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The Influence of Human Behavior on Heroin Dynamics

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Abstract In this paper, considering the influence of human behavior on heroin abuse, we establish a mathematical model to describe the spread of heroin. When the basic reproduction number is less than one, the heroin-free abuse equilibrium point is globally asymptotically stable. When the basic reproduction number is greater than one, the model has a unique heroin abuse equilibrium point which is globally asymptotically stable, and the heroin-free abuse equilibrium point is unstable. Finally, based on the partial rank correlation coefficients (PRCCs) and numerical simulations, the dynamic behavior of the model is further revealed. Our results show that human behavior can reduce the heroin abuse level.

Keywords Nonlinear incidence rate, Heroin model, Human behavior, Global asymptotic stability.

MSC(2010) 34D20, 34D23.

1. Introduction

Heroin is an opioid drug made from morphine, a natural substance extracted from the seed pods of various opium poppy plants grown in Southeast and South-West Asia, Mexico and Colombia [1]. Heroin has become the most widely abused drug in the world. Among all drugs, drug crimes involving heroin manufacture, smuggling and abuse rank first. It is called the king of drugs in the world. Once one consumes heroin, the drug treatment can be divided into three stages: detoxification, rehabilitation and social return. Buprenorphine and methadone maintenance therapy are effective in the treatment of physical detoxification. Some studies have found that despite the relatively advanced conditions and methods for drug treatment, the relapse rate is still hovering 80%∼90% [2]. While social problems such as alcohol and drug use have been referred to in terms of epidemics, little has been published on the application of mathematical modelling methods to such problems [3]. Therefore, the problem of heroin abuse has attracted more and more attention of the society.

In the last decades, various mathematical modelling techniques have been extended for the purpose of understanding and combating heroin addiction prob-

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lems [3–13]. Using the idea of warehouse modeling in infectious disease dynamics, White and Comiskey applied infectious disease model to the study of drug users for the first time and established a kind of ordinary differential equation mathematical model of heroin drug transmission [3]. Subsequently, based on the literature [3], Samanta suggested that the coefficients of the model should depend on some long-term trends and seasonal changes of heroin epidemics. A non-autonomous heroin drug transmission model with distributed delays was established, and the sufficient conditions for the global asymptotic stability of the model were derived (see [4]). In [5], Fang and Li studied global asymptotic properties for an age-structured model of heroin use based on the principles of mathematical epidemiology where the incidence rate depends on the age of susceptible individuals. In [6], Wangari and Stone developed a model to test how heroin addiction spreads in society by using saturation functions to represent limited therapeutic effects and successful detoxification phenomena. The global stability of the model and an inherent backward bifurcation are obtained. In [14], Gao and Wang studied the impact of human behavior on cholera infection, local and global dynamics of the model are analyzed with respect to the basic reproduction number. Then they extend the ODE model to a reaction convection diffusion partial differential equation (PDE) model that accounts for the movement of both human hosts and bacteria. However, the influence of human behavior on heroin drug transmission and successful drug detoxification is rarely investigated in mathematical models.

To better describe the dynamic behavior of heroin drug transmission, in this paper, we consider the influence of human behavior on the transmission of heroin drugs and the successful detoxification of heroin addicts. This paper is organized as follows: In Section 1, we will establish a heroin abuse model. In Section 2, we will prove the local stability and global asymptotic stability of the equilibrium point. In Section 3, we will do numerical simulations and analyze the influence of parameters on the model. Finally, we will make a summary for this article.

2. Model description

The total population is divided into three compartments: heroin susceptible individuals, untreated heroin addicts, and treated heroin addicts, whose population size at time t was recorded as $S(t)$, $U_1(t)$, $U_2(t)$. The model is established as follows:

$$\begin{cases} \frac{\mathrm{d}S}{\mathrm{d}t} = \Lambda - \beta(U_1)SU_1 - \mu S + \varepsilon U_1 + \eta U_2, \\ \frac{\mathrm{d}U_1}{\mathrm{d}t} = \beta(U_1)SU_1 + kU_2 - (p + \mu + \varepsilon + \delta_1)U_1, \\ \frac{\mathrm{d}U_2}{\mathrm{d}t} = pU_1 - (\mu + k + \eta + \delta_2)U_2. \end{cases} \tag{2.1}$$

In (2.1), the total population satisfies $N(t) = S(t) + U_1(t) + U_2(t)$. Λ is the number of susceptible individuals to heroin from the general population. k is the proportion of heroin addicts who relapse after cessation of treatment. η is the proportion of successful detoxification among heroin addicts receiving treatment. According to biological meaning we know $k > \eta$, because the rate of heroin relapse is higher than the rate of successful detoxification. p is the proportion of heroin addicts receiving treatment in detoxification centers. ε is the proportion of self-healing

among untreated heroin addicts. μ is the natural mortality rate of individuals in population. δ_1, δ_2 are the removal rates of untreated heroin addicts and treated heroin addicts. Assume that all parameters are non-negative.

The function $\beta(U_1)$ is the proportion of a susceptible individual who is successfully infected by untreated heroin addicts to become a new heroin addict. The incorporation of heroin abuse is dependent on contact rates and host shedding rate. We assume $\beta(U_1) = a - bm(U_1)$, where a is the usual contact rate without considering the influence of human behavior, b is the maximum reduced contact rate due to behavior change, and $m(U_1)$ is a saturation function to measure the impact of the number of heroin user individuals. These parameters satisfy

$$a > b \geq 0, m(U_1) \in C^1\left(0, \frac{\Lambda}{\mu}\right),$$

$$\text{with } m'(U_1) \geq 0, m(0) = 0, 0 \leq m\left(\frac{\Lambda}{\mu}\right) \leq 1.$$

In the epidemic model, we know that the number of people in the model cannot be negative. Therefore, we will prove this practical conclusion by theoretical deduction. First of all, we assume that $S(0) \geq 0, U_1(0) > 0, U_2(0) > 0$ holds. The following theorem shows the positivity of solutions.

