Journal of Nonlinear Modeling and Analysis
Volume 2, Number 1, March 2020, 95–110
http://jnma.ca; http://jnma.ijournal.cn
DOI:10.12150/jnma.2020.95

Applications of Variational Method to Impulsive Fractional Differential Equations with Two Control Parameters*

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Abstract In this paper, we study the existence of the impulsive fractional differential equation. Based on a previous paper [2], we give more accurate condition to guarantee the impulsive fractional differential equation has at least three solutions under certain assumptions by using variational methods and critical point theory. Moreover, some recent results are generalized and significantly improved.

Keywords Impulsive fractional differential equations, Variational methods, Critical point theory, Three solutions.

MSC(2010) 34A60, 34B37, 58E05.

1. Introduction

In this paper, we will consider the following fractional differential equation with impulsive effects

$$\begin{cases} \frac{d}{dt}\{ {}_0\mathrm{D}_t^{\alpha-1}({}^c{}_0\mathrm{D}_t^{\alpha}u(t)) - {}_t\mathrm{D}_T^{\alpha-1}({}^c{}_t\mathrm{D}_T^{\alpha}u(t))\} \\ +\lambda f(t,u(t)) + \mu g(t,u(t)) = 0, t \in [0,T], t \neq t_k, \\ \Delta(\ \mathrm{D}_t^{\alpha}u)(t_k) = I_k(u(t_k)), t = t_k, k = 1,2,...l, \\ u(0) = u(T) = 0, \end{cases} \tag{1.1}$$

where $\alpha \in (\frac{1}{2}, 1]$, ${}_0\mathrm{D}_t^{\alpha-1}$ and ${}_t\mathrm{D}_T^{\alpha-1}$ represent the left and right Riemann-Liouville fractional integrals of order $1-\alpha$, ${}^c{}_0\mathrm{D}_t^{\alpha}$ and ${}^c{}_t\mathrm{D}_T^{\alpha}$ represent the left and right Caputo fractional derivative of order α, respectively. $f, g : [0,T] \times R \to R$ are given continuous functions, λ and μ are positive parameters, $I_k : R \to R$, $k = 1,2,...l$ are continuous functions and

$$(\ \mathrm{D}_t^{\alpha}u)(t) = \Big\{ {}_0\mathrm{D}_t^{\alpha-1}({}^c{}_0\mathrm{D}_t^{\alpha}u) - {}_t\mathrm{D}_T^{\alpha-1}({}^c{}_t\mathrm{D}_T^{\alpha}u)\Big\}(t),$$

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*The authors were supported by the NSFC(11571088, 11471109), the Zhejiang Provincial Natural Science Foundation of China (LY14A010024), and Scientific Research Fund of Hunan Provincial Education Department (14A098).

$$\Delta(\ \mathrm{D}_t^{\alpha}u)(t_k) = \Big\{\ {}_0\mathrm{D}_t^{\alpha-1}({}^c{}_0\mathrm{D}_t^{\alpha}u) - \ {}_t\mathrm{D}_T^{\alpha-1}({}^c{}_t\mathrm{D}_T^{\alpha}u)\Big\}(t_k^+)$$
$$- \Big\{\ {}_0\mathrm{D}_t^{\alpha-1}({}^c{}_0\mathrm{D}_t^{\alpha}u) - \ {}_t\mathrm{D}_T^{\alpha-1}({}^c{}_t\mathrm{D}_T^{\alpha}u)\Big\}(t_k^-),$$

$$\Big\{\ {}_0\mathrm{D}_t^{\alpha-1}({}^c{}_0\mathrm{D}_t^{\alpha}u) - \ {}_t\mathrm{D}_T^{\alpha-1}({}^c{}_t\mathrm{D}_T^{\alpha}u)\Big\}(t_k^+) = \lim_{t\to t_k^+}\Big\{\ {}_0\mathrm{D}_t^{\alpha-1}({}^c{}_0\mathrm{D}_t^{\alpha}u) - \ {}_t\mathrm{D}_T^{\alpha-1}({}^c{}_t\mathrm{D}_T^{\alpha}u)\Big\}(t_k),$$

$$\Big\{\ {}_0\mathrm{D}_t^{\alpha-1}({}^c{}_0\mathrm{D}_t^{\alpha}u) - \ {}_t\mathrm{D}_T^{\alpha-1}({}^c{}_t\mathrm{D}_T^{\alpha}u)\Big\}(t_k^-) = \lim_{t\to t_k^-}\Big\{\ {}_0\mathrm{D}_t^{\alpha-1}({}^c{}_0\mathrm{D}_t^{\alpha}u) - \ {}_t\mathrm{D}_T^{\alpha-1}({}^c{}_t\mathrm{D}_T^{\alpha}u)\Big\}(t_k),$$

for $k = 1, \cdots, l$.

In recent years, more and more attention have been paid to the fractional differential equations have obtained by many authors. By using variational methods and some critical point theory, some interesting results on fractional differential equations which have been presented to our vision, see [2–16] and the references therein.

More precisely, in a recent paper [2], the authors have considered the following fractional boundary problem without impulsive effects

$$\begin{cases} \frac{d}{dt}\{\ {}_0\mathrm{D}_t^{\alpha-1}({}^c{}_0\mathrm{D}_t^{\alpha}u(t)) - \ {}_t\mathrm{D}_T^{\alpha-1}({}^c{}_t\mathrm{D}_T^{\alpha}u(t))\} \\ +\lambda f(t,u(t)) + \mu g(t,u(t)) = 0, t \in [0,T], \\ u(0) = u(T) = 0, \end{cases} \tag{1.2}$$

the main result is as follows:

Theorem 1.1. *[Theorem 3.1, [2]] Assume that there exist positive constants c, d with*

$$c < \Big(\frac{4d\Omega}{T\Gamma(2-\alpha)}\Big)\sqrt{C(T,\alpha)}, \tag{1.3}$$

such that

(A1) $F(t,\xi) \geq 0$, for each $(t,\xi) \in ([0, \frac{T}{4}] \bigcup [\frac{3T}{4}, T]) \times [0, d]$;

(A2) $\frac{\int_0^T \max_{|\xi|\leq c} F(t,\xi)dt}{c^2} < \frac{|\cos(\pi\alpha)| \int_{\frac{T}{4}}^{\frac{3T}{4}} F(t,d)dt}{\Omega^2 \omega_{\alpha,d}}$;

(A3) $\limsup_{|\xi|\to+\infty} \frac{\sup_{t\in[0,T]} F(t,\xi)}{\xi^2} < \frac{\int_0^T \max_{|\xi|\leq c} F(t,\xi)dt}{2c^2T}$.

