Journal of Nonlinear Modeling and Analysis
Volume 2, Number 2, June 2020, 161–171
http://jnma.ca; http://jnma.ijournal.cn
DOI:10.12150/jnma.2020.161

On the Number of Zeros of Abelian Integrals for a Class of Quadratic Reversible Centers of Genus One*

Lijun Hong[1], Junliang Lu[1] and Xiaochun Hong[1,†]

Abstract In this paper, using the method of Picard-Fuchs equation and Riccati equation, for a class of quadratic reversible centers of genus one, we research the upper bound of the number of zeros of Abelian integrals for the system $(r10)$ under arbitrary polynomial perturbations of degree n. Our main result is that the upper bound is $21n - 24$ $(n \geq 3)$, and the upper bound depends linearly on n.

Keywords Abelian integral, Quadratic reversible center, Weakened Hilbert's 16th problem, Limit cycle.

MSC(2010) 34C07, 34C08, 37G15.

1. Introduction and Main Results

We research the planar polynomial system

$$\dot{x} = \frac{H_y(x,y)}{\mu(x,y)} + \varepsilon f(x,y), \quad \dot{y} = -\frac{H_x(x,y)}{\mu(x,y)} + \varepsilon g(x,y), \tag{1.1}$$

where ε $(0 < \varepsilon \ll 1)$ is a real parameter, $H_y(x,y)/\mu(x,y)$, $H_x(x,y)/\mu(x,y)$, $f(x,y)$, $g(x,y)$ are all polynomials of x and y, with $\max\{\deg(f(x,y)), \deg(g(x,y))\} = n$, $\max\{\deg(H_y(x,y)/\mu(x,y)), \deg(H_x(x,y)/\mu(x,y))\} = m$. We suppose that when $\varepsilon = 0$, the system (1.1) is an integrable system, it has at least one center. The function $H(x,y)$ is a first integral with the integrating factor $\mu(x,y)$. That is, we can define a continuous family of periodic orbits

$$\{\Gamma_h\} \subset \left\{(x,y) \in \mathbb{R}^2 : H(x,y) = h, h \in \Delta\right\},$$

which are defined on a maximal open interval $\Delta = (h_1, h_2)$. The problem which needs to be solved in this paper is: for any small number ε, how many limit cycles in the system (1.1) can be bifurcated from the periodic orbits $\{\Gamma_h\}$. It is well known that in any compact region of the periodic orbits, the number of limit cycles of the system (1.1) is no more than the number of isolated zeros of the following Abelian

† the corresponding author.
Email address: lia_hong0920@163.com(L. Hong), wmb0@163.com(J. Lu), xchong@ynufe.edu.cn(X. Hong)
[1] School of Statistics and Mathematics, Yunnan University of Finance and Economics, Kunming, Yunnan 650221, China
* The authors were supported by National Natural Science Foundation of China (No. 11761075).

integrals $A(h)$,

$$A(h) = \oint_{\Gamma_h} \mu(x,y)\,[g(x,y)\,dx - f(x,y)\,dy]\,, \quad h \in \Delta. \tag{1.2}$$

A) If the $\mu(x,y)$ is constant, i.e., the system (1.1) is a Hamiltonian system when $\varepsilon = 0$, then the function $H(x,y)$ is a polynomial of x and y, with $\deg(H(x,y)) = m+1$. Finding an upper bound $Z(m,n)$ for the number of isolated zeros of A-belian integrals $A(h)$ is a significant and important problem, where the upper bound $Z(m,n)$ only depends on m, n, and does not depend on the concrete forms of $H(x,y)$, $f(x,y)$, and $g(x,y)$. It is called the weakened Hilbert's 16th problem by Arnold in [1]. This problem has been studied extensively, such as, for some specially planar systems, researchers obtain plentiful important results [2, 10, 13, 15], further, for some special three-dimensional differential systems, researchers obtain important results too [14], more details can be found in the review article [11] and the books [3, 5].

B) If the $\mu(x,y)$ is not constant, i.e., the system (1.1) is an integrable non-Hamiltonian system when $\varepsilon = 0$, researchers consider this problem by starting from the simplest case: m is low. For the specific case of $m = 2$, people conjecture that the upper bound of the number of zeros of Abelian integrals $A(h)$ depends linearly on n. Unfortunately, this conjecture is still far from being solved.

For quadratic reversible centers of genus one, Gautier et al. [4] showed that there are essentially 22 types, namely $(r1)$–$(r22)$. The linear dependence of case $(r1)$ was studied in [16]; cases $(r3)$–$(r6)$ were studied in [12]; cases $(r9)$, $(r13)$, $(r17)$, $(r19)$ were studied in [9]; cases $(r11)$, $(r16)$, $(r18)$, $(r20)$ were studied in [8]; cases $(r12)$, $(r21)$ were studied in [7]; and case $(r22)$ was studied in [6]. All of these upper bounds depend linearly on n. In this paper, we research the case $(r10)$, and obtain that its upper bound is $21n - 24$ $(n \geq 3)$. Our result shows that the upper bound depends linearly on n.