Theorem 2.1. *Any solution $(S(t), U_1(t), U_2(t))$ of model (2.1) is positive for all $t > 0$.*

Proof. Let $(S(t), U_1(t), U_2(t))$ be a solution of model (2.1). According to the existence and uniqueness of the solution, it is easy to see that on the maximum existence interval $(0, t^*)$ of solutions, there is $S(t) > 0, U_1(t) > 0, U_2(t) > 0$. Suppose the conclusion is not true, then there is a $t_1 > 0$ such that $\min\{S(t_1), U_1(t_1), U_2(t_1)\} = 0$, and $S(t) > 0$, $U_1(t) > 0$, $U_2(t) > 0$ for all $t \in (0, t^*)$. Next, we will discuss it in three cases:

(1)$S(t_1)$=0,
(2)$U_1(t_1)$=0,
(3)$U_2(t_1)$=0.

If (1) is valid, then we can get $S'(t_1) \leq 0$. However $S'(t_1) = \Lambda + \varepsilon U_1(t_1) + \eta U_2(t_1) > 0$ can be obtained from the first equation of model (2.1), which leads to a contradiction. If (2) is valid, from the second equation of model (2.1), we can get

$$\begin{aligned} U_1(t) =& U_1(0) \exp\left[\int_0^t [\beta(U_1(\theta)) S(\theta) - (p + \mu + \varepsilon + \delta_1)] \mathrm{d}\theta\right] \\ &+ \exp\left[\int_0^t [\beta(U_1(\theta)) S(\theta) - (p + \mu + \varepsilon + \delta_1)] \mathrm{d}\theta\right] \\ &\cdot \int_0^t k U_2(\varphi) \exp\left[-\int_0^t [\beta(U_1(\theta)) S(\theta) - (p + \mu + \varepsilon + \delta_1)] \mathrm{d}\theta\right] \mathrm{d}\varphi, \end{aligned}$$

then $U_1(t_1) = U_1(0) \exp\left[\int_0^{t_1} [\beta(U_1(\theta)) S(\theta) - (p + \mu + \varepsilon + \delta_1)] \mathrm{d}\theta\right] > 0$. If (3) is valid, from the third equation of model (2.1), we can get

$$\begin{aligned} U_2(t) =& U_2(0) \exp[-(k + \mu + \eta + \delta_2) t] + \exp[-(k + \mu + \eta + \delta_2) t] \\ &\cdot \int_0^t p U_1(\theta) \exp[(k + \mu + \eta + \delta_2) t] \mathrm{d}\theta, \end{aligned}$$

then $U_2(t_1) = U_2(0)\exp\left[-\left(k+\mu+\eta+\delta_2\right)t_1\right] > 0$.

In summary, t_1 does not exist. Thus we get that $S(t)$, $U_1(t)$ and $U_2(t)$ are positive on $(0, t^*)$.

The total population of model (2.1) satisfies the equation

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \Lambda - \mu N - \delta_1 U_1 - \delta_2 U_2 \leq \Lambda - \mu N,$$

then

$$\lim_{t\to+\infty} \sup N\left(t\right) \leq \frac{\Lambda}{\mu}.$$

This proves that $t^* = \infty$. Then completes the proof. □

Remark 2.1. When assuming initial data $S(0) \geq 0, U_1(0) \geq 0, U_2(0) > 0$ and $S(0) \geq 0, U_1(0) > 0, U_2(0) \geq 0$ the same conclusion can be drawn. However, when assuming initial data $S(0) \geq 0, U_1(0) = U_2(0) = 0$, we will obtain a disease-free solution $\left(S\left(t\right), 0, 0\right)$.

Therefore, the region

$$\Omega = \left\{ (S, U_1, U_2) \in \mathbb{R}^3_+ \middle| 0 \leq S + U_1 + U_2 \leq \frac{\Lambda}{\mu} \right\},$$

is a positively invariant and attractive set for model (2.1).

3. Threshold and equilibrium point of the model

Obviously, model (2.1) always has a heroin-free abuse equilibrium point $E^0\left(\frac{\Lambda}{\mu}, 0, 0\right)$. According to the notation in [15], the matrices $\mathbf{F}$ (new infection terms) and $\mathbf{V}$ (remaining transition terms) are given [16], as follows

$$\mathbf{F} = \begin{bmatrix} \frac{a\Lambda}{\mu} & k \\ 0 & 0 \end{bmatrix}, \mathbf{V} = \begin{bmatrix} p+\mu+\delta_1+\varepsilon & 0 \\ -p & \mu+\delta_2+k+\eta \end{bmatrix},$$

$$\mathbf{V}^{-1} = \begin{bmatrix} \frac{1}{p+\mu+\delta_1+\varepsilon} & 0 \\ \frac{p}{(p+\mu+\delta_1+\varepsilon)(\mu+\delta_2+k+\eta)} & \frac{1}{\mu+\delta_2+k+\eta} \end{bmatrix}.$$

It follows that the spectral radius of $\mathbf{F}\mathbf{V}^{-1}$ is

$$\rho\left(\mathbf{F}\mathbf{V}^{-1}\right) = \frac{a\Lambda}{\mu\left(p+\mu+\delta_1+\varepsilon\right)} + \frac{pk}{\left(p+\mu+\delta_1+\varepsilon\right)\left(\mu+\delta_2+k+\eta\right)}.$$

Thus, the basic reproduction number of model (2.1) is given by

$$R_0 = \frac{a\Lambda}{\mu\left(p+\mu+\delta_1+\varepsilon\right)} + \frac{pk}{\left(p+\mu+\delta_1+\varepsilon\right)\left(\mu+\delta_2+k+\eta\right)}.$$

Remark 3.1. Note that the basic reproduction number R_0 is independent of b. This is due to our model assumption that behavior change only starts when the heroin has already started and R_0 is calculated at the heroin-free abuse state. An

implication is that behavior change alone is usually not sufficient to terminate heroin abuse outbreak. Nevertheless, the behavior change alone cannot drive an endemic disease extinct, but it plays a significant role in reducing the number of infected persons and its proportion to the total population (see [17]). We will demonstrate this for our heroin model in the numerical simulation.