Then, for every $\lambda \in \Lambda$ and for every continuous function $g : [0,T] \times R \to R$ such that

$$\limsup_{|\xi|\to+\infty} \frac{\sup_{t\in[0,T]} G(t,\xi)}{\xi^2} < +\infty,$$

where $F(t,\xi) = \int_0^{\xi} f(t,s)ds$ and $G(t,\xi) = \int_0^{\xi} f(t,s)ds$. Then there exists $\bar{\delta}$ such that for each $\mu \in [0, \bar{\delta}]$, problem (1.2) admits at least three solutions.

In fact, Theorem 1.1 is not valid. In [2], the authors fixed $c, d > 0$ such that

$$\frac{\omega_{\alpha,d}}{\int_{\frac{T}{4}}^{\frac{3T}{4}} F(t,d)dt} < \frac{\frac{|\cos(\pi\alpha)|}{\Omega_2^2}c^2}{\int_0^T \max_{|u|\leq c} F(t,u)dt}, \tag{1.4}$$

holds, where

$$\omega_{\alpha,d} = \frac{16d^2}{T^2\Gamma^2(2-\alpha)|\cos(\pi\alpha)|}C(T,\alpha)$$

and

$$C(T,\alpha):=\int_0^{\frac{T}{4}} t^{2-2\alpha}dt+\int_{\frac{T}{4}}^{\frac{3T}{4}}[t^{1-\alpha}-(t-\frac{T}{4})^{1-\alpha}]^2dt$$
$$+\int_{\frac{3T}{4}}^{T}[t^{1-\alpha}-(t-\frac{T}{4})^{1-\alpha}+(t-\frac{3T}{4})^{1-\alpha}]^2dt.$$

However, on the one hand, by (1.3), we have

$$\omega_{\alpha,d}-\frac{|\cos(\pi\alpha)|}{\Omega^2}c^2>\frac{16d^2}{T^2\Gamma^2(2-\alpha)}C(T,\alpha)(\frac{1}{|\cos(\pi\alpha)|}-|\cos(\pi\alpha)|),$$

since $C(T,\alpha)>0$ and $\frac{1}{|\cos(\pi\alpha)|}-|\cos(\pi\alpha)|\geq 0$, so we immediately have

$$\omega_{\alpha,d}>\frac{|\cos(\pi\alpha)|}{\Omega^2}c^2. \tag{1.5}$$

On the other hand, we intend to give a sharper estimate as shown by the following counterexamples.

Example 1.1. Let $\alpha=1,T=1$, by simple calculations, we know $\Omega=\frac{T^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)\sqrt{2\alpha-1}}=1$, $C(T,\alpha)=\frac{1}{2}$, then (1.4) becomes $c<2\sqrt{2}d$. If we take $c>d$, then we have

$$\int_{\frac{T}{4}}^{\frac{3T}{4}} F(t,d)dt<\int_0^T \max_{|u|\leq c} F(t,u)dt. \tag{1.6}$$

Thus, by (1.5) and (1.6), we have

$$\frac{\omega_{\alpha,d}}{\int_{\frac{T}{4}}^{\frac{3T}{4}} F(t,d)dt}>\frac{\frac{|\cos(\pi\alpha)|}{\Omega_2^2}c^2}{\int_0^T \max_{|u|\leq c}F(t,u)dt},$$

which is a contradiction with (1.4).

Motivated by the above facts, in this paper, on the one hand, we give more accurate condition (1.7) than (1.3). On the other hand, we take impulsive effects into the system (1.2). Then by applying the variational methods and some critical point theory, at least three solutions for the system (1.1) have also been obtained. It is worth pointing out that the systems of [2] with impulsive effects have not been considered yet. Now, we state our main results as follow:

Theorem 1.2. *Assume that there exist positive constants c,d with*

$$c<\min\{1,\frac{4M_2}{T}\sqrt{M(T,\alpha)}\}d, \tag{1.7}$$

such that (A1) holds and

(A2)*

$$\limsup_{|u|\to+\infty}\frac{\sup_{t\in[0,T]}F(t,u)}{u^2}<\frac{M_2^2\int_0^T \max_{|u|\leq c}F(t,u)dt}{2c^2M_1^2|\cos(\pi\alpha)|}.$$

(A3)* $uI_k(u) \geq 0$ *and there exists* $e > 0$ *such that*

$$J_k(u) \leq \frac{e}{|\cos(\pi\alpha)|} \tag{1.8}$$

for all $u \in R, k = 1, 2, ..., l$.

Then, for every $\lambda \in \Lambda_1$ *and for every continuous function* $g : [0, T] \times R \to R$ *such that*

$$\limsup_{|u|\to+\infty} \frac{\sup_{t\in[0,T]} G(t, u)}{u^2} < +\infty,$$

there exists $\bar{\delta}_1$ *such that for each* $\mu \in [0, \bar{\delta}_1]$, *problem (1.1) admits at least three solutions.*

The arrangement of the rest paper is as follows. In Section 2, some preliminaries and results which are applied in the later paper are presented. In Section 3, the main proof of theorems will be vividly showed.

2. Preliminaries

In this section, we recall some basic knowledge of the fractional calculus theory and lemmas that we shall use in the rest of the paper. For more details, please refer to the references [1, 4].

Definition 2.1. [1] Let f be a function defined on $[a, b]$ and $\alpha > 0$. The left and right Riemann-Liouville fractional derivatives of order α for function f are denoted by ${}_a\mathrm{D}_t^{-\alpha} f(t)$ and ${}_t\mathrm{D}_b^{-\alpha} f(t)$, respectively, are defined by

$${}_a\mathrm{D}_t^{-\alpha} f(t) = \frac{d^n}{dt^n}\, {}_a\mathrm{D}_t^{\alpha-n} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f(s)ds$$

and

$${}_t\mathrm{D}_b^{-\alpha} f(t) = (-1)^n \frac{d^n}{dt^n}\, {}_t\mathrm{D}_b^{\alpha-n} f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b (s-t)^{n-\alpha-1} f(s)ds,$$

provided the right-hand sides are pointwise defined on $[a, b]$, where Γ is the gamma function.

