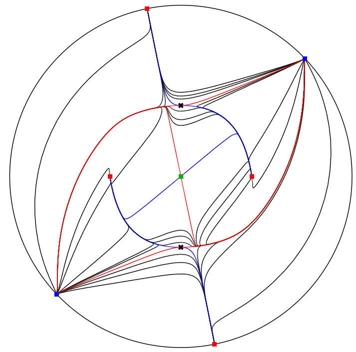

Figure 2. Phase portrait showing $(0, \pm 1)$ to be saddle-node points for $A_{50} = A_{41} = A_{32} = A_{14} = B_{50} = B_{41} = B_{32} = B_{14} = 1$, $B_{23} = A_{23} = 0$.

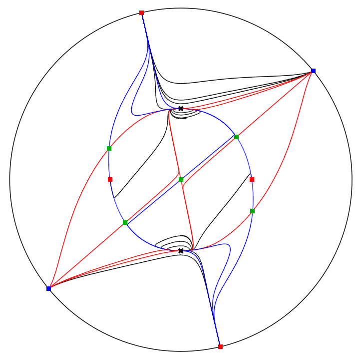

Figure 3. Phase portrait showing $(0, \pm 1)$ to be degenerate singular points for $A_{50} = A_{41} = A_{32} = A_{23} = A_{14} = B_{50} = B_{41} = B_{32} = B_{14} = 1$, $B_{23} = 0$.

as follows:

$$\begin{cases} \alpha_5 > 0, \ \textit{saddle}, \\ \alpha_5 < 0, \begin{cases} \beta_1 \neq 0, \ \textit{degenerate point}, \\ \beta_1 = 0 \begin{cases} \beta_2 = 0, \ \textit{center or focus}, \\ \beta_2 \neq 0, \begin{cases} \lambda < 0 \, \textit{center or focus}, \\ \lambda \geq 0 \, \textit{node}. \end{cases} \end{cases} \end{cases} \end{cases}$$

Proposition 2.5. *The nilpotent singular points* $(0, \pm 1)$ *of* (2.7) *are saddle-node points when the following conditions hold:*

$$B_{23} = 0, \quad \alpha_3 = 0, \quad \alpha_4 = \alpha_5 = \beta_1 = 0, \quad \alpha_6 \beta_2 \neq 0;$$

and degenerate singular points when

$$B_{23} = 0, \quad \alpha_3 = 0, \quad \alpha_4 = \alpha_5 = \beta_1 = \beta_2 = 0, \quad \alpha_6 \neq 0.$$

Example 2.3. When $A_{50} = A_{41} = A_{32} = A_{23} = A_{14} = B_{50} = B_{41} = B_{32} = B_{14} = 1$, $B_{23} = 0$, we have $\alpha_2 = B_{23} = 0$, $\alpha_3 = -1 > 0$, $\lambda = -1 < 0$, which implies that $(0, \pm 1)$ are foci, see Figure 4.

When $A_{50} = A_{41} = A_{32} = A_{14} = B_{50} = B_{41} = B_{32} = B_{14} = 1$, $B_{23} = A_{23} = 0$, we obtain $\alpha_2 = B_{23} = 0$, $\alpha_3 = -1 > 0$, $\lambda = -1 < 0$, and so $(0, \pm 1)$ are centers, as shown in Figure 5.

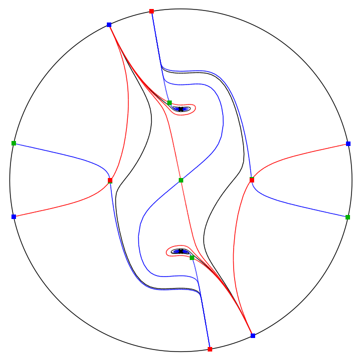

Figure 4. Phase portrait showing $(0, \pm 1)$ to be foci for $A_{50} = A_{41} = A_{32} = A_{14} = B_{50} = B_{41} = B_{14} = 1$, $B_{23} = 0$, $A_{23} = -2$, $B_{32} = -20$.

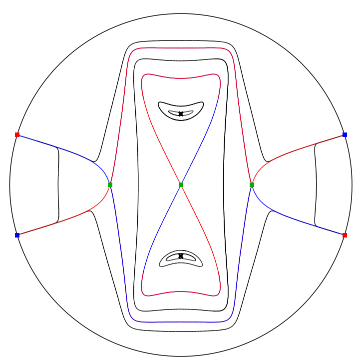

Figure 5. Phase portrait showing $(0, \pm 1)$ to be centers for $A_{50} = A_{41} = A_{32} = A_{14} = B_{50} = B_{41} = B_{32} = B_{14} = 1$, $B_{23} = 0$, $A_{23} = -2$, $B_{32} = -10$.

Proposition 2.6. *The multiplicity of the nilpotent singular points* $(0, \pm 1)$ *of* (2.7) *is at most six.*

Proof. Using (2.10) and setting $\alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 0$, we obtain

$$
\begin{aligned}
&B_{23} = 0, \quad B_{32} = 4A_{23}B_{14}, \\
&B_{41} = -4(4A_{14}A_{23} - a_{32})B_{14}, \\
&B_{50} = 4(16A_{14}^2 A_{23}B_{14} - 4A_{14}A_{32}B_{14} + A_{41}B_{14}),
\end{aligned}
$$

and $\alpha_6 = 4(64A_{14}^3A_{23} - 16A_{14}^2A_{32} + 4A_{14}A_{41} - A_{50})B_{14}$. For $\alpha_6 = 0$, there are two cases.

If $B_{14} = 0$, $\Psi(\xi,\eta) \equiv 0$, indicating that $(0,\pm1)$ are not isolated.