Again, if model (2.1) has a heroin abuse equilibrium point $E^*(S^*, U_1^*, U_2^*)$, then S^*, U_1^* and U_2^* satisfy the following algebraic equations

$$\begin{cases} \Lambda - \beta(U_1^*) S^* U_1^* - \mu S^* + \varepsilon U_1^* + \eta U_2^* = 0, \\ \beta(U_1^*) S^* U_1^* + k U_2^* - (p + \mu + \varepsilon + \delta_1) U_1^* = 0, \\ p U_1^* - (\mu + k + \eta + \delta_2) U_2^* = 0. \end{cases} \tag{3.1}$$

From the last equation of (3.1), we have

$$U_2^* = \frac{p U_1^*}{\mu + k + \eta + \delta_2}. \tag{3.2}$$

Adding up the first two equations of (3.1) yield

$$S^* = \frac{\Lambda - (p + \mu + \delta_1) U_1^* + (\eta + k) U_2^*}{\mu},$$

thus,

$$S^* = \frac{\Lambda}{\mu} - \frac{(p + \mu + \delta_1)(\mu + \delta_2) + (\mu + \delta_1)(\eta + k)}{\mu(\mu + k + \eta + \delta_2)} U_1^*.$$

To ensure $S^* > 0$, we need

$$U_1^* < \frac{\Lambda(\mu + k + \eta + \delta_2)}{(p + \mu + \delta_1)(\mu + \delta_2) + (\mu + \delta_1)(\eta + k)}.$$

When $U_1^* \neq 0$, from the second equation of (3.1) and (3.2), we get

$$S^* = \frac{(p + \mu + \varepsilon + \delta_1) U_1^* - k U_2^*}{\beta(U_1^*) U_1^*} = \frac{(p + \mu + \varepsilon + \delta_1)(\mu + k + \eta + \delta_2) - pk}{\beta(U_1^*)(\mu + k + \eta + \delta_2)}.$$

So

$$\begin{aligned} &\frac{\Lambda}{\mu} - \frac{(p + \mu + \delta_1)(\mu + \delta_2) + (\mu + \delta_1)(\eta + k)}{\mu(\mu + k + \eta + \delta_2)} U_1^* \\ &= \frac{(p + \mu + \varepsilon + \delta_1)(\mu + k + \eta + \delta_2) - pk}{\beta(U_1^*)(\mu + k + \eta + \delta_2)}. \end{aligned} \tag{3.3}$$

Set

$$\begin{aligned} F(U_1) &= \frac{\Lambda}{\mu} - \frac{(p + \mu + \delta_1)(\mu + \delta_2) + (\mu + \delta_1)(\eta + k)}{\mu(\mu + k + \eta + \delta_2)} U_1^*, \\ G(U_1) &= \frac{(p + \mu + \varepsilon + \delta_1)(\mu + k + \eta + \delta_2) - pk}{\beta(U_1)(\mu + k + \eta + \delta_2)}. \end{aligned}$$

When $0 < U_1 < \frac{\Lambda(\mu+k+\eta+\delta_2)}{(p+\mu+\delta_1)(\mu+\delta_2)+(\mu+\delta_1)(\eta+k)}$, $F(U_1)$ is strictly decreasing and

$$F(0) = \frac{\Lambda}{\mu}, F\left(\frac{\Lambda(\mu+k+\eta+\delta_2)}{(p+\mu+\delta_1)(\mu+\delta_2)+(\mu+\delta_1)(\eta+k)}\right) = 0.$$

According to the monotonicity of $\beta(U_1)$, function $G(U_1)$ is strictly increasing and

$$\lim_{U_1^* \to 0} G(U_1^*) = \frac{(p+\mu+\varepsilon+\delta_1)(\mu+k+\eta+\delta_2) - pk}{a(\mu+k+\eta+\delta_2)}.$$

Therefore, when $R_0 > 1$, we have

$$\frac{\Lambda}{\mu} > \frac{(p+\mu+\varepsilon+\delta_1)(\mu+k+\eta+\delta_2) - pk}{a(\mu+k+\eta+\delta_2)}.$$

Equation (3.3) exists a unique positive root U_1^* in the interval

$$0 < U_1^* < \frac{\Lambda(\mu+k+\eta+\delta_2)}{(p+\mu+\delta_1)(\mu+\delta_2)+(\mu+\delta_1)(\eta+k)} < \frac{\Lambda}{\mu}.$$

When $R_0 \leq 1$, Equation (3.3) does not have any positive root. Then model (2.1) does not have heroin abuse equilibrium point, so we have the following result.

Theorem 3.1. *Model (2.1) has a heroin-free abuse equilibrium point* $E^0\left(\frac{\Lambda}{\mu}, 0, 0\right)$. *When* $R_0 > 1$, *there is only one heroin abuse equilibrium point* $E^*(S^*, U_1^*, U_2^*)$ *besides* E^0, *where* $U_2^* = \frac{pU_1^*}{\mu+k+\eta+\delta_2}$, $S^* = \frac{\Lambda}{\mu} - \frac{(p+\mu+\delta_1)(\mu+\delta_2)+(\mu+\delta_1)(\eta+k)}{\mu(\mu+k+\eta+\delta_2)}U_1^*$, U_1^* *is the unique positive root of equation (3.3) in the interval* $\left(0, \frac{\Lambda(\mu+k+\eta+\delta_2)}{(p+\mu+\delta_1)(\mu+\delta_2)+(\mu+\delta_1)(\eta+k)}\right)$.

4. Stability of the heroin-free abuse equilibrium

In this section, we investigate the global asymptotic stability of the heroin-free abuse equilibrium point E^0.

Theorem 4.1. *If* $R_0 < 1$, *the heroin-free abuse equilibrium point* E^0 *is locally asymptotically stable. If* $R_0 > 1$, E^0 *is unstable.*

Proof. Linearizing model (2.1) at E^0, we get

$$\mathbf{J}\left(E^0\right) = \begin{bmatrix} -\mu & -\frac{a\Lambda}{\mu} + \varepsilon & \eta \\ 0 & \frac{a\Lambda}{\mu} - (p+\mu+\varepsilon+\delta_1) & k \\ 0 & p & -(\eta+\mu+k+\delta_2) \end{bmatrix}.$$

Let λ_1, λ_2 and λ_3 denote the eigenvalues of $\mathbf{J}\left(E^0\right)$, where $\lambda_1 = -\mu < 0$. It is easy to verify that λ_2 and λ_3 satisfy the equation

$$\lambda^2 - c_1\lambda - c_2 = 0, \tag{4.1}$$

where

$$c_1 = \frac{a\Lambda}{\mu} - (p+\mu+\varepsilon+\delta_1) - (\eta+\mu+k+\delta_2),$$

$$c_2 = \left[\frac{a\Lambda}{\mu} - (p+\mu+\varepsilon+\delta_1)\right](\eta+\mu+k+\delta_2) + pk.$$