Definition 2.2. [1] Let f be a function defined on $[a, b]$. The left and right Riemann-Liouville fractional derivatives of order α for function f are denoted by ${}_a\mathrm{D}_t^{-\alpha} f(t)$ and ${}_t\mathrm{D}_b^{-\alpha} f(t)$, respectively, are defined by

$${}_a\mathrm{D}_t^{-\alpha} f(t) = \frac{d^n}{dt^n}\, {}_a\mathrm{D}_t^{\alpha-n} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f(s)ds$$

and

$${}_t\mathrm{D}_b^{-\alpha} f(t) = (-1)^n \frac{d^n}{dt^n}\, {}_t\mathrm{D}_b^{\alpha-n} f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b (s-t)^{n-\alpha-1} f(s)ds,$$

where $t \in [a, b]$ and $\alpha > 0$.

Definition 2.3. [1] If $\alpha \in (n-1,n)$ and $f \in AC^n([a,b],R)$, then the left and right Caputo fractional derivatives of order of a function f are denoted by ${}^c_a\mathrm{D}_t^\alpha f(t)$ and ${}^c_t\mathrm{D}_b^\alpha f(t)$, respectively, and are defined by

$$ {}^c_a\mathrm{D}_t^\alpha f(t) = {}_a\mathrm{D}_t^{\alpha-n}\frac{d^n}{dt^n}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}f^{(n)}(s)ds $$

and

$$ {}^c_t\mathrm{D}_b^\alpha f(t) = (-1)^n\ {}_t\mathrm{D}_b^{\alpha-n}\frac{d^n}{dt^n}f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_t^b (s-t)^{n-\alpha-1}f^{(n)}(s)ds, $$

where $t \in [a,b]$ and $\alpha > 0$.

Proposition 2.1. *[1] We have the following property of fractional integration*

$$ \int_a^b [\ {}_aD_t^{-\gamma}f(t)]g(t)dt = \int_a^b [\ {}_tD_b^{-\gamma}g(t)]f(t)dt, \gamma > 0, \tag{2.1} $$

provided that $f \in L^p([a,b],R), g \in L^q([a,b],R)$ *and* $p \geq 1, q \geq 1, \frac{1}{p}+\frac{1}{q} \leq 1+\gamma$ *or* $p \neq 1, q \neq 1, \frac{1}{p}+\frac{1}{q} = 1+\gamma$.

Definition 2.4. [1] The left and right Riemann-Liouville fractional integral operators have the property of a semigroup as follow

$$ {}_a\mathrm{D}_t^{-\gamma_1}(\ {}_a\mathrm{D}_t^{-\gamma_2}f(t)) = \ {}_a\mathrm{D}_t^{-\gamma_1-\gamma_2}f(t) $$

and

$$ {}_t\mathrm{D}_b^{-\gamma_1}(\ {}_t\mathrm{D}_b^{-\gamma_2}f(t)) = \ {}_t\mathrm{D}_b^{-\gamma_1-\gamma_2}f(t), $$

where $\gamma_1, \gamma_2 > 0, t \in [a,b]$ and $f \in L^1([a,b],R)$.

Proposition 2.2. *[1] Let* $n \in N$ *and* $n-1 < \gamma \leq n$. *If* $f \in AC^n([a,b],R)$ *or* $f \in C^n([a,b],R)$, *then*

$$ {}_aD_t^{-\gamma}({}^c_aD_t^\gamma f(t)) = f(t) - \sum_{j=0}^{n-1}\frac{f^{(j)}(a)}{j!}(t-a)^j $$

and

$$ {}_tD_b^{-\gamma}({}^c_tD_b^\gamma f(t)) = f(t) - \sum_{j=0}^{n-1}\frac{(-1)^j f^{(j)}(b)}{j!}(b-t)^j, $$

for $t \in [a,b]$. *In particular, if* $0 < \gamma \leq 1$ *and* $f \in AC([a,b],R)$ *or* $f \in C^1([a,b],R)$, *then*

$$ {}_aD_t^{-\gamma}({}^c_aD_t^\gamma f(t)) = f(t) - f(a) $$

and

$$ {}_tD_b^{-\gamma}({}^c_tD_b^\gamma f(t)) = f(t) - f(b). $$

Then, by Definition 2.1 and Definition 2.4, we can transform problem (1.1) into following equivalent form:

$$ \begin{cases} \frac{d}{dt}\{\ {}_0\mathrm{D}_t^{-\beta}(u'(t)) -\ {}_t\mathrm{D}_T^{-\beta}(u'(t))\} + \lambda f(t,u) + \mu g(t,u) = 0, t \in [0,T], t \neq t_k, \\ \Delta(\ \mathrm{D}_t^\alpha u)(t_k) = I_k(u(t_k)), t = t_k, k = 1,2,...l, \\ u(0) = u(T) = 0, \end{cases} $$

where $\beta = 2(1-\alpha) \in [0,1)$. We denote that the function u $AC([0,T],R)$ is a solution of the above system if :

(i) the map $t \mapsto {}_0\mathrm{D}_t^{-\beta}(u'(t)) - {}_t\mathrm{D}_T^{-\beta}(u'(t))$ is differentiable for almost every $t \in [0,T]$, and

(ii)the function u satisfies the equations of the above system.

Next, we will introduce some basic notations and lemmas which used in later paper. For any fixed $t \in [0,T]$ and $1 \le p < \infty$, define

$$\|u\|_\infty = \max_{t\in[0,T]} |u(t)|, \|u\|_{L^p([0,t])} = (\int_0^t |u(s)|^p ds)^{\frac{1}{p}}, \|u\|_{L^p} = (\int_0^T |u(s)|^p ds)^{\frac{1}{p}}. \tag{2.2}$$

In order to establish a variational structure for the system (1.1), it is necessary to construct suitable function spaces. Denote by $C_0^\infty([0,T],R)$ the set of all functions $u \in C_0^\infty([0,T],R)$ and $u(0) = u(T) = 0$.