If $A_{50} = 4(16A_{14}^3A_{23} - 4A_{14}^2A_{32} + A_{14}A_{41})$, system (2.8) becomes

$$\begin{aligned}\frac{d\xi}{dt} &= \frac{1}{4}\eta(1 + 4A_{14}\xi + \eta)f(\xi,\eta),\\ \frac{d\eta}{dt} &= B_{14}\xi f(\xi,\eta),\end{aligned}$$

where

$$\begin{aligned}f(\xi,\eta) = {}& 4A_{41}\xi^4 + 4A_{32}\xi^3(1 - 4A_{14}\xi + \eta)\\ &+4A_{23}\xi^2\big[16A_{14}^2\xi^2 - 4A_{14}\xi(1+\eta)\\ &+(1+\eta)^2\big] + \eta(2+\eta)\big[2 + \eta(2+\eta)\big].\end{aligned}$$

Hence there exists a common factor $f(\xi,\eta)$ in $\Phi(\xi,\eta)$ and $\Psi(\xi,\eta)$, implying that $(0,\pm1)$ are not isolated. $\square$

Since the multiplicity of a nilpotent focus or center is an odd positive integer greater than 1, Proposition 2.6 implies that the multiplicity of $(0,\pm1)$ is 3 or 5 if $(0,\pm1)$ are nilpotent foci or centers of system (2.7). In particular, $(0,\pm1)$ are nilpotent foci or centers of system (2.7) with multiplicity 3 if and only if

$$\alpha_2 = 0, \quad \alpha_3 < 0, \quad \Delta = \beta_1^2 + 8\alpha_3 < 0,$$

namely,

$$B_{23} = 0, \quad B_{32} - 2A_{23}B_{14} < 0, \quad 4(A_{23} - 2B_{14})^2 + 8B_{32} < 0.$$

By a simple scaling, we can assume that $\alpha_3 = -2$, yielding $B_{32} = -2 + 4A_{23}B_{14}$. Now, combining the above results, we get our first main theorem.

Theorem 2.1. *Suppose* $(0,\pm1)$ *are the nilpotent foci or centers of system* (1.3) *with multiplicity* 3. *By proper linear transformation and time rescaling, system* (1.3) *can be changed to*

$$\begin{aligned}\frac{dx}{dt} &= -A_{14}x - \frac{1}{4}y + \frac{1}{4}y^5 + A_{50}x^5 + A_{41}x^4y + A_{32}x^3y^2 + A_{23}x^2y^3,\\ \frac{dy}{dt} &= -B_{14}x + B_{50}x^5 + B_{41}x^4y + (-2 + 4A_{23}B_{14})x^3y^2 + B_{14}xy^4,\end{aligned} \tag{2.11}$$

where the following condition is satisfied:

$$\Delta = 4(A_{23} + 2B_{14})^2 - 16 < 0.$$

Furthermore, we can obtain that if

$$\alpha_2 = \alpha_3 = \alpha_4 = 0, \alpha_5 < 0, \quad \Delta = \beta_2^2 + 12\alpha_5 < 0,$$

namely

$$\begin{aligned}&B_{23} = 0, \quad B_{32} = 2A_{23}B_{14}, \quad B_{41} = -4(4A_{14}A_{23}B_{14} - A_{32}B_{14}),\\ &A_{23} = -2B_{14}, \quad \beta_2^2 + 12\alpha_5 < 0,\end{aligned}$$

the singular points $(0, \pm 1)$ are isolated nilpotent foci or centers of system (2.7) with multiplicity 5.

Similarly, with a simple scaling, we can assume that $\alpha_3 = -\frac{3}{4}$, yielding

$$B_{50} = -\frac{3}{4} - (16A_{14}A_{32}B_{14} - 4A_{41}B_{14} + 128A_{14}^2 B_{14}^2). \tag{2.12}$$

Now, combining the above results, we have our second main theorem.

Theorem 2.2. *Suppose* $(0, \pm 1)$ *are isolated nilpotent foci or centers of system* (1.3) *with multiplicity 5. By proper linear transformation and time rescaling, system* (1.3) *can be rewritten as*

$$\begin{aligned} \frac{dx}{dt} &= \frac{1}{4}(-4A_{14}x + 4A_{50}x^5 - y + 4A_{41}x^4y + 4A_{32}x^3y^2 \\ &\quad + 4A_{23}x^2y^3 + 4A_{14}xy^4 + y^5), \\ \frac{dy}{dt} &= -x(B_{14} - B_{50}x^4 + 16A_{14}A_{23}B_{14}x^3y - 4A_{32}B_{14}x^3y \\ &\quad - 4A_{23}B_{14}x^2y^2 - B_{14}y^4), \end{aligned} \tag{2.13}$$

where B_{50} *is given in* (2.12) *and* $\Delta = (3A_{32} + 8A_{14}B_{14})^2 - 9 < 0$.

3. Center problem of $(0, \pm 1)$ with multiplicity 3

Consider the following system,

$$\begin{aligned} \frac{dx}{dt} &= y + a_{20}x^2 + \sum_{k+2j=3}^{\infty} a_{kj}x^k y^j, \\ \frac{dy}{dt} &= b_{11}xy + b_{30}x^3 + \sum_{k+2j=4}^{\infty} b_{kj}x^k y^j. \end{aligned} \tag{3.1}$$

The origin of system (3.1) is a nilpotent singular point with multiplicity 3 if and only if $b_{30} - a_{20}b_{11} \neq 0$. Especially, when $b_{30} - a_{20}b_{11} < 0$, the origin of system (3.1) is a nilpotent focus or center if $(2a_{20} - b_{11})^2 + 8b_{30} < 0$, or a degenerate singular point if $(2a_{20} - b_{11})^2 + 8b_{30} \geq 0$.

For system (3.1) with a nilpotent focus or center, an inverse integrating factor method developed by Liu and Li in [14] for computing the Lyapunov constants of the system, as stated in the following theorem. However, note that this method is restricted to the case when the nilpotent focus or center has multiplicity 3.

Theorem 3.1. *For system* (3.1)*, there exists a power series of the form*

$$M(x, y) = \left[y^2 + \frac{1}{2}(2a_{20} - b_{11})x^2y - \frac{1}{2}b_{30}x^4\right] + \sum_{k+2j=5}^{\infty} c_{kj}x^k y^j,$$

such that

$$\frac{\partial}{\partial x}\left(\frac{X}{M^{s+1}}\right) + \frac{\partial}{\partial y}\left(\frac{Y}{M^{s+1}}\right) = \frac{1}{M^{s+2}}\sum_{m=1}^{\infty} \nu_m(2m - 4s - 3)x^{2m+4},$$

for an integer s.

The recursive formulas for computing c_{kj} and ν_m can be found in Theorem 4.5 of [14], ν_m is the mth Lyapunov constant of system (3.1) at the origin.

Now we compute the first seven Lyapunov constants at $(0, \pm 1)$ of system (2.11). For simplicity, we denote

$$A_{23} = 2\mu - 2B_{14}.$$

According to Theorem 2.1, we detect that $(0, \pm 1)$ of system (2.11) are foci or centers if $\mu^2 < 1$ and degenerate singular points if $\mu^2 \geq 1$. But it is very difficult to compute the Lyapunov constants when $\mu \neq 0$. In this paper we only consider the case when $\mu = 0$, namely $A_{23} = -2B_{14}$.