If $R_0 \leq 1$, then $c_1 < 0$ and

$$\begin{aligned} c_2 &= \left[\frac{a\Lambda}{\mu} - (p+\mu+\varepsilon+\delta_1)\right](\eta+\mu+k+\delta_2) + pk \\ &= (\eta+\mu+k+\delta_2)(p+\mu+\varepsilon+\delta_1)(R_0-1) \leq 0. \end{aligned}$$

Thus if $R_0 < 1$, characteristic equation (4.1) has no roots with non-negative real parts. According to the Routh – Hurwitz criterion [18], we obtain the heroin-free abuse equilibrium point E^0 is locally asymptotically stable in the interior of Ω. If $R_0 > 1$, we have $c_2 > 0$. That is, characteristic equation (4.1) has a root with positive real part. So, the heroin-free abuse equilibrium point E^0 is unstable. This completes the proof of Theorem 4.1. □

Theorem 4.2. *If $R_0 < 1$, the heroin-free abuse equilibrium point E^0 is globally asymptotically stable. If $R_0 = 1$, the heroin-free abuse equilibrium point E^0 is globally attractive.*

Proof. Define the following Lyapunov function

$$V(U_1, U_2) = U_1 + \frac{k}{(\eta+\mu+k+\delta_2)}U_2.$$

The derivative along model (2.1) is

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}t} &= \frac{\mathrm{d}U_1}{\mathrm{d}t} + \frac{k}{(\eta+\mu+k+\delta_2)}\frac{\mathrm{d}U_2}{\mathrm{d}t} \\ &= \beta(U_1)SU_1 - \left[(p+\mu+\varepsilon+\delta_1) - \frac{pk}{(\eta+\mu+k+\delta_2)}\right]U_1 \\ &\leq U_1\left[\frac{a\Lambda}{\mu} + \frac{pk}{(\eta+\mu+k+\delta_2)} - (p+\mu+\varepsilon+\delta_1)\right], \end{aligned}$$

thus

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}t} &\leq U_1\left[\frac{a\Lambda}{\mu} + \frac{pk}{(\eta+\mu+k+\delta_2)} - (p+\mu+\varepsilon+\delta_1)\right] \\ &= U_1(p+\mu+\varepsilon+\delta_1)(R_0-1). \end{aligned}$$

If $R_0 < 1$, then $\frac{\mathrm{d}V}{\mathrm{d}t} \leq 0$. So omega limit point of trajectory is in the set $M_1 = \left\{(S,U_1,U_2)\,\middle|\,\frac{\mathrm{d}V}{\mathrm{d}t} = 0\right\} = \{(S,U_1,U_2)\,|U_1 = 0\}$. The solution of model (2.1) in set M_1 satisfies $U_2(t) = U_2(0)e^{-(\eta+\mu+k+\delta_2)t}$. When $t \to +\infty$, we have $U_2 \to 0$. In the similar way we have $S \to \frac{\Lambda}{\mu}$ as $t \to +\infty$. Thus $\left\{E^0\right\}$ is maximum omega limit point of model (2.1) in the interior M_1. According to the LaSalle's Invariant Principle [19], E^0 is globally attractive. Next, we discuss the case of $R_0 = 1$,

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}t} &= \frac{\mathrm{d}U_1}{\mathrm{d}t} + \frac{k}{(\eta+\mu+k+\delta_2)}\frac{\mathrm{d}U_2}{\mathrm{d}t} \\ &= \beta(U_1)SU_1 - \left[(p+\mu+\varepsilon+\delta_1) - \frac{pk}{(\eta+\mu+k+\delta_2)}\right]U_1 \end{aligned}$$

$$\leq \beta(U_1) SU_1 - \frac{\Lambda}{\mu}\beta(U_1) U_1 + (R_0 - 1) U_1$$
$$= \beta(U_1) U_1 \left(S - \frac{\Lambda}{\mu}\right) \leq 0.$$

Thus $M_2 = \left\{(S, U_1, U_2) \,|\, \frac{\mathrm{d}V}{\mathrm{d}t} = 0\right\} = \left\{(S, U_1, U_2) \,\middle|\, S = \frac{\Lambda}{\mu} \text{ or } U_1{=}0\right\}$. We know that $\left\{E^0\right\}$ is maximum omega limit point of model (2.1) in the interior M_2. In summary, by the LaSalle's Invariant Principle, E^0 is globally attractive. Thus, if $R_0 \leq 1$, E^0 is globally attractive in Ω. This completes the proof of Theorem 4.2. □

5. Stability of the heroin abuse equilibrium

Now we investigate the global asymptotic stability of E^* when $R_0 > 1$.

Theorem 5.1. *If $R_0 > 1$, the heroin abuse equilibrium point E^* is locally asymptotically stable.*

Proof. Linearizing model (2.1) at E^*, we get

$$\mathbf{J}(E^*) = \begin{bmatrix} -\beta(U_1^*) U_1^* - \mu & -\beta'(U_1^*) U_1^* S^* - \beta(U_1^*) S^* + \varepsilon & \eta \\ \beta(U_1^*) U_1^* & \zeta & k \\ 0 & p & -(\eta + \mu + k + \delta_2) \end{bmatrix},$$

where

$$\zeta = \beta'(U_1^*) U_1^* S^* + \beta(U_1^*) S^* - (p + \mu + \varepsilon + \delta_1).$$

From the equations (3.1), we have

$$\beta(U_1^*) S^* = (p + \mu + \varepsilon + \delta_1) - \frac{pk}{(\mu + k + \eta + \delta_2)}.$$