Definition 2.5. Let $0 < \alpha \le 1$ and $1 \le p < \infty$. The fractional derivative space $E_0^{\alpha,p}$ is defined by the closure of $C_0^\infty([0,T],R)$ with respect to the weighted norm

$$\|u\|_{\alpha,p} = (\int_0^T |{}^c_0\mathrm{D}_t^\alpha u(t)|^p dt + \int_0^T |u(t)|^p dt)^{\frac{1}{p}}, u \in E_0^{\alpha,p}.$$

Lemma 2.1. *Let $0 < \alpha \le 1, 1 \le p < \infty$ and $f \in L^p([0,T],R)$. Then we have*

$$\| {}_0D_\xi^{-\alpha} f\|_{L^p([0,t])} \le M^* \|f\|_{L^p([0,t])}, \xi \in [0,t], t \in [0,T], \tag{2.3}$$

where ${}_0D_t^{-\alpha}$ is left Riemann-Liouville fractional integral of order α and

$$M^* = \begin{cases} \frac{t^\alpha}{\Gamma(\alpha+1)}, \alpha \le \frac{1}{p}, \\ \frac{t^\alpha}{\Gamma(\alpha)[q(\alpha-1)+1]^{\frac{1}{q}}}, \alpha > \frac{1}{p}, \end{cases}$$

where q satisfies $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. If $\alpha > \frac{1}{p}$, from $\frac{1}{p} + \frac{1}{q} = 1$, we immediately obtain $q(\alpha-1)+1 = \frac{1}{p-1}(p\alpha - 1) > 0$ and by (2.3), we have

$$\| {}_0\mathrm{D}_\xi^{-\alpha} f\|_{L^p([0,t])} = \frac{1}{\Gamma(\alpha)} \Big(\int_0^t | \int_0^\xi (\xi-\tau)^{\alpha-1} f(\tau) d\tau|^p d\xi \Big)^{\frac{1}{p}},$$

since

$$\begin{aligned} |\int_0^\xi (\xi-\tau)^{\alpha-1} f(\tau)d\tau| &\le \int_0^\xi (\xi-\tau)^{\alpha-1}|f(\tau)|d\tau \\ &\le (\int_0^\xi [(\xi-\tau)^{\alpha-1}]^q d\tau)^{\frac{1}{q}} \cdot \Big(\int_0^\xi |f(\tau)|^p d\tau\Big)^{\frac{1}{p}} \\ &= \Big[\frac{1}{q(\alpha-1)+1} \cdot \xi^{q(\alpha-1)+1}\Big]^{\frac{1}{q}} \cdot \Big(\int_0^\xi |f(\tau)|^p d\tau\Big)^{\frac{1}{p}} \\ &\le \frac{t^{\alpha-1+\frac{1}{q}}}{[q(\alpha-1)+1]^{\frac{1}{q}}} \cdot \Big(\int_0^t |f(\tau)|^p d\tau\Big)^{\frac{1}{p}}, \end{aligned}$$

so

$$\begin{aligned}
\| {}_0\mathrm{D}_\xi^{-\alpha} f\|_{L^p([0,t])} &\le \frac{1}{\Gamma(\alpha)} \cdot \frac{t^{\alpha-1+\frac{1}{q}}}{[q(\alpha-1)+1]^{\frac{1}{q}}} \cdot \Big[\int_0^t (\int_0^t |f(\tau)|^p d\tau) d\xi \Big]^{\frac{1}{p}} \\
&= \frac{1}{\Gamma(\alpha)} \cdot \frac{t^{\alpha-1+\frac{1}{q}}}{[q(\alpha-1)+1]^{\frac{1}{q}}} \cdot t^{\frac{1}{p}} \cdot \Big(\int_0^t |f(\tau)|^p d\tau \Big)^{\frac{1}{p}} \\
&= \frac{t^\alpha}{\Gamma(\alpha)[q(\alpha-1)+1]^{\frac{1}{q}}} \|f\|_{L^p([0,t])}.
\end{aligned}$$

If $\alpha \le \frac{1}{p}$, by Lemma 3.1 of [4], we have

$$\| {}_0\mathrm{D}_\xi^{-\alpha} f\|_{L^p([0,t])} \le \frac{t^\alpha}{\Gamma(\alpha+1)} \|f\|_{L^p([0,t])}.$$

Let

$$M^* = \begin{cases} \frac{t^\alpha}{\Gamma(\alpha+1)}, \alpha \le \frac{1}{p}, \\ \frac{t^\alpha}{\Gamma(\alpha)[q(\alpha-1)+1]^{\frac{1}{q}}}, \alpha > \frac{1}{p}, \end{cases}$$

we obtain $\| {}_0\mathrm{D}_\xi^{-\alpha} f\|_{L^p([0,t])} \le M^* \|f\|_{L^p([0,t])}$.

□

Remark 2.1. (i) When $\frac{1}{2} < \alpha \le 1$ and $p \ge 2$, we have $M^* = \frac{t^\alpha}{\Gamma(\alpha)[q(\alpha-1)+1]^{\frac{1}{q}}}$.

(ii) When $\alpha > \frac{1}{p}$, it is clear to see $\frac{1}{[q(\alpha-1)+1]^{\frac{1}{q}}} < \frac{1}{\alpha}$. So M^* in our paper is better than the Lemma 3.1 of [4], which is defined as $M^* = \frac{t^\alpha}{\Gamma(\alpha+1)}$, thus we improve and extend some previous results.

Lemma 2.2. *Let $\frac{1}{2} < \alpha \le 1, p \ge 2$ and $\frac{1}{p} + \frac{1}{q} = 1$, for any $u \in E_0^{\alpha,p}$, we have*

$$\|u\|_{L^p} \le \frac{T^\alpha}{\Gamma(\alpha)[q(\alpha-1)+1]^{\frac{1}{q}}} \Big(\int_0^T |{}^c{}_0D_t^\alpha u(t)|^p dt \Big)^{\frac{1}{p}} \tag{2.4}$$

and

$$\|u\|_\infty \le \frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)[q(\alpha-1)+1]^{\frac{1}{q}}} \Big(\int_0^T |{}^c{}_0D_t^\alpha u(t)|^p dt \Big)^{\frac{1}{p}}. \tag{2.5}$$

Proof. By Lemma 2.1, as the similar proof of Proposition 3.2 of [4], we immediately know (2.4) and (2.5) hold. □

Corollary 2.1. *Let $u \in E_0^{\alpha,2}, p = 2$, then we have*

$$\|u\|_{L^2} \le \frac{T^\alpha}{\Gamma(\alpha)(2\alpha-1)^{\frac{1}{2}}} \Big(\int_0^T |{}^c{}_0D_t^\alpha u(t)|^2 dt \Big)^{\frac{1}{2}} = M_1 \Big(\int_0^T |{}^c{}_0D_t^\alpha u(t)|^2 dt \Big)^{\frac{1}{2}} \tag{2.6}$$

and

$$\|u\|_\infty \le \frac{T^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)(2\alpha-1)^{\frac{1}{2}}} \Big(\int_0^T |{}^c{}_0D_t^\alpha u(t)|^2 dt \Big)^{\frac{1}{2}} = M_2 \Big(\int_0^T |{}^c{}_0D_t^\alpha u(t)|^2 dt \Big)^{\frac{1}{2}}, \tag{2.7}$$

where $M_1 = \frac{T^\alpha}{\Gamma(\alpha)(2\alpha-1)^{\frac{1}{2}}}$ and $M_2 = \frac{T^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)(2\alpha-1)^{\frac{1}{2}}}$.