3.1. Lyapunov constants for the case $\mu = 0$

Using the formulas in Theorem 3.1, we obtain the Lyapunov constants as follows.

Proposition 3.1. *The first seven Lyapunov constants at* $(0, \pm 1)$ *of system* (2.11) *are*

$$\begin{aligned}
\nu_1 &= 3A_{32} + 8A_{14}B_{14},\\
\nu_2 &= \frac{H_2}{9},\\
\nu_3 &= \frac{H_3}{675},\\
\nu_4 &= \frac{H_4}{2835000},\\
\nu_5 &= \frac{H_5}{30618000000},\\
\nu_6 &= \frac{H_6}{848730960000000},\\
\nu_7 &= \frac{H_7}{18536284166400000000},
\end{aligned}$$

where

$$\begin{aligned}
H_2 = 45A_{50} + 4A_{14}\big\{&-3A_{41}(3 + 32B_{14}^2)\\
&+2(3 + 8B_{14}^2)\big[-3 + 8B_{14}(4A_{14}^2 + 3B_{14})\big]\big\}\\
&-9B_{41} + 6\big[3A_{41} - 4B_{14}(4A_{14}^2 + 3B_{14})\big]B_{41},
\end{aligned}$$

and H_3, H_4, H_5, H_6 *and* H_7 *are polynomials in* B_{50}, A_{14}, B_{14}, A_{41}, B_{41} *and* μ, *which contain* 47, 174, 481, 1070, *and* 2133 *terms, respectively.*

3.2. Bi-center conditions of system (2.11) for $\mu = 0$

In this subsection, we apply the inverse integrating factor method to study the bi-center problem of system (2.11). We have the following result.

Theorem 3.2. *The first seven Lyapunov constants at the two singular points* $(0, \pm 1)$ *of system* (2.11) *are all zero if and only if one of the following fifteen conditions* *holds:*

$$
\begin{aligned}
I_1:\quad & A_{32}=0,\ A_{14}=0,\ A_{50}=0,\ B_{41}=0;\\
I_2:\quad & A_{32}=0,\ A_{14}=0,\ A_{50}=-\frac{1}{5}B_{41},\ A_{41}=1+4B_{14}^2;\\
I_3:\quad & A_{32}=0,\ A_{14}=0,\ A_{50}=-\frac{1}{5}B_{41}(-1+2A_{41}),\ B_{14}=0, B_{50}=0;\\
I_4:\quad & A_{32}=0,\ A_{50}=2A_{14},\ B_{50}=4(8A_{14}+B_{41})A_{14},\ A_{41}=\frac{1}{2}, B_{14}=0;\\
I_5:\quad & A_{32}=0,\ A_{50}=4A_{14}A_{41},\ B_{50}=0,\ B_{41}=-8A_{14},\ B_{14}=0;\\
I_6:\quad & A_{32}=-8A_{14}^3,\ A_{50}=-4A_{14}(64A_{14}^4-A_{41}),\ B_{14}=3A_{14}^2,\\
& B_{50}=-12(64A_{14}^4-2-A_{41})A_{14}^2,\ B_{41}=8A_{14}(-1+24A_{14}^4);\\
I_7:\quad & A_{32}=-\frac{8}{3}A_{14}B_{14},\ A_{50}=\frac{16}{105}A_{14}(-91B_{14}^2+168A_{14}^2B_{14}-15),\\
& A_{41}=-2+\frac{416}{15}A_{14}^2B_{14}-\frac{52}{15}B_{14}^2,\ B_{41}=\frac{16}{21}A_{14}(28B_{14}^2-3),\\
& B_{50}=\frac{16}{15}B_{14}^2(-13B_{14}+24A_{14}^2);\\
I_8:\quad & A_{32}=-\frac{2B_{41}B_{14}}{9+16B_{14}^2},\ A_{50}=-\frac{3}{25(9+16B_{14}^2)}B_{41}(12B_{14}^2-5),\\
& A_{41}=\frac{196}{45}B_{14}^2+\frac{4}{5},\ B_{50}=\frac{56}{125}(14B_{14}^2+5)B_{14},\\
& A_{14}=\frac{3B_{41}}{4(9+16B_{14}^2)},\ B_{41}=\frac{4}{45}\sqrt{15B_{14}}(16B_{14}^2+9), B_{14}>0;\\
I_9:\quad & A_{32}=-\frac{4B_{41}B_{14}}{3+32B_{14}^2},\ A_{50}=\frac{3B_{41}(8B_{14}^2+1)}{3+32B_{14}^2},\\
& A_{41}=-\frac{28}{9}B_{14}^2+\frac{1}{3},\ B_{50}=-\frac{28}{9}(1+4B_{14}^2)B_{14},\\
& A_{14}=-\frac{3B_{41}}{2(3+32B_{14}^2)},\ B_{41}=\frac{4}{9}\sqrt{-3B_{14}}(32B_{14}^2+3), B_{14}<0;\\
I_{10}:\quad & A_{32}=-\frac{2B_{41}B_{14}}{16B_{14}^2-1},\ A_{50}=\frac{B_{41}(412B_{14}^2-25)}{25(16B_{14}^2-1)},\\
& A_{41}=-\frac{28}{5}B_{14}^2-2,\ B_{50}=-\frac{13104}{25}B_{14}^3,\ A_{14}=\frac{3}{4(16B_{14}^2-1)},\\
& B_{41}=\frac{4}{5}\sqrt{-5B_{14}}(16B_{14}^2-1),\ B_{14}<0;\\
I_{11}:\quad & A_{32}=-\frac{8}{3}A_{14}B_{14},\\
& A_{50}=-\frac{1}{3(-3B_{41}-24A_{14}+64B_{14}^2A_{14})}f_5,\\
& B_{50}=-\frac{16(256A_{14}^3B_{14}^2-12B_{41}A_{14}^2-64A_{14}B_{14}^3-8A_{14}B_{14}+3B_{14}B_{41})B_{14}^2}{-3B_{41}-24A_{14}+64B_{14}^2A_{14}},\\
& A_{41}=\frac{2}{3(-3B_{41}-24A_{14}+64B_{14}^2A_{14})}f_6;
\end{aligned}
$$