The form of the case $(r10)$ as follows:

$$(r10) \quad \dot{x} = -xy, \quad \dot{y} = -\frac{1}{3}y^2 + \frac{1}{3^2 \cdot 2^4}x - \frac{1}{3^2 \cdot 2^4}. \tag{1.3}$$

The (1.3) is an integrable non-Hamiltonian system. It has a center $(1,0)$, an integral curve $x = 0$, a family of periodic orbits $\{\Gamma_h\}$ $(1/2^5 < h < +\infty)$(see Figure 1), and a first integral as follows:

$$H(x,y) = x^{-\frac{2}{3}}\left(\frac{1}{2}y^2 + \frac{1}{3 \cdot 2^4}x + \frac{1}{3 \cdot 2^5}\right) = h, \quad h \in \left(\frac{1}{2^5}, +\infty\right), \tag{1.4}$$

with an integrating factor $\mu(x,y) = x^{-5/3}$.

In this paper, our main results include the following theorem.

Theorem 1.1. *If $f(x,y)$ and $g(x,y)$ are any polynomials of x and y, then the upper bound of the number of zeros of Abelian integrals $A(h)$ for the system $(r10)$ depends linearly on n. More concretely, the upper bound is $21n - 24$ for $n \geq 3$; the upper bound is 3 for $n = 1, 2$; and the upper bound is 0 for $n = 0$.*

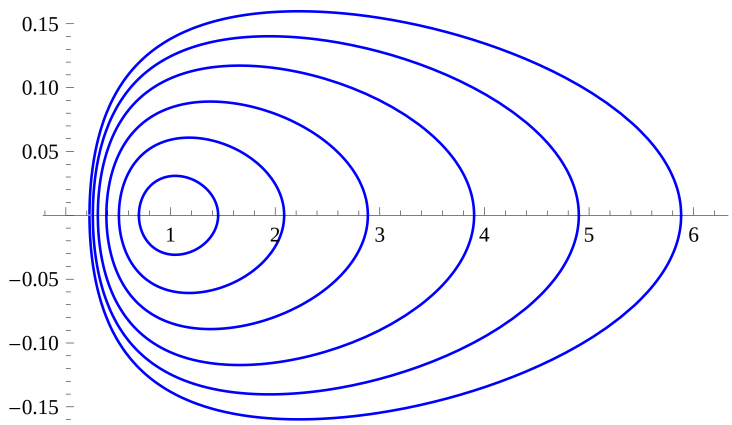

Figure 1. The periodic orbits of the system $(r10)$

The rest of this paper is organized as follows. In Section 2, we search a simple expression of Abelian integrals $A(h)$, prove the Proposition 2.1. In Section 3, we study relation among functions $J_s(h)$ and their derivatives $J_s'(h)$ for $s=-1/3,0,1/3$; relation among $J_s'(h)$ and the second derivatives $J_s''(h)$ for $s=-1/3,0,1/3$; relation between $J_{-1/3}'(h)$ and $J_{1/3}'(h)$, obtain two Picard-Fuchs equations and a Riccati equation. In Section 4, we consider relation between $A(h)$ and its derivative $A'(h)$, obtain a variable coefficient first order linear ordinary differential equation. At last, we prove the Theorem 1.1 using the method of Picard-Fuchs equation and Riccati equation. In Section 5, we give a short conclusion.

2. Simple Expression of Abelian Integrals $A(h)$

In this section, we give a simple expression of Abelian integrals $A(h)$, obtaining Proposition 2.1.

Suppose $f(x,y)=\sum_{0\le i+j\le n} a_{i,j}x^iy^j$, and $g(x,y)=\sum_{0\le i+j\le n} b_{i,j}x^iy^j$ are two arbitrary polynomials. From (1.2), Abelian integrals $A(h)$ in Theorem 1.1 have the form

$$A(h)=\oint_{\Gamma_h} x^{-\frac{5}{3}}\left(\sum_{0\le i+j\le n} b_{i,j}x^iy^j\,dx-\sum_{0\le i+j\le n} a_{i,j}x^iy^j\,dy\right),\quad h\in\left(\frac{1}{2^5},+\infty\right),$$

where $x^{-5/3}$ is an integrating factor.

For conciseness, we introduce function $I_{s,j}(h)$ as follows:

$$I_{s,j}(h)=\oint_{\Gamma_h} x^{s-\frac{5}{3}}y^j\,dx,$$

where $s=t/3$ $(t=-3,-2,-1,\cdots,3n-1,3n)$; $j=0,1,2,\cdots,n,n+1$. When $j=1$, we write $I_{s,1}(h)$ as $J_s(h)$.

Note that

$$\oint_{\Gamma_h} x^{i-\frac{5