Then we write the characteristic equation as

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0, \tag{5.1}$$

where

$$\begin{aligned} a_1 =& \beta(U_1^*) U_1^* + \mu - \beta'(U_1^*) U_1^* S^* + \frac{pk}{(\mu + k + \eta + \delta_2)} + (\eta + \mu + k + \delta_2), \\ a_2 =& (\beta(U_1^*) U_1^* + \mu) \left(\frac{pk}{(\mu + k + \eta + \delta_2)} - \beta'(U_1^*) U_1^* S^*\right) \\ &+ \left(\frac{pk}{(\mu + k + \eta + \delta_2)} - \beta'(U_1^*) U_1^* S^*\right)(\eta + \mu + k + \delta_2) \\ &+ (\beta(U_1^*) U_1^* + \mu)(\eta + \mu + k + \delta_2) - pk \\ &+ \beta(U_1^*) U_1^* \left[\beta'(U_1^*) U_1^* S^* + (p + \mu + \delta_1) - \frac{pk}{(\mu + k + \eta + \delta_2)}\right] \\ =& -\mu\beta'(U_1^*) U_1^* S^* + \beta(U_1^*) U_1^* (p + \mu + \delta_1) + \frac{\mu pk}{(\mu + k + \eta + \delta_2)} \\ &- \beta'(U_1^*) U_1^* S^* (\eta + \mu + k + \delta_2) + (\beta(U_1^*) U_1^* + \mu)(\eta + \mu + k + \delta_2), \end{aligned}$$

$$
\begin{aligned}
a_3 =& \left(\beta\left(U_1^*\right)U_1^* + \mu\right)\left(\frac{pk}{(\mu+k+\eta+\delta_2)} - \beta'\left(U_1^*\right)U_1^*S^*\right)(\eta+\mu+k+\delta_2) \\
&+ p\eta\beta\left(U_1^*\right)U_1^* - pk\left(\beta\left(U_1^*\right)U_1^* + \mu\right) + \beta\left(U_1^*\right)U_1^* \\
&(\eta+\mu+k+\delta_2)\left[\beta'\left(U_1^*\right)U_1^*S^* + (p+\mu+\delta_1) - \frac{pk}{(\mu+k+\eta+\delta_2)}\right] \\
=& -\mu\beta'\left(U_1^*\right)U_1^*S^*(\eta+\mu+k+\delta_2) + p\eta\beta\left(U_1^*\right)U_1^* \\
&+ \beta\left(U_1^*\right)U_1^*(\eta+\mu+k+\delta_2)\left[(p+\mu+\delta_1) - \frac{pk}{(\mu+k+\eta+\delta_2)}\right],
\end{aligned}
$$

and

$$
\begin{aligned}
a_1a_2 - a_3 =& \left[\beta\left(U_1^*\right)U_1^* + \mu - \beta'\left(U_1^*\right)U_1^*S^* + \frac{pk}{(\mu+k+\eta+\delta_2)}\right. \\
&\left.+ (\eta+\mu+k+\delta_2)\right]\left[-\mu\beta'\left(U_1^*\right)U_1^*S^* + \frac{\mu pk}{(\mu+k+\eta+\delta_2)}\right. \\
&- \beta'\left(U_1^*\right)U_1^*S^*(\eta+\mu+k+\delta_2) + \beta\left(U_1^*\right)U_1^*(p+\mu+\delta_1) \\
&\left.+ \left(\beta\left(U_1^*\right)U_1^* + \mu\right)(\eta+\mu+k+\delta_2)\right] + \mu\beta'\left(U_1^*\right)U_1^*S \\
&(\eta+\mu+k+\delta_2) - p\eta\beta\left(U_1^*\right)U_1^* - \beta\left(U_1^*\right)U_1^* \\
&(\eta+\mu+k+\delta_2)\left[(p+\mu+\delta_1) - \frac{pk}{(\mu+k+\eta+\delta_2)}\right] \\
=& \left[\beta\left(U_1^*\right)U_1^* - \beta'\left(U_1^*\right)U_1^*S^* + (\eta+\mu+k+\delta_2)\right] \\
&\left[-\mu\beta'\left(U_1^*\right)U_1^*S^* - \beta'\left(U_1^*\right)U_1^*S^*(\eta+\mu+k+\delta_2)\right. \\
&\left.+ \frac{\mu pk}{(\mu+k+\eta+\delta_2)} + \left(\beta\left(U_1^*\right)U_1^* + \mu\right)(\eta+\mu+k+\delta_2)\right] \\
&+ \mu\left[-\mu\beta'\left(U_1^*\right)U_1^*S^* + \beta\left(U_1^*\right)U_1^*(p+\mu+\delta_1) + \frac{\mu pk}{(\mu+k+\eta+\delta_2)}\right. \\
&\left.+ \left(\beta\left(U_1^*\right)U_1^* + \mu\right)(\eta+\mu+k+\delta_2)\right] + \beta\left(U_1^*\right)U_1^*(p+\mu+\delta_1) \\
&\left[\beta\left(U_1^*\right)U_1^* - \beta'\left(U_1^*\right)U_1^*S^* + \frac{pk}{(\mu+k+\eta+\delta_2)}\right] + \beta\left(U_1^*\right)U_1^* \\
&(\eta+\mu+k+\delta_2)\frac{pk}{(\mu+k+\eta+\delta_2)} + \frac{pk}{(\mu+k+\eta+\delta_2)}\left[-\mu\beta'\left(U_1^*\right)U_1^*S^*\right. \\
&\left.- \beta'\left(U_1^*\right)U_1^*S^*(\eta+\mu+k+\delta_2) + \frac{\mu pk}{(\mu+k+\eta+\delta_2)}\right] \\
&+ p\beta\left(U_1^*\right)U_1^*(k-\eta) + \mu pk.
\end{aligned}
$$

We know

$$
\begin{aligned}
&\beta'\left(U_1^*\right) = -bm'\left(U_1^*\right) < 0, k > \eta, \\
&(p+\mu+\delta_1) - \tfrac{pk}{(\mu+k+\eta+\delta_2)} = \tfrac{(\mu+\delta_1)(\mu+k+\eta+\delta_2)+p(\mu+\eta+\delta_2)}{(\mu+k+\eta+\delta_2)} > 0.
\end{aligned}
$$

Thus, $a_1 > 0$, $a_2 > 0$, $a_3 > 0$, $a_1a_2 - a_3 > 0$. According to the Routh – Hurwitz criterion, we obtain E^* is locally asymptotically stable. This completes the proof of Theorem 5.1. □

Theorem 5.2. *If $R_0 > 1$, the heroin abuse equilibrium point E^* is globally asymptotically stable in the interior of Ω, provided that*

$$\sup\left\{S\left(\beta\left(U_1\right)U_1\right)' : S \geq 0, U_1 \geq 0, S + U_1 \leq N\right\} \leq \varepsilon.$$