$$
\begin{aligned}
I_{12}:\quad & A_{32} = -\frac{8}{3}A_{14}B_{14},\\
& A_{50} = \frac{128}{15}A_{41}B_{14}^2A_{14} + \frac{32}{15}B_{41}A_{14}^2B_{14} + \frac{1}{5}B_{41} + \frac{8}{5}A_{14} - \frac{2}{5}A_{41}B_{41} + \frac{4}{5}A_{14}A_{41}\\
&\qquad - \frac{256}{15}A_{14}^3B_{14} - \frac{2048}{45}B_{14}^3A_{14}^3 + \frac{8}{5}B_{14}^2B_{41} - \frac{512}{15}B_{14}^4A_{14} - \frac{128}{15}B_{14}^2A_{14},\\
& B_{50} = -\frac{4f_1}{15g_1},\\
& A_{41} = \frac{1}{3B_{14}}(16B_{14}^2A_{14}^2 + 12B_{14}^3 + 3B_{14} + 3A_{14}^2 \pm 3\sqrt{(A_{14}^2B_{14} + A_{14}^4)});\\
I_{13}:\quad & A_{32} = -\frac{8}{3}A_{14}B_{14},\ \ B_{41} = \frac{16}{21}A_{14}(28B_{14}^2 - 3),\\
& A_{50} = -\frac{4}{105}A_{14}(576A_{14}^2B_{14} + 208B_{14}^2 - 30 - 45A_{41}),\\
& B_{50} = \frac{120}{7}B_{14} - \frac{212992}{315}A_{14}^4B_{14}^3 - \frac{26624}{63}B_{14}^4A_{14}^2 + \frac{6656}{105}B_{14}^5\\
&\qquad + \frac{64}{7}A_{14}^2A_{41} - \frac{26624}{105}A_{14}^4B_{14} - \frac{4608}{35}B_{14}^2A_{14}^2 + \frac{5504}{105}B_{14}^3\\
&\qquad + \frac{128}{7}A_{14}^2 + \frac{256}{105}A_{41}B_{14}^3 - \frac{32}{7}A_{41}^2B_{14} + \frac{15872}{105}A_{14}^2B_{14}^2A_{41} - \frac{4}{7}B_{14}A_{41},\\
& A_{41} = \frac{1}{3B_{14}}(3B_{14} + 16B_{14}^2A_{14}^2 + 3A_{14}^2 + 12B_{14}^3 \pm 3A_{14}\sqrt{B_{14} + A_{14}^2});\\
I_{14}:\quad & A_{32} = -\frac{8}{3}A_{14}B_{14},\\
& A_{50} = \frac{128}{15}A_{41}B_{14}^2A_{14} + \frac{32}{15}B_{41}A_{14}^2B_{14} + \frac{1}{5}B_{41} + \frac{8}{5}A_{14} - \frac{2}{5}A_{41}B_{41}\\
&\qquad + \frac{4}{5}A_{14}A_{41} - \frac{256}{15}A_{14}^3B_{14} - \frac{2048}{45}B_{14}^3A_{14}^3 + \frac{8}{5}B_{14}^2B_{41}\\
&\qquad - \frac{512}{15}B_{14}^4A_{14} - \frac{128}{15}B_{14}^2A_{14},\\
& A_{41} = \frac{1}{3(448B_{14}^2A_{14} - 21B_{41} - 48A_{14})}f_7,\\
& B_{41} = \frac{4f_2}{21g_2},\\
& B_{50} = -\frac{512}{5}A_{14}^4B_{14} - \frac{448}{15}B_{14}^2A_{14}^2 + \frac{172}{7}A_{14}^2 + 16B_{14}^3 + \frac{132}{7}B_{14} \pm \frac{4}{105}\sqrt{f_3};\\
I_{15}:\quad & A_{32} = -\frac{8}{3}A_{14}B_{14},\\
& A_{50} = \frac{128}{15}A_{41}B_{14}^2A_{14} + \frac{32}{15}B_{41}A_{14}^2B_{14} + \frac{1}{5}B_{41} + \frac{8}{5}A_{14} - \frac{2}{5}A_{41}B_{41} + \frac{4}{5}A_{14}A_{41}\\
&\qquad - \frac{256}{15}A_{14}^3B_{14} - \frac{2048}{45}B_{14}^3A_{14}^3 + {}_{8}5B_{14}^2B_{41} - \frac{512}{15}B_{14}^4A_{14} - \frac{128}{15}B_{14}^2A_{14},\\
& A_{41} = \frac{1}{3(448B_{14}^2A_{14} - 21B_{41} - 48A_{14})}f_8,\\
& B_{41} = -8A_{14}\frac{f_4}{g_4},\\
& B_{50} = (\frac{16}{15}B_{14}^2 + 7 - \frac{96}{5}A_{14}^2B_{14} \pm \frac{1}{15}\sqrt{g_3})B_{14},
\end{aligned}
$$

where f_i and g_i are given in Appendix.

If the condition I_1 in Theorem 3.2 holds, system (2.11) can be rewritten as

$$\begin{aligned}\frac{dx}{dt} &= \frac{1}{4}(1+y)(8B_{14}x^2 - 4A_{41}x^4 - 4y + 16B_{14}x^2y\\ &\quad - 6y^2 + 8B_{14}x^2y^2 - 4y^3 - y^4),\\ \frac{dy}{dt} &= x(-2x^2 - 8B_{14}^2x^2 + B_{50}x^4 + 4B_{14}y - 4x^2y - 16B_{14}^2x^2y\\ &\quad + 6B_{14}y^2 - 2x^2y^2 - 8B_{14}^2x^2y^2 + 4B_{14}y^3 + B_{14}y^4),\end{aligned}$$

which is symmetric with the x-axis.

If the condition I_2 in Theorem 3.2 is satisfied, system (2.11) is reduced to

$$\begin{aligned}\frac{dx}{dt} &= \frac{1}{20}(-4B_{41}x^5 - 5y + 20x^4y + 80B_{14}^2x^4y - 40B_{14}x^2y^3 + 5y^5),\\ \frac{dy}{dt} &= -x(B_{14} - B_{50}x^4 - B_{41}x^3y + 2x^2y^2 + 8B_{14}^2x^2y^2 - B_{14}y^4),\end{aligned}$$

which is a Hamiltonian system with the first integral,

$$\begin{aligned}F_1 = \frac{1}{120}\big\{&20B_{50}x^6 + 240B_{14}^2x^4y^2 - 60B_{14}x^2(1+y^4)\\ &+y\big[-24B_{41}x^5 + 5y(-3+12x^4+y^4)\big]\big\}.\end{aligned}$$