In this paper, we treat the system (1.1) in the reflexive and separable Banach space $E^\alpha = E_0^{\alpha,2}$ for $0 < \alpha \leq 1$ and consider E^α with respect to the norm

$$\|u\|_\alpha := (\int_0^T |{}^c{}_0\mathrm{D}_t^\alpha u(t)|^2 dt)^{\frac{1}{2}} = \|{}^c{}_0\mathrm{D}_t^\alpha u(t)\|_{L^2},$$

which is equivalent to the norm $\|u\|_{\alpha,2} = (\int_0^T |{}^c{}_0\mathrm{D}_t^\alpha u(t)|^2 dt + \int_0^T |u(t)|^2 dt)^{\frac{1}{2}}$ for $u \in E^\alpha$.

Lemma 2.3 (Proposition 4.1, [4])**.** *If $\frac{1}{2} < \alpha \leq 1$, then for any $u \in E_0^\alpha$, one has*

$$|\cos(\pi\alpha)|\|u\|_\alpha^2 \leq -\int_0^T {}^c{}_0D_t^\alpha u(t) {}^c{}_tD_T^\alpha u(t) dt \leq \frac{1}{|\cos(\pi\alpha)|}\|u\|_\alpha^2. \tag{2.8}$$

Lemma 2.4. *Let $0 < \alpha \leq 1$ and $1 \leq p < \infty$. The fractional derivative space $E_0^{\alpha,p}$ is a reflexive and separable Banach space.*

Definition 2.6. A function $u \in E_0^\alpha$ is called a weak solution of the system (1.1) if

$$\int_0^T -{}^c{}_0\mathrm{D}_t^\alpha u(t) {}^c{}_t\mathrm{D}_T^\alpha u(t) dt + \sum_{k=1}^l I_k(u(t_k))v(t_k) - \lambda \int_0^T f(t, u(t)) dt - \mu \int_0^T g(t, u(t)) dt = 0,$$

for all $v(t) \in E_0^\alpha$.

Lemma 2.5. *The function $u(t) \in E_0^\alpha$ is a classical solution of (1.1) if and only if u is a weak solution of (1.1).*

Proof. The proof is similar to the proof of Lemma 2.1 in [2], we omit it here. □

Then, since $f, g : [0, T] \times R \to R$ are continuous functions, we let

$$F(t, u) = \int_0^u f(t, u) ds, G(t, u) = \int_0^u g(t, u) ds, J_k(u) = \int_0^u I_k(s) ds$$

for all $(t, u) \in [0, T] \times R$. Here, $G_c := \int_0^T \max_{|u| \leq c} G(t, u) dt$ for $c > 0$ and $G_d := \inf_{[0,T]\times[0,d]} G$ for $d > 0$. Obviously, $G_c \geq 0$ and $G_d \leq 0$.

Consider the functional $\Upsilon_\lambda : E_0^\alpha \to R$, defined by

$$\Upsilon_\lambda(u) := \Phi(u) - \lambda\Psi(u), u \in E_0^\alpha, \tag{2.9}$$

where

$$\Phi(u) := -\int_0^T {}^c{}_0\mathrm{D}_t^\alpha u(t) {}^c{}_t\mathrm{D}_T^\alpha u(t) dt + \sum_{k=1}^l J_k(u(t_k)) \tag{2.10}$$

and

$$\Psi(u) := \int_0^T [F(t, u(t)) + \frac{\mu}{\lambda} G(t, u(t))] dt. \tag{2.11}$$

Obviously, Φ and Ψ are *Gâteaux* differentiable functionals whose *Gâteaux* derivatives at the point $u \in E_0^\alpha$ are as follow:

$$\Phi'(u)(v) = -\int_0^T \left({}^c{}_0\mathrm{D}_t^\alpha u(t) {}^c{}_t\mathrm{D}_T^\alpha v(t) + {}^c{}_t\mathrm{D}_T^\alpha u(t) {}^c{}_0\mathrm{D}_t^\alpha v(t)\right) dt + \sum_{k=1}^l I_k(u(t_k))v(t_k),$$

$$\Psi^{'}(u)(v) = \int_0^T [f(t,u(t)) + \frac{\mu}{\lambda} g(t,u(t))]dt.$$

Then the critical point of $\Phi - \lambda\Psi$ is a exactly solution of the system of (1.1).

Now, we give the following critical point theorem which is powerful tool to verify our main results in this paper.

Theorem 2.1. *[Theorem 3.1, [17]] Let X be a reflexive real Banach space; $\Phi : X \to R$ be a coercive, continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*; $\Psi : X \to R$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that $\Phi(0) = \Psi(0) = 0$. Assume that there exist $r > 0$ and $\bar{x} \in X$, with $r < \Phi(\bar{x})$, such that*

(B1) $\frac{\sup_{\Phi(x)\le r}\Psi(x)}{r} < \frac{\Psi(\bar{x})}{\Phi(\bar{x})}$;

(B2) for each $\lambda \in \Lambda_r :=]\frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \frac{r}{\sup_{\Phi(x)\le r}\Psi(x)}[$ the functional $\Upsilon_\lambda := \Phi - \lambda\Psi$ is coercive.

Then, for each $\lambda \in \Lambda_r$ the functional Υ_r has at least three distinct critical points in X.