Proof. The Jacobian matrix associated with the linearized system of (2.1) is

$$\mathbf{J} = \begin{bmatrix} -\beta\left(U_1\right)U_1 - \mu & -\beta'\left(U_1\right)U_1 S - \beta\left(U_1\right)S + \varepsilon & \eta \\ \beta\left(U_1\right)U_1 & \beta'\left(U_1\right)U_1 S + \beta\left(U_1\right)S - (p + \mu + \varepsilon + \delta_1) & k \\ 0 & p & -(\eta + \mu + k + \delta_2) \end{bmatrix},$$

and its second additive compound matrix is

$$\mathbf{J}^{[2]} = \begin{bmatrix} j_{11} & k & -\eta \\ p & j_{22} & -\beta'\left(U_1\right)U_1 S - \beta\left(U_1\right)S + \varepsilon \\ 0 & \beta\left(U_1\right)U_1 & j_{33} \end{bmatrix},$$

where

$$\begin{aligned} j_{11} &= -\beta\left(U_1\right)U_1 - \mu + \beta'\left(U_1\right)U_1 S + \beta\left(U_1\right)S - (p + \mu + \varepsilon + \delta_1), \\ j_{22} &= -\beta\left(U_1\right)U_1 - \mu - (\eta + \mu + k + \delta_2), \\ j_{33} &= \beta'\left(U_1\right)U_1 S + \beta\left(U_1\right)S - (p + \mu + \varepsilon + \delta_1) - (\eta + \mu + k + \delta_2). \end{aligned}$$

We now take $\mathbf{P} = diag\left[1, \frac{U_1}{U_2}, \frac{U_1}{U_2}\right]$, then $\mathbf{P}_f\mathbf{P}^{-1} = diag\left[0, \frac{U_1'}{U_1} - \frac{U_2'}{U_2}, \frac{U_1'}{U_1} - \frac{U_2'}{U_2}\right]$, where f denotes the vector field of (2.1). Thus, we have

$$\mathbf{P}\mathbf{J}^{[2]}\mathbf{P}^{-1} = \begin{bmatrix} j_{11} & k\frac{U_2}{U_1} & -\eta\frac{U_2}{U_1} \\ p\frac{U_1}{U_2} & j_{22} & -\beta'\left(U_1\right)U_1 S - \beta\left(U_1\right)S + \varepsilon \\ 0 & \beta\left(U_1\right)U_1 & j_{33} \end{bmatrix}.$$

So the matrix $\mathbf{Q} = \mathbf{P}_f\mathbf{P}^{-1} + \mathbf{P}\mathbf{J}^{[2]}\mathbf{P}^{-1}$ can be written in the following block form

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} \\ \mathbf{Q}_{21} & \mathbf{Q}_{22} \end{bmatrix},$$

where

$$\begin{aligned} \mathbf{Q}_{11} &= -\beta\left(U_1\right)U_1 - \mu + \beta'\left(U_1\right)U_1 S + \beta\left(U_1\right)S - (p + \mu + \varepsilon + \delta_1), \\ \mathbf{Q}_{12} &= \begin{bmatrix} k\frac{U_2}{U_1} & -\eta\frac{U_2}{U_1} \end{bmatrix}, \mathbf{Q}_{21} = \begin{bmatrix} p\frac{U_1}{U_2} \\ 0 \end{bmatrix}, \mathbf{Q}_{22} = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}, \end{aligned}$$

with

$$\begin{aligned} q_{11} &= -\beta\left(U_1\right)U_1 - \mu - (\eta + \mu + k + \delta_2) + \tfrac{U_1'}{U_1} - \tfrac{U_2'}{U_2}, \\ q_{12} &= -\beta'\left(U_1\right)U_1 S - \beta\left(U_1\right)S + \varepsilon, \\ q_{21} &= \beta\left(U_1\right)U_1, \\ q_{22} &= \beta'\left(U_1\right)U_1 S + \beta\left(U_1\right)S - (p + \mu + \varepsilon + \delta_1) - (\eta + \mu + k + \delta_2) + \tfrac{U_1'}{U_1} - \tfrac{U_2'}{U_2}. \end{aligned}$$

The vector norm $|\cdot|$ in $\mathbb{R}^3$ is chosen as $|(x_1, x_2, x_3)| = \max\{|x_1|, |x_2|, |x_3|\}$. One can verify that the Lozinskii measure $\mathcal{M}(Q)$ with respect to this norm can be estimated as

$$\mathcal{M}(Q) \leq \sup\{g_1, g_2\},$$

where

$$g_1 = \mathcal{M}_1(Q_{11}) + |Q_{12}|,$$
$$g_2 = |Q_{21}| + \mathcal{M}_1(Q_{22}).$$

Here $|Q_{12}|$ and $|Q_{21}|$ are matrix norms induced by the l_1 vector norm, $\mathcal{M}_1$ denotes the Lozinskii measure with respect to the l_1 vector norm (see [20]). For a generic matrix, we have $\mathbf{A} = (a_{ij})$, then

$$|\mathbf{A}| = \max_{1\leq k\leq n} \sum_{j=1}^{n} |a_{jk}|,$$

$$\mathcal{M}_1(\mathbf{A}) = \max_{1\leq k\leq n} \left(a_{kk} + \sum_{j=1(j\neq k)}^{n} |a_{jk}| \right).$$

If $S(\beta(U_1)U_1)' < \varepsilon$, we have $q_{12} = -\beta'(U_1)U_1S - \beta(U_1)S + \varepsilon = -S(\beta(U_1)U_1)' + \varepsilon > 0$, and hence $|q_{12}| = q_{12}$. More specifically,

$$\begin{aligned} g_1 =& Q_{11} + |Q_{12}| \\ =& -\beta(U_1)U_1 - \mu + \beta'(U_1)U_1S + \beta(U_1)S \\ & - (p + \mu + \varepsilon + \delta_1) + k\frac{U_2}{U_1}, \\ g_2 =& |Q_{21}| + \max\{q_{11} + |q_{21}|, |q_{12}| + q_{22}\} \\ \leq& p\frac{U_1}{U_2} + \max\left\{-\mu - (\eta + \mu + k + \delta_2) + \frac{U_1'}{U_1} - \frac{U_2'}{U_2}, \right. \\ & \left. - (p + \mu + \delta_1) - (\eta + \mu + k + \delta_2) + \frac{U_1'}{U_1} - \frac{U_2'}{U_2}\right\}. \end{aligned}$$