If the condition I_3 in Theorem 3.2 holds, system (2.11) becomes

$$\begin{aligned}\frac{dx}{dt} &= \frac{1}{20}(4B_{41}x^5 - 8A_{41}B_{41}x^5 - 5y + 20A_{41}x^4y + 5y^5),\\ \frac{dy}{dt} &= x^3(B_{41}x - 2y)y,\end{aligned}$$

which has an integrating factor,

$$M_1 = 2(-1 + A_{41})y.$$

If the condition I_4 in Theorem 3.2 holds, system (2.11) takes the form

$$\begin{aligned}\frac{dx}{dt} &= \frac{1}{4}(4A_{14}x + y)(-1 + 2x^4 + y^4),\\ \frac{dy}{dt} &= x^3(8A_{14}x + B_{41}x - 2y)(4A_{14}x + y),\end{aligned}$$

for which an integrating factor is obtained,

$$M_2 = (4A_{14}x + y).$$

If the condition I_5 in Theorem 3.2 holds, system (2.11) is transformed to

$$\begin{aligned}\frac{dx}{dt} &= \frac{1}{4}(4A_{14}x + y)(-1 + 4A_{41}x^4 + y^4),\\ \frac{dy}{dt} &= -2x^3y(4A_{14}x + y),\end{aligned}$$

which is symmetric with the x-axis.

If the condition I_6 in Theorem 3.2 holds, system (2.11) takes the form

$$\begin{aligned}\frac{dx}{dt} &= \frac{1}{4}(4A_{14}x+y)(1+256A_{14}^4x^4-4A_{41}x^4-64A_{14}^3x^3y+24A_{14}^2x^2y^2-y^4),\\ \frac{dy}{dt} &= -x(3A_{14}^2-24A_{14}^2x^4+768A_{14}^6x^4-12A_{14}^2A_{41}x^4+8A_{14}x^3y\\ &\quad -192A_{14}^5x^3y+2x^2y^2+72A_{14}^4x^2y^2-3A_{14}^2y^4),\end{aligned} \tag{3.2}$$

which has two algebraic invariant curves:

$$f_1=-2A_{14}x+y,\quad f_2=6A_{14}x+y.$$

Then an inverse integrating factor for (3.2) is obtained as

$$M_3=(-2A_{14}x+y)^{\frac{1}{2}+39A_{14}^4-\frac{3}{4}A_{41}}(6A_{14}x+y)^{\frac{1}{4}(-2-52A_{14}^4+A_{41})}.$$

If the condition I_7 in Theorem 3.2 holds, system (2.11) takes the form

$$\begin{aligned}\frac{dx}{dt} &= \frac{1}{420}(-420A_{14}x-960A_{14}x^5+10752A_{14}^3B_{14}x^5\\ &\quad -5824A_{14}B_{14}^2x^5-105y-840x^4y+11648A_{14}^2B_{14}x^4y\\ &\quad -1456B_{14}^2x^4y-1120A_{14}B_{14}x^3y^2-840B_{14}x^2y^3\\ &\quad +420A_{14}xy^4+105y^5),\\ \frac{dy}{dt} &= -\frac{1}{105}x(105B_{14}-2688A_{14}^2B_{14}^2x^4\\ &\quad +1456B_{14}^3x^4+240A_{14}x^3y-2240A_{14}B_{14}^2x^3y+210x^2y^2\\ &\quad +840B_{14}^2x^2y^2-105B_{14}y^4)\end{aligned} \tag{3.3}$$

which has two algebraic invariant curves:

$$\begin{aligned}f_3 &= -4B_{14}x^2+y(4A_{14}x+y),\\ f_4 &= 840B_{14}(-3A_{14}^2+B_{14})+16(24A_{14}^2-13B_{14})B_{14}(-15\\ &\quad +56(3A_{14}^2-B_{14})B_{14})x^4+320A_{14}B_{14}(-15+56(3A_{14}^2\\ &\quad -B_{14})B_{14})x^3y+120B_{14}(15+56B_{14}(-3A_{14}^2+B_{14}))x^2y^2\\ &\quad +15(-15+56(3A_{14}^2-B_{14})B_{14})y^4,\end{aligned}$$

which in turn yield an inverse integrating factor $M_3=f_3f_4$, for system (3.3).

Remark 3.1. If the condition I_8 in Theorem 3.2 holds, let $B_{14}=15r^2$, system (2.11) takes the form

$$\begin{aligned}\frac{dx}{dt} &= \frac{1}{20}(-20rx+16rx^5-8640r^5x^5-5y+16x^4y\\ &\quad +19600r^4x^4y-800r^3x^3y^2-600r^2x^2y^3+20rxy^4+5y^5).\\ \frac{dy}{dt} &= \frac{1}{5}x(-75r^2+168r^2x^4+105840r^6x^4+60rx^3y\\ &\quad +24000r^5x^3y-10x^2y^2-9000r^4x^2y^2+75r^2y^4),\end{aligned}$$

which has an algebraic invariant curve,

$$f_5 = -6rx + y.$$

Although we have tried up to 13th-degree curves, we did not obtain more algebraic invariant curves and thus we cannot construct an integrating factor and so cannot prove that this condition is sufficient for the origin of this system to be a center. Similarly, it is very hard to prove the sufficiency for the conditions I_9-I_{15}.

Summarizing the above results, we have the following theorem.

Theorem 3.3. *The conditions* I_1-I_{15} *in Theorem 3.2 are necessary conditions for* $(0, \pm 1)$ *of system* (2.11) *to be centers. Moreover, the conditions* I_1-I_7 *are also sufficient.*

Conjecture 3.1. *When* $\mu = 0$*, the conditions* I_8-I_{15} *in Theorem 3.2 are sufficient for* $(0, \pm 1)$ *of system* (2.11) *to be centers.*

4. Center problem of $(0, \pm 1)$ with multiplicity 5

Suppose $(0, \pm 1)$ are nilpotent foci or centers of system (1.3) with multiplicity 5. By proper linear transformation and time recalling, system (1.3) can be changed to

$$\begin{aligned}
\frac{dx}{dt} &= \frac{1}{4}(-4A_{14}x + 4A_{50}x^5 - y + 4A_{41}x^4y + 4A_{32}x^3y^2 \\
&\qquad + 4A_{23}x^2y^3 + 4A_{14}xy^4 + y^5), \\
\frac{dy}{dt} &= -x(B_{14} - B_{50}x^4 + 16A_{14}A_{23}B_{14}x^3y - 4A_{32}B_{14}x^3y \\
&\qquad - 4A_{23}B_{14}x^2y^2 - B_{14}y^4),
\end{aligned} \tag{4.1}$$

where B_{50} is given in (2.12). Denote

$$A_{32} = \mu - \frac{8}{3}A_{14}B_{14}.$$

Theorem 2.2 shows that $(0, \pm 1)$ of system (4.1) are foci or centers if $\mu^2 < 1$ and degenerate singular points if $\mu^2 > 1$.