3. Main results

In this section, we will prove our main results. For convenience, we let

$$\begin{aligned} M(T,\alpha) := & \int_0^{\frac{T}{4}} t^{2-2\alpha}dt + \int_{\frac{T}{4}}^{\frac{3T}{4}} [t^{1-\alpha} - (t-\frac{T}{4})^{1-\alpha}]^2 dt \\ & + \int_{\frac{3T}{4}}^{T} [t^{1-\alpha} - (t-\frac{T}{4})^{1-\alpha} + (t-\frac{3T}{4})^{1-\alpha}]^2 dt, \end{aligned}$$

$$\omega^*_{\alpha,d} := \frac{1}{|\cos(\pi\alpha)|}\Big[\frac{16d^2}{T^2}M(T,\alpha) + el\Big],$$

$$r := \frac{|\cos(\pi\alpha)|}{M_2^2}c^2,$$

Assume that

$$\frac{\omega^*_{\alpha,d}}{\int_{\frac{T}{4}}^{\frac{3T}{4}} F(t,d)dt} < \frac{\frac{|\cos(\pi\alpha)|}{M_2^2}c^2}{\int_0^T \max_{|u|\le c} F(t,u)dt} \tag{3.1}$$

and let

$$\lambda \in \Lambda_1 := \Big]\frac{\omega^*_{\alpha,d}}{\int_{\frac{T}{4}}^{\frac{3T}{4}} F(t,d)dt}, \frac{c^2|\cos(\pi\alpha)|}{M_2^2\int_0^T \max_{|u|\le c} F(t,u)dt}\Big[, \tag{3.2}$$

put

$$\delta := \min\Big\{\frac{c^2|\cos(\pi\alpha)| - \lambda\int_0^T \max_{|u|\le c} F(t,u)dt}{M_2^2 G_c}, \frac{\omega^*_{\alpha,d} - \lambda\int_{\frac{T}{4}}^{\frac{3T}{4}} F(t,d)dt}{TG_d}\Big\} \tag{3.3}$$

and

$$\bar{\delta}_1 := \min\Big\{\delta, \frac{1}{\max\{0, \frac{2M_1^2}{|\cos(\pi\alpha)|}\limsup_{u\to\infty}\frac{\sup_{t\in[0,T]}G(t,u)}{u^2}\}}\Big\}. \tag{3.4}$$

Proof of Theorem 1.2. To complete this proof, we will apply Theorem 2.1 to prove Theorem 1.2. Firstly, we can clearly see $\Phi(0) = \Psi(0) = 0$, Φ is a nonnegative *Gâteaux* differentiable and its *Gâteaux* derivative has a continuous inverse on $(E_0^\alpha)^*$. Indeed, let $u_n \rightharpoonup u$ weakly in E_0^α, we easily have that

$$\begin{aligned}\liminf_{n\to\infty} \Phi(u_n) &= \liminf_{n\to\infty} \Big[-\int_0^T {}^c_0\mathrm{D}_t^\alpha u_n(t) {}^c_t\mathrm{D}_T^\alpha u_n(t)dt + \sum_{k=1}^{l} J_k(u_n(t_k))\Big] \\ &\geq -\int_0^T {}^c_0\mathrm{D}_t^\alpha u(t) {}^c_t\mathrm{D}_T^\alpha u(t)dt + \sum_{k=1}^{l} J_k(u(t_k)) \\ &= \Phi(u),\end{aligned}$$

so Φ is weakly sequentially lower semicontinuous. Moreover, by $(A3^*)$, we have

$$0 \leq J_k(u) \leq \frac{e}{|\cos(\pi\alpha)|}. \tag{3.5}$$

So, by (2.8) and (3.5), we have

$$\begin{aligned}\Phi(u) &= -\int_0^T {}^c_0\mathrm{D}_t^\alpha u(t) {}^c_t\mathrm{D}_T^\alpha u(t)dt + \sum_{k=1}^{l} J_k(u(t_k)) \\ &\geq |\cos(\pi\alpha)| \|u\|_\alpha^2 \to +\infty\end{aligned}$$

as $\|u\|_\alpha \to +\infty$, that is to say, Φ is coercive. By similar asserts, Ψ defined in (2.11) is a continuously *Gâteaux* differentiable functional whose *Gâteaux* derivative is compact.

Next, we intend to verify (B1) and (B2) of Theorem 2.1 hold.

Let $\tau(t)$ be the function given by

$$\tau(t) := \begin{cases} \frac{4\Gamma(2-\alpha)d}{T} t, & t \in [0, \frac{T}{4}], \\ \Gamma(2-\alpha)d, & t \in [\frac{T}{4}, \frac{3T}{4}], \\ \frac{4\Gamma(2-\alpha)d}{T}(T-t), & t \in [\frac{3T}{4}, T]. \end{cases} \tag{3.6}$$

It is obvious that $\tau(0) = \tau(T) = 0$ and $\tau \in L^2([0, T])$. Moreover, we have

$$\tau'(t) := \begin{cases} \frac{4\Gamma(2-\alpha)d}{T}, & t \in (0, \frac{T}{4}), \\ 0, & t \in (\frac{T}{4}, \frac{3T}{4}), \\ -\frac{4\Gamma(2-\alpha)d}{T}, & t \in (\frac{3T}{4}, T). \end{cases} \tag{3.7}$$

Then we can directly calculate the left Caputo fractional derivative of order α for $\tau(t)$.

If $t \in [0, \frac{T}{4}]$, we have

$$\begin{aligned}{}^c_0\mathrm{D}_t^\alpha \tau(t) &= \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \tau'(s)ds \\ &= \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{4\Gamma(2-\alpha)d}{T} ds\end{aligned}$$

$$= \frac{4d}{T} t^{1-\alpha}.$$

If $t \in (\frac{T}{4}, \frac{3T}{4}]$, we have

$$\begin{aligned}
{}^c_0\mathrm{D}_t^\alpha \tau(t) &= \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \tau'(s) ds \\
&= \frac{1}{\Gamma(1-\alpha)} \Big[\int_0^{\frac{T}{4}} (t-s)^{-\alpha} \frac{4\Gamma(2-\alpha)d}{T} ds + \int_{\frac{T}{4}}^t (t-s)^{-\alpha} 0 ds \Big] \\
&= \frac{4d(t^{1-\alpha} - (t-\frac{T}{4})^{1-\alpha})}{T}.
\end{aligned}$$

If $t \in (\frac{3T}{4}, T]$, we have

$$\begin{aligned}
{}^c_0\mathrm{D}_t^\alpha \tau(t) &= \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \tau'(s) ds \\
&= \frac{1}{\Gamma(1-\alpha)} \Big[\int_0^{\frac{T}{4}} (t-s)^{-\alpha} \frac{4\Gamma(2-\alpha)d}{T} ds + \int_{\frac{T}{4}}^{\frac{3T}{4}} (t-s)^{-\alpha} 0 ds \\
&- \int_{\frac{3T}{4}}^t (t-s)^{-\alpha} \frac{4\Gamma(2-\alpha)d}{T} ds \Big] \\
&= \frac{4d[t^{1-\alpha} - (t-\frac{T}{4})^{1-\alpha} - (t-\frac{3T}{4})^{1-\alpha}]}{T}.
\end{aligned}$$