Since

$$U_1' = \beta(U_1)SU_1 + kU_2 - (p + \mu + \varepsilon + \delta_1)U_1,$$
$$U_2' = pU_1 - (\mu + k + \eta + \delta_2)U_2,$$

we have

$$-(p + \mu + \varepsilon + \delta_1) = \frac{U_1'}{U_1} - \frac{kU_2}{U_1} - \beta(U_1)S,$$
$$-(\mu + k + \eta + \delta_2) = \frac{U_2'}{U_2} - \frac{pU_1}{U_2}.$$

This leads to

$$\begin{aligned} g_1 &= Q_{11} + |Q_{12}| \\ &= -\beta(U_1)U_1 - \mu + \beta'(U_1)U_1S + \frac{U_1'}{U_1} \\ &\leq \frac{U_1'}{U_1} - \mu, \end{aligned} \tag{5.2}$$

and

$$\begin{aligned} g_2 &= |Q_{21}| + \max\left\{q_{11} + |q_{21}|, |q_{12}| + q_{22}\right\} \\ &\leq p\frac{U_1}{U_2} + \max\left\{-\mu - \frac{pU_1}{U_2} + \frac{U_1'}{U_1}, -(p+\mu+\delta_1) - \frac{pU_1}{U_2} + \frac{U_1'}{U_1}\right\} \\ &\leq \frac{U_1'}{U_1} - \mu. \end{aligned} \tag{5.3}$$

Thus, (5.2) and (5.3) yield

$$\mathcal{M}(Q) \leq \frac{U_1'}{U_1} - \mu.$$

It follows from $0 \leq U_1 \leq \frac{\Lambda}{\mu}$ that

$$\frac{\ln(U_1(t) - \ln U_1(0))}{t} \leq \frac{\mu}{2},$$

for t large sufficiently. We then obtain

$$\begin{aligned} \frac{1}{t}\int_0^t \mathcal{M}(Q)\,\mathrm{d}s &\leq \frac{1}{t}\int_0^t \left(\frac{U_1'}{U_1} - \mu\right)\mathrm{d}s \\ &= \frac{\ln(U_1(t)) - \ln(U_1(0))}{t} - \mu \\ &\leq -\frac{\mu}{2}, \end{aligned}$$

if t is large enough. According to Theorem D.1 in [14], we have come to the conclusion that E^* is globally asymptotically stable in the interior of Ω. This completes the proof of Theorem 5.2. □

6. Numerical simulation

In this section, some numerical results of model (2.1) are presented for supporting the analytic results obtained above. Some typical examples of $m(U_1)$ with such properties are $1 - c/(c + U_1^n)$ with $c > 0$ and $n > 0$, $1 - e^{-cU_1}$ with $c > 0$, and $\frac{\mu U_1}{\Lambda}$. Then we choose $m(U_1) = \frac{\mu U_1}{\Lambda}$, and $S(0) = 3$, $U_1(0) = 2$, $U_2(0) = 3$. Further numerical simulation of the model.

6.1 Numerical simulation

(1) The model parameters are taken as: $\Lambda = 0.6$, $a = 0.5$, $b = 0.2$, $\mu = 0.3$, $\varepsilon = 0.3$, $\eta = 0.2$, $k = 0.8$, $\delta_1 = 0.6$, $\delta_2 = 0.3$, $p = 0.6$, then $R_0 = 0.7222 < 1$. The solution curve of model (2.1) tends to be heroin-free abuse equilibrium point $E^0(1.97, 0.00, 0.00)$, Theorem 4.2 is verified (see Figure 1(a)).

(2) The model parameters are taken as: $\Lambda = 1.6$, $a = 0.6$, $b = 0.3$, $\mu = 0.3$, $\varepsilon = 0.3$, $\eta = 0.2$, $k = 0.8$, $\delta_1 = 0.6$, $\delta_2 = 0.3$, $p = 0.6$, then $R_0 = 1.944 > 1$. The solution curve of model (2.1) tends to be heroin abuse equilibrium point $E^*(1.96, 1.62, 0.61)$, Theorem 5.2 is verified (see Figure 1(b)).

(3) We take $b = 0$ re-simulate the data. Then $R_0 = 0.7222 < 1$, heroin-free abuse equilibrium point E^0 is the same as Figure 1(a)(see Figure 1(c)). What's more $R_0 = 1.944 > 1$, the solution curve of model (2.1) tends to be heroin abuse equilibrium point $E^*(1.67, 1.76, 0.66)$, (see Figure 1(d)). According to Figure 1, we can find that human behavior change alone cannot eliminate the heroin abuse, but can significantly reduce the heroin outbreak level and larger behavior change leads to increase heroin susceptible individuals and decrease the number of heroin addicts.

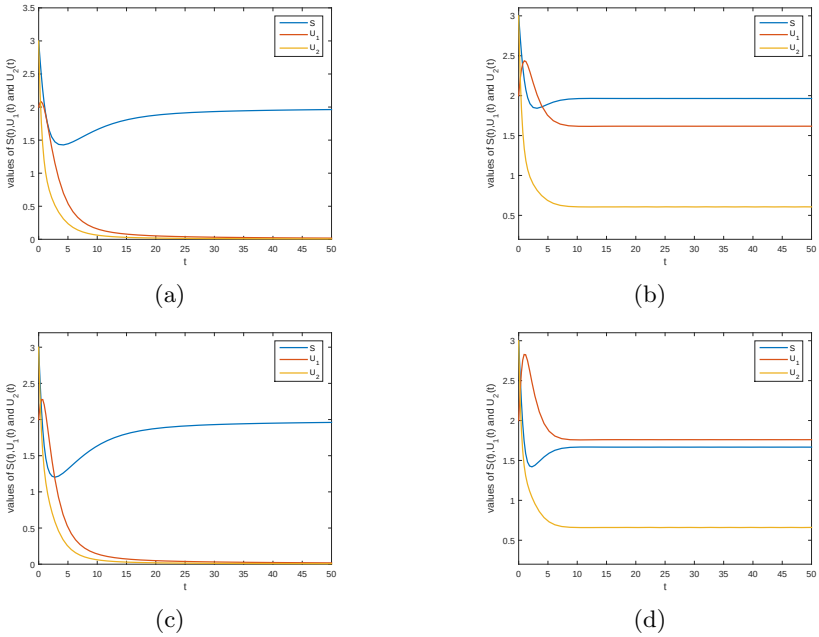

Figure 1. Illustration of time series of the solution of the model.