4.1. Lyapunov constants at $(0, \pm 1)$ of system (4.1)

In this section, we study the integrability of $(0, \pm 1)$ of system (4.1) when $(0, \pm 1)$ are nilpotent singular points with multiplicity 5. We first compute the Lyapunov constants at $(0, \pm 1)$ by using the method of normal forms developed in [15].

Proposition 4.1. *The first five Lyapunov constants at* $(0, \pm 1)$ *of system* (4.1) *are*

$$\begin{aligned}
\nu_1 &= \mu, \\
\nu_2 &= \frac{1}{3}(60A_{14}A_{41} + 15A_{50} - 128A_{14}^3B_{14} - 224A_{14}B_{14}^2), \\
\nu_3 &= \frac{A_{14}}{90}(675 - 40320A_{14}^2A_{41} + 104448A_{14}^4B_{14} \\
&\quad + 3840A_{41}B_{14} + 177664A_{14}^2B_{14}^2 - 15360B_{14}^3).
\end{aligned}$$

Case 1. *For* $B_{14} \neq \frac{21}{2}A_{14}^2$,

$$\begin{aligned}\nu_4 &= -\frac{2A_{14}}{225(21A_{14}^2 - 2B_{14})}G_1,\\ \nu_5 &= -\frac{A_{14}}{9676800(21A_{14}^2 - 2B_{14})^2}G_2.\end{aligned}$$

Case 2. *For* $B_{14} = \frac{21}{2}A_{14}^2$,

$$\begin{aligned}\nu_4 &= \frac{3}{2}A_{14}(5 + 21504A_{14}^6),\\ \nu_5 &= 0.\end{aligned}$$

where G_1, G_2 *are given in Appendix.*

Theorem 4.1. *The first five Lyapunov constants at the two singular points* $(0, \pm 1)$ *of system* (4.1) *are all zero if and only if the following condition holds:*

$$I_{16}: \quad A_{14} = A_{50} = \mu = 0.$$

Proof. When the condition I_{16} holds, system (4.1) can be rewritten as

$$\begin{aligned}\frac{dx}{dt} &= -2B_{14}x^2 + A_{41}x^4 + y - 6B_{14}x^2y + A_{41}x^4y + \frac{5}{2}y^2\\ &\quad - 6B_{14}x^2y^2 + \frac{5}{2}y^3 - 2B_{14}x^2y^3 + \frac{5}{4}y^4 + \frac{1}{4}y^5,\\ \frac{dy}{dt} &= -8B_{14}^2x^3 - \frac{3}{4}x^5 + 4A_{41}B_{14}x^5 + 4B_{14}xy\\ &\quad - 16B_{14}^2x^3y + 6B_{14}xy^2 - 8B_{14}^2x^3y^2 + 4B_{14}xy^3 + B_{14}xy^4,\end{aligned}$$

which is symmetric with the x-axis. □

Hence, we have the following theorem.

Theorem 4.2. *The singular points* $(0, \pm 1)$ *of system* (4.1) *with multiplicity 5 are centers if and only if* $A_{14} = A_{50} = \mu = 0$.

5. Conclusion

In this paper, we have studied quintic Z_2-equivariant vector fields with two isolated nilpotent singular points, with particular attention on the case $\mu = 0$. We first introduce some transformations to simplify the system, and get a general form of the system which has two isolated nilpotent foci or centers at $(0, \pm 1)$. Then we compute the first seven Lyapunov constants of the system by using the inverse integrating factor method when $(0, \pm 1)$ are nilpotent singular points with multiplicity 3. Moreover, the integrability of the system is discussed, leading to fifteen center conditions, all of them are necessary and seven of them, I_1-I_7, are also proved to be sufficient. In addition, by using the method of normal forms, the first five Lyapunov constants of the system are obtained when $(0, \pm 1)$ are nilpotent singular points with multiplicity 5, and the integrability of the system is discussed. Two problems are still open, left for future study: (1) proving sufficiency of the other eight conditions, I_8-I_{15} for the system when $(0, \pm 1)$ are order-3 nilpotent singular points; and (2) computing the Lyapunov constants for the system when $\mu \neq 0$.

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Appendix

$$
\begin{aligned}
f_1 = {} & 5760B_{14}^3A_{14}A_{41} + 2160B_{14}A_{41}^2A_{14} - 76032B_{14}^2A_{41}A_{14}^3 - 540A_{14}B_{14}A_{41} \\
& +11520A_{14}B_{14}^5A_{41} - 328704B_{14}^4A_{41}A_{14}^3 + 8640B_{14}^3A_{14}A_{41}^2 - 9720A_{14}B_{14} \\
& +860160A_{14}^5B_{14}^3 + 503808B_{14}^4A_{14}^3 - 138240A_{14}B_{14}^5 - 5184A_{14}^3A_{41} \\
& +110592A_{14}^5B_{14} + 36864A_{14}^3B_{14}^2 - 40320A_{14}B_{14}^3 + 884736A_{14}^3B_{14}^6 \\
& +1507328B_{14}^5A_{14}^5 + 5400B_{14}^3B_{41} + 378A_{14}B_{41}^2 + 1728B_{41}A_{14}^2 \\
& +810B_{14}B_{41} - 10368A_{14}^3 + 11376A_{14}^2A_{41}B_{14}^2B_{41} - 540A_{41}B_{14}^3B_{41} \\
& -28800A_{14}^4B_{14}B_{41} - 49152A_{14}^4B_{14}^3B_{41} + 189B_{41}^2A_{14}A_{41} + 864A_{14}^2A_{41}B_{41} \\
& -405A_{41}^2B_{14}B_{41} - 1008A_{14}^3B_{14}B_{41}^2 - 405B_{14}A_{41}B_{41} - 756B_{41}^2B_{14}^2A_{14} \\
& -25344B_{41}B_{14}^4A_{14}^2 - 24480B_{41}B_{14}^2A_{14}^2 + 8640B_{14}^5B_{41} - 184320A_{14}B_{14}^7,
\end{aligned}
$$

$$
\begin{aligned}
g_1 = {} & 36A_{14} - 63B_{41} - 336B_{41}A_{14}^2B_{14} - 1344A_{41}B_{14}^2A_{14} - 252B_{14}^2B_{41} \\
& +7168B_{14}^3A_{14}^3 + 63A_{41}B_{41} + 144A_{14}A_{41} + 5376B_{14}^4A_{14} \\
& -768A_{14}^3B_{14} + 768B_{14}^2A_{14},
\end{aligned}
$$