That is

$$ {}^c_0\mathrm{D}_t^\alpha \tau(t) = \begin{cases} \frac{4d}{T} t^{1-\alpha}, & t \in [0, \frac{T}{4}], \\ \frac{4d(t^{1-\alpha} - (t-\frac{T}{4})^{1-\alpha})}{T}, & t \in (\frac{T}{4}, \frac{3T}{4}], \\ \frac{4d[t^{1-\alpha} - (t-\frac{T}{4})^{1-\alpha} - (t-\frac{3T}{4})^{1-\alpha}]}{T}, & t \in (\frac{3T}{4}, T]. \end{cases} \tag{3.8}$$

Then

$$\begin{aligned}
\|\tau(t)\|_\alpha^2 &= \int_0^T ({}^c_0\mathrm{D}_t^\alpha \tau(t))^2 dt \\
&= \int_0^{\frac{T}{4}} \Big(\frac{4d}{T} t^{1-\alpha}\Big)^2 dt + \int_{\frac{T}{4}}^{\frac{3T}{4}} \Big[\frac{4d(t^{1-\alpha} - (t-\frac{T}{4})^{1-\alpha})}{T}\Big]^2 dt \\
&+ \int_{\frac{3T}{4}}^T \Big[\frac{4d[t^{1-\alpha} - (t-\frac{T}{4})^{1-\alpha} - (t-\frac{3T}{4})^{1-\alpha}]}{T}\Big]^2 dt \\
&= \frac{16d^2}{T^2} \Big\{ \int_0^{\frac{T}{4}} t^{2-2\alpha} dt + \int_{\frac{T}{4}}^{\frac{3T}{4}} [t^{1-\alpha} - (t-\frac{T}{4})^{1-\alpha}]^2 dt \\
&+ \int_{\frac{3T}{4}}^T [t^{1-\alpha} - (t-\frac{T}{4})^{1-\alpha} + (t-\frac{3T}{4})^{1-\alpha}]^2 dt \Big\} \\
&= \frac{16d^2}{T^2} M(T, \alpha).
\end{aligned}$$

Next, by (2.10) and (3.5), we have

$$\Phi(\tau) = -\int_0^T {}^c_0\mathrm{D}_t^\alpha \tau(t) {}^c_t\mathrm{D}_T^\alpha \tau(t) dt + \sum_{k=1}^l J_k(\tau(t_k))$$

$$\le \Big(\frac{1}{|\cos(\pi\alpha)|}\Big)\|\tau\|_\alpha^2 + \frac{el}{|\cos(\pi\alpha)|}$$
$$= \frac{1}{|\cos(\pi\alpha)|}\Big[\frac{16d^2}{T^2}M(T,\alpha) + el\Big] := \omega^*_{\alpha,d}.$$

Since $0 \le \tau(t) \le d$ for $t \in [0,T]$, by (A1) and (2.11), we have

$$\begin{aligned}\Psi(\tau) &= \int_0^T [F(t,\tau(t)) + \frac{\mu}{\lambda}G(t,\tau(t))]dt \\ &\ge \int_{\frac{T}{4}}^{\frac{3T}{4}} F(t,d)dt + \frac{\mu}{\lambda}\int_0^T G(t,\tau(t))dt \\ &\ge \int_{\frac{T}{4}}^{\frac{3T}{4}} F(t,d)dt + \frac{\mu}{\lambda}TG_d.\end{aligned}$$

So, we have

$$\frac{\Psi(\tau)}{\Phi(\tau)} \ge \frac{\int_{\frac{T}{4}}^{\frac{3T}{4}} F(t,d)dt + \frac{\mu}{\lambda}TG_d}{\omega^*_{\alpha,d}}.$$

Hence, if $G_d = 0$, that is

$$\frac{\Psi(\tau)}{\Phi(\tau)} > \frac{1}{\lambda}. \tag{3.9}$$

If $G_d < 0$, then the same relation holds since

$$\mu < \frac{\omega^*_{\alpha,d} - \lambda\int_{\frac{T}{4}}^{\frac{3T}{4}} F(t,d)dt}{TG_d}.$$

On the other hand, from (1.7), we have

$$\Phi(\tau) \ge |\cos(\pi\alpha)|\|\tau\|_\alpha^2 = \frac{16d^2|\cos(\pi\alpha)|}{T^2}M(T,\alpha) > r. \tag{3.10}$$

Then for all $u \in E_0^\alpha$, with $\Phi(u) \le r$, by (2.8) and (3.5), we have

$$|\cos(\pi\alpha)|\|u\|_\alpha^2 \le \Phi(u) \le r,$$

which yields

$$\|u\|_\alpha^2 \le \frac{r}{|\cos(\pi\alpha)|}.$$

Then, by (2.2) and (2.7), we have

$$|u| \le M_2\sqrt{\frac{r}{|\cos(\pi\alpha)|}} = c, t \in [0,T].$$

By (2.11), we have

$$\begin{aligned}\Psi(u) &= \int_0^T [F(t,u(t)) + \frac{\mu}{\lambda}G(t,u(t))]dt \\ &\le \int_0^T \max_{|u|\le c} F(t,u)dt + \frac{\mu}{\lambda}\int_0^T \max_{|u|\le c} G(t,u)dt.\end{aligned}$$

Therefore

$$\frac{\sup_{\Phi(u)\le r}\Psi(u)}{r} \le \frac{\int_0^T \max_{|u|\le c} F(t,u)dt + \frac{\mu}{\lambda}\int_0^T \max_{|u|\le c} G(t,u)dt}{\frac{|\cos(\pi\alpha)|}{M_2^2}c^2}$$
$$\le \frac{M_2^2}{c^2|\cos(\pi\alpha)|}\Big(\int_0^T \max_{|u|\le c} F(t,u)dt + \frac{\mu}{\lambda}G_c\Big).$$

If $G_c = 0$, it is obvious that

$$\frac{\sup_{\Phi(u)\le r}\Psi(u)}{r} < \frac{1}{\lambda}. \tag{3.11}$$

If $G_c > 0$, we also know (3.11) holds since

$$\mu < \frac{c^2|\cos(\pi\alpha)| - \lambda\int_0^T \max_{|u|\le c} F(t,u)dt}{M_2^2 G_c}.$$

Together with (3.9) and (3.11), the condition (B1) of Theorem 2.1 holds. Then, the coercive of function Υ_λ will be verified as follow.