6.2 Sensitivity analysis of parameters

(1)The theoretical analysis shows that the basic reproduction number is the main factor affecting the epidemic of the disease. The basic reproduction number of the heroin transmission model established in this paper

$$R_0 = \frac{a\Lambda}{\mu\left(p+\mu+\delta_1+\varepsilon\right)} + \frac{pk}{\left(p+\mu+\delta_1+\varepsilon\right)\left(\mu+\delta_2+k+\eta\right)}.$$

In order to detect its dependence on parameters, based on the partial rank correlation coefficients (PRCCs) to make sensitivity analysis of parameters to determine the impact of parameters on basic reproduction number (see [21]). As can be seen from Figure 2, the proportion of heroin addicts receiving treatment in detoxification centers p has the greatest impact on R_0. The second is the contact rate a which normally without take into account human behavior. The third is the proportion of untreated heroin addicts recovering from their own illness ε. The fourth is the proportion of heroin addicts who relapse after cessation of treatment k.

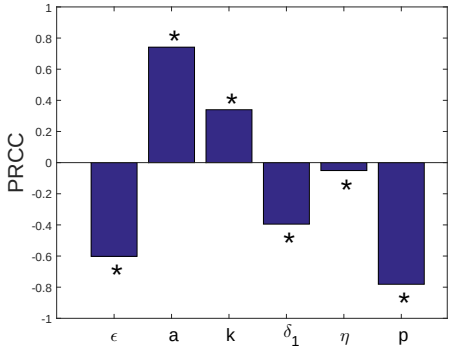

Figure 2. PRCCs results for the dependence of R_0 on each parameter.

(2)Figure 3(a) represents the effect of varying parameter a on the heroin user $U_1(t)$. Figure 3(b) represents the influence of parameter b on the value of $U_1(t)$. The effect of varying parameter p on the heroin user $U_1(t)$ as shown in Figure 3(c). It can be seen from Figure 3(d) that the effect of varying parameter ε on the value of $U_1(t)$. All other parameter values in each figure are the same as Figure 1(b).

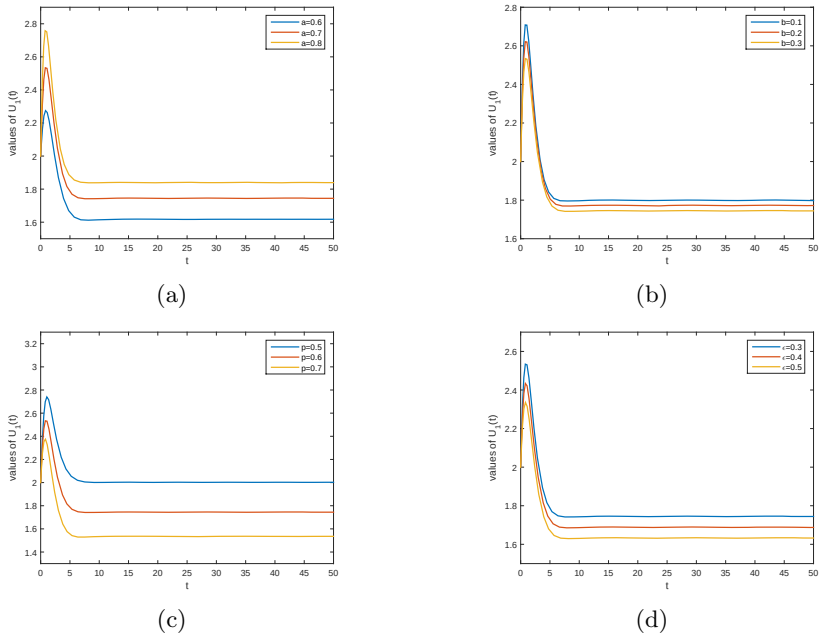

Figure 3. The effect of varying parameter's change on the heroin user $U_1(t)$.

In real life, some strategies are given to control the heroin transmission in this model.

(i)Fundamental in our assumption is that people are well informed of the development and severity of the heroin outbreak, made possible by the media coverage and reports from various resources, thus will do a good job of protective measures

for the susceptible people in order to reduce contact with other individuals, especially with those high-risk groups. Generally, it is to minimize the risk of heroin infection among susceptible people as far as possible, which decrease contact rate a and increase the maximum reduced contact rate b, so this result can be reflected from Figure 3(a) and Figure 3(b).

(ii)With the development of society, the medical facilities in drug rehabilitation centers have been improved to a certain extent, the family and social environment of heroin addicts has been improved. So that the heroin addicts can get better treatment, that is, we can enhance the rate of addicts receiving treatment in detoxification centers p to control the spread of heroin. This result can be reflected in Figure 3(c).

(iii)In the early stage of drug abuse, some heroin addicts will realize through themselves or the outside world that continuing to use drugs will lose their labor force, cause economic constraints and even lead to the separation of their wife and children. Then they will be strongly persuaded by their hearts, under the actively understand attitude by family members and the guidance of public opinion to detoxify themselves which has played a role in controlling the spread of heroin. That is, we can enhance the rate of untreated heroin addicts recovering from their own illness ε to reduce the transmission of heroin. This result can be reflected in Figure 3(d).

7. Discussion

In this paper, we focused on a deterministic system of heroin transmission with treatment. A goal of this article is to improve our quantitative understanding of the impact of human behavior on disease dynamics. Particularly, we incorporate human behavior into mathematical modeling of heroin abuse. Therefore, in our system, we consider not only the influence of human behavior on heroin transmission and the influence of heroin addict's self-healing and successful drug treatment on heroin transmission. The existence and stability of heroin-free abuse equilibrium and heroin abuse equilibrium were analyzed theoretically, and the theoretical results were verified by numerical simulation. Through partial rank correlation coefficients (PRCCs), the effects of different parameters on heroin transmission were obtained. Therefore, we can get a conclusion that increasing addicts receiving treatment in detoxification centers p, untreated heroin addicts recovering from their own illness ε and decreasing contact rate a which normally without take into account human behavior, heroin addicts who relapse after cessation of treatment k can control the spread of the heroin abuse. Hence, through science and technology information technology to increase people's awareness of heroin, so that people can understand the details of heroin transmission and the harmful of abuse to the body can inhibit the spread and outbreak of heroin. What's more, we can raise the level of medical care as soon as possible. Our results show that human behavior can reduce the number of heroin abuse. In practice, the movement of humans is not random but strongly affected by socioeconomic factors, age structure factors and gender differences factors. The model did not include such factors, which is the direction of our follow up study.

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