$$
\begin{aligned}
f_2 = {} & 241266060B_{14}^3 - 202713840B_{14}^5 - 390620160B_{14}^7 + 185932800A_{14}^6 \\
& +21244317696B_{14}^4A_{14}^6 + 7774470144B_{14}^3A_{14}^8 - 832978944A_{14}^8B_{14} \\
& -4532161536A_{14}^6B_{14}^2 - 3284508416A_{14}^2B_{14}^6 + 8318679040A_{14}^4B_{14}^5 \\
& -7645236480A_{14}^4B_{14}^3 + 701367840A_{14}^4B_{14} - 3860832864B_{14}^4A_{14}^2 \\
& -4067280A_{14}^4B_{50} + 24413760B_{14}^4B_{50} - 9128385B_{14}^2B_{50} \\
& +761100660B_{14}^2A_{14}^2 + 95032560A_{14}^2B_{14}^3B_{50} \\
& -12724110A_{14}^2B_{14}B_{50} + 37961280A_{14}^4B_{14}^2B_{50},
\end{aligned}
$$

$$
\begin{aligned}
g_2 = {} & -4110480B_{14}^3 - 3487680B_{14}^5 - 12724110A_{14}^2B_{14}B_{50} - 15494400A_{14}^6 \\
& +69414912A_{14}^8B_{14} + 189681408A_{14}^6B_{14}^2 - 12724110A_{14}^2B_{14}B_{50} \\
& +74273920A_{14}^4B_{14}^3 - 48763320A_{14}^4B_{14} - 29325968B_{14}^4A_{14}^2 \\
& -12724110A_{14}^2B_{14}B_{50} + 338940A_{14}^4B_{50} + 217980B_{14}^2B_{50} \\
& -35144880B_{14}^2A_{14}^2 + 848505A_{14}^2B_{14}B_{50},
\end{aligned}
$$

$$
\begin{aligned}
f_3 = {} & 308025A_{14}^4 - 4014080A_{14}^4B_{14}^4 - 1989120A_{14}^4B_{14}^2 + 308025A_{14}^2B_{14} \\
& -2983680A_{14}^6B_{14} + 2408448A_{14}^6B_{14}^3 + 802816B_{14}^5A_{14}^2 \\
& +308025A_{14}^2B_{14} + 994560B_{14}^3A_{14}^2 + 7225344A_{14}^8B_{14}^2,
\end{aligned}
$$

$$
\begin{aligned}
g_3 = {} & 50176B_{14}^4 + 33600B_{14}^2 - 301056B_{14}^3A_{14}^2 + 11025 - 100800A_{14}^2B_{14} \\
& +451584A_{14}^4B_{14}^2,
\end{aligned}
$$

$$
\begin{aligned}
f_4 = {} & 24647490000B_{14}^2 - 184104576000B_{14}^4 - 2097506016000B_{14}^6 + 151723125B_{50}^2 \\
& -6665641113600B_{14}^8 - 8943530803200B_{14}^{10} - 4369888051200B_{14}^{12} \\
& -237456011427840A_{14}^6B_{14}^5 + 120010151854080A_{14}^4B_{14}^6 \\
& +29883528806400A_{14}^6B_{14}^3 + 11934755758080B_{14}^7A_{14}^2 \\
& +25735457894400A_{14}^4B_{14}^4 - 3984368486400B_{14}^5A_{14}^2 \\
& -2801288448000A_{14}^4B_{14}^2 - 1444283136000B_{14}^3A_{14}^2 \\
& +102876480000A_{14}^2B_{14} - 17069875200B_{14}^6B_{50}^2 + 546236006400B_{14}^9B_{50} \\
& -1454355000B_{14}^2B_{50}^2 - 9601804800B_{14}^4B_{50}^2 + 289494777600B_{14}^5B_{50} \\
& +712599552000B_{14}^7B_{50} - 7372620000A_{14}^2B_{50} - 3870990000B_{14}B_{50} \\
& +33083964000B_{14}^3B_{50} - 42433693876224A_{14}^8B_{14}^5B_{50} \\
& -2755620000A_{14}^2B_{14}B_{50}^2 + 2342283116544A_{14}^4B_{14}^7B_{50} \\
& +26231813701632A_{14}^6B_{14}^6B_{50} + 13289141698560A_{14}^6B_{14}^4B_{50} \\
& -3014888325120A_{14}^4B_{14}^5B_{50} + 149751504000A_{14}^2B_{14}^2B_{50} \\
& -1995466752000A_{14}^4B_{14}^3B_{50} + 172642752000A_{14}^4B_{14}B_{50} \\
& -115828531200A_{14}^2B_{14}^4B_{50} - 1470895718400A_{14}^6B_{14}^2B_{50} \\
& +23318668800A_{14}^2B_{14}^3B_{50}^2 - 3698374213632A_{14}^2B_{14}^8B_{50} \\
& -2794541137920A_{14}^2B_{14}^6B_{50} + 16460236800A_{14}^4B_{14}^2B_{50}^2 \\
& +4998616842240A_{14}^8B_{14}^3B_{50} + 102419251200A_{14}^2B_{14}^5B_{50}^2 \\
& +598476849152A_{14}^2B_{14}^10B_{50} - 7181722189824A_{14}^4B_{14}^9B_{50} \\
& -153628876800A_{14}^4B_{14}^4B_{50}^2 + 32317749854208A_{14}^6B_{14}^8B_{50} \\
& -64635499708416A_{14}^8B_{14}^7B_{50} + 48476624781312A_{14}^{10}B_{14}^6B_{50} \\
& -5193924083712A_{14}^{10}B_{14}^4B_{50} + 3102503986003968B_{14}^8A_{14}^{12} \\
& -332411141357568B_{14}^6A_{14}^{12} + 153210073382912B_{14}^{12}A_{14}^4 \\
& -976714217816064B_{14}^11A_{14}^6 + 3102503986003968B_{14}^1A_{14}^8 \\
& -4912297977839616B_{14}^9A_{14}^{10} + 3762665054797824B_{14}^8A_{14}^8 \\
& -3214373120114688B_{14}^7A_{14}^{10} + 1225295206023168A_{14}^8B_{14}^6 \\
& -1031122351816704A_{14}^6B_{14}^7 - 155511957749760A_{14}^8B_{14}^4 \\
& +43036204990464B_{14}^9A_{14}^2 + 144789717123072B_{14}^8A_{14}^4 \\
& -1391757090619392A_{14}^6B_{14}^9 + 131010014478336B_{14}^{10}A_{14}^4 \\
& +382238566907904A_{14}^{10}B_{14}^5 + 25772936921088B_{14}^{11}A_{14}^2 \\
& -9575629586432B_{14}^{13}A_{14}^2,
\end{aligned}
$$