Case 1. If

$$\limsup_{|u|\to+\infty} \frac{\sup_{t\in[0,T]} F(t,u)}{u^2} > 0,$$

there exist a $\varepsilon > 0$ such that

$$\limsup_{|u|\to+\infty} \frac{\sup_{t\in[0,T]} F(t,u)}{u^2} < \varepsilon < \frac{M_2^2\int_0^T \max_{|u|\le c} F(t,u)dt}{2c^2M_1^2}, \tag{3.12}$$

so, there exists a function $f_\varepsilon \in L^1([0,T])$ such that $F(t,u) \le \varepsilon u^2 + f_\varepsilon(t)$ for each $(t,u) \in [0,T]\times R$. Since

$$\lambda < \frac{c^2|\cos(\pi\alpha)|}{M_2^2\int_0^T \max_{|u|\le c} F(t,u)dt},$$

by (2.6), we immediately have

$$\begin{aligned}\lambda\int_0^T F(t,u)dt &\le \lambda\Big(\varepsilon\int_0^T u(t)^2dt + \int_0^T f_\varepsilon(t)dt\Big)\\ &< \frac{c^2|\cos(\pi\alpha)|}{M_2^2\int_0^T \max_{|u|\le c} F(t,u)dt}\Big(\varepsilon M_1^2\|u\|_\alpha^2 + \|f_\varepsilon\|_{L^1([0,T])}\Big).\end{aligned} \tag{3.13}$$

Since $\mu < \bar{\delta}_1$, that is

$$\limsup_{|u|\to+\infty} \frac{\sup_{t\in[0,T]} G(t,u)}{u^2} < \frac{|\cos(\pi\alpha)|}{2\mu M_1^2},$$

then, there exist a function $f_\mu \in L^1([0,T])$ such that

$$G(t,u) \le \frac{|\cos(\pi\alpha)|}{2\mu M_1^2}u^2 + f_\mu(t)$$

for each $(t,u)\in[0,T]\times R$. By (2.6), we have

$$\begin{aligned}\int_0^T G(t,u)dt &\le \tfrac{|\cos(\pi\alpha)|}{2\mu M_1^2}\int_0^T u(t)^2dt+\int_0^T f_\mu(t)dt\\ &\le \tfrac{|\cos(\pi\alpha)|}{2\mu}\|u\|_\alpha^2+\|f_\mu\|_{L^1([0,T])}.\end{aligned}\tag{3.14}$$

Combining (2.11), (3.5) and (3.12)-(3.14), we immediately have

$$\begin{aligned}\Upsilon_\lambda(u)&=\Phi(u)-\lambda\Psi(u)\\ &=-\int_0^T {}^c_0\mathrm{D}_t^\alpha u(t){}^c_t\mathrm{D}_T^\alpha u(t)dt+\sum_{k=1}^l J_k(u(t_k))-\lambda\Big(\int_0^T[F(t,u(t))+\frac{\mu}{\lambda}G(t,u(t))]dt\Big)\\ &\ge|\cos(\pi\alpha)|\|u\|_\alpha^2-\lambda\int_0^T F(t,u(t))dt-\mu\int_0^T G(t,u(t))dt\\ &\ge|\cos(\pi\alpha)|\|u\|_\alpha^2-\frac{c^2|\cos(\pi\alpha)|}{M_2^2\int_0^T\max_{|u|\le c}F(t,u)dt}\Big(\varepsilon M_1^2\|u\|_\alpha^2+\|f_\varepsilon\|_{L^1([0,T])}\Big)\\ &-\frac{|\cos(\pi\alpha)|}{2}\|u\|_\alpha^2-\mu\|f_\mu\|_{L^1([0,T])}\\ &=|\cos(\pi\alpha)|\Big(\frac12-\frac{c^2M_1^2}{M_2^2\int_0^T\max_{|u|\le c}F(t,u)dt}\varepsilon\Big)\|u\|_\alpha^2\\ &-\frac{c^2|\cos(\pi\alpha)|}{M_2^2\int_0^T\max_{|u|\le c}F(t,u)dt}\|f_\varepsilon\|_{L^1([0,T])}-\mu\|f_\mu\|_{L^1([0,T])}.\end{aligned}$$

Case 2. If

$$\limsup_{|u|\to+\infty}\frac{\sup_{t\in[0,T]}F(t,u)}{u^2}\le 0,$$

there exist a function $f_\varepsilon\in L^1([0,T])$ such that $F(t,u)\le\varepsilon u^2+f_\varepsilon(t)$ for each $(t,u)\in[0,T]\times R$. By the same proof as the Case 1., we obtain

$$\begin{aligned}\Upsilon_\lambda(u)&\ge|\cos(\pi\alpha)|\|u\|_\alpha^2\\ &-\frac{c^2|\cos(\pi\alpha)|}{M_2^2\int_0^T\max_{|u|\le c}F(t,u)dt}\|f_\varepsilon\|_{L^1([0,T])}-\mu\|f_\mu\|_{L^1([0,T])}.\end{aligned}$$

The above two cases imply that $\lim_{\|u\|_\alpha\to+\infty}(\Phi(u)-\lambda\Psi(u))=+\infty$, so $\Phi(u)-\lambda\Psi(u)$ is coercive and the condition (B2) of Theorem 2.1 holds. Thus, we deduce that for each $\lambda\in\Lambda_1$, Theorem 2.1 ensures the functional Υ_λ has at least three solutions in E^α.

Remark 3.1. If $g(t,u)=0$ and $I_k(u(t_k))=0$, then the system (1.1) become the following form

$$\begin{cases}\frac{d}{dt}\{\ {}_0\mathrm{D}_t^{\alpha-1}({}^c_0\mathrm{D}_t^\alpha u(t))-\ {}_t\mathrm{D}_T^{\alpha-1}({}^c_t\mathrm{D}_T^\alpha u(t))\}\\ +\lambda f(t,u(t))=0, t\in[0,T],\\ u(0)=u(T)=0,\end{cases}\tag{1.1*}$$

Our result is a generalization of [3] and [4]. In this case, the functionals are defines as follow:

$$\Phi(u):=-\int_0^T {}^c_0\mathrm{D}_t^\alpha u(t){}^c_t\mathrm{D}_T^\alpha u(t)dt$$

and

$$\Psi(u) := \int_0^T [F(t, u(t)) - \lambda \sum_{k=1}^{l} J_k(u(t_k))].$$

Then, by the similar proof as Theorem 1.2, we will obtain at least three solutions for the system (1.1*).

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