$$
\begin{aligned}
g_4 = {} & 147884940000B_{14}^2 + 1038282624000B_{14}^4 + 2848372128000B_{14}^6 \\
& +889835625B_{50}^2 + 3529399910400B_{14}^8 + 1638708019200B_{14}^{10} \\
& +442736545628160A_{14}^6B_{14}^5 - 59382170910720A_{14}^4B_{14}^6 \\
& +130807797350400A_{14}^6B_{14}^3 - 17197283819520B_{14}^7A_{14}^2 \\
& -50355377049600A_{14}^4B_{14}^4 - 6218257766400B_{14}^5A_{14}^2 \\
& -13987531008000A_{14}^4B_{14}^2 + 944270784000B_{14}^3A_{14}^2 \\
& +617258880000A_{14}^2B_{14} + 4286520000B_{14}^2B_{50}^2 \\
& +6401203200B_{14}^4B_{50}^2 - 289171814400B_{14}^5B_{50} \\
& -204838502400B_{14}^7B_{50} - 44114220000A_{14}^2B_{50} \\
& -23007240000B_{14}B_{50} - 135869076000B_{14}^3B_{50} \\
& +24238312390656A_{14}^8B_{14}^5B_{50} - 12859560000A_{14}^2B_{14}B_{50}^2 \\
& +2693145821184A_{14}^4B_{14}^7B_{50} - 12119156195328A_{14}^6B_{14}^6B_{50} \\
& -11135411159040A_{14}^6B_{14}^4B_{50} - 589804830720A_{14}^4B_{14}^5B_{50} \\
& +157022064000A_{14}^2B_{14}^2B_{50} + 1031159808000A_{14}^4B_{14}^3B_{50} \\
& +837179712000A_{14}^4B_{14}B_{50} + 1197997516800A_{14}^2B_{14}^4B_{50} \\
& -6067818086400A_{14}^6B_{14}^2B_{50} - 38407219200A_{14}^2B_{14}^3B_{50}^2 \\
& -224428818432A_{14}^2B_{14}^8B_{50} + 1362844385280A_{14}^2B_{14}^6B_{50} \\
& +57610828800A_{14}^4B_{14}^2B_{50}^2 + 18509597245440A_{14}^8B_{14}^3B_{50} \\
& -18178734292992A_{14}^{10}B_{14}^4B_{50} - 1163438994751488B_{14}^6A_{14}^{12} \\
& -1163438994751488B_{14}^8A_{14}^8 + 184211171689856B_{14}^7A_{14}^{10} \\
& -1535653573558272A_{14}^8B_{14}^6 + 561151890948096A_{14}^6B_{14}^7 \\
& -615324828303360A_{14}^8B_{14}^4 - 9280116228096B_{14}^9A_{14}^2 \\
& -55284517306368B_{14}^8A_{14}^4 + 366267831681024A_{14}^6B_{14}^9 \\
& -57453777518592B_{14}^{10}A_{14}^4 + 1402759035224064A_{14}^{10}B_{14}^5 \\
& +3590861094912B_{14}^{11}A_{14}^2,
\end{aligned}
$$

$$
\begin{aligned}
f_5 = {} & 9B_{41}^2 + 72B_{41}A_{14} + 12288B_{14}^3A_{14}^4 + 1024B_{14}^4A_{14}^2 - 1920B_{14}^2A_{14}^2 \\
& -576B_{41}A_{14}^3B_{14} - 240B_{41}B_{14}^2A_{14},
\end{aligned}
$$

$$
\begin{aligned}
f_6 = {} & -768A_{14}^3B_{14} - 144B_{14}^2A_{14} + 512B_{14}^3A_{14}^3 + 72A_{14} + 9B_{41} \\
& -24B_{41}A_{14}^2B_{14} - 18B_{14}^2B_{41} + 384B_{14}^4A_{14},
\end{aligned}
$$

$$
\begin{aligned}
f_7 = {} & 36A_{14} - 63B_{41} - 336B_{41}A_{14}^2B_{14} - 252B_{14}^2B_{41} + 7168B_{14}^3A_{14}^3 \\
& +768B_{14}^2A_{14} + 5376B_{14}^4A_{14} - 768A_{14}^3B_{14},
\end{aligned}
$$

$$
\begin{aligned}
f_8 = {} & 36A_{14} - 63B_{41} - 336B_{41}A_{14}^2B_{14} - 252B_{14}^2B_{41} + 7168B_{14}^3A_{14}^3 \\
& +768B_{14}^2A_{14} + 5376B_{14}^4A_{14} - 768A_{14}^3B_{14},
\end{aligned}
$$

$$\begin{aligned}
G_1 = {} & 8847360A_{14}^8B_{14} - 279281664A_{14}^6B_{14}^2 + 88375296A_{14}^4B_{14}^3 \\
& +4987744A_{14}^2B_{14}^4 - 707520B_{14}^5 - 3645000A_{14}^4 + 170100A_{14}^2B_{14} \\
& +105300B_{14}^2,
\end{aligned}$$

$$\begin{aligned}
G_2 = {} & 106542032486400A_{14}^12B_{14} - 9381533303439360A_{14}^10B_{14}^2 \\
& +863284903280640A_{14}^8B_{14}^3 + 1618159372861440A_{14}^6B_{14}^4 \\
& +8847360A_{14}^8B_{14} - 455881204695040A_{14}^4B_{14}^5 \\
& -110125080576000A_{14}^8 - 279281664A_{14}^6B_{14}^2 \\
& +44573995827200A_{14}^2B_{14}^6 - 24409340928000A_{14}^6B_{14} \\
& +88375296A_{14}^4B_{14}^3 - 1494173614080B_{14}^7 \\
& +18695570688000A_{14}^4B_{14}^2 + 4987744A_{14}^2B_{14}^4 \\
& -2317135824000A_{14}^2B_{14}^3 - 707520B_{14}^5 - 3645000A_{14}^4 \\
& +87395616000B_{14}^4 + 170100A_{14}^2B_{14} - 11577431250A_{14}^2 \\
& +105300B_{14}^2 + 1929571875B_{14}.
\end{aligned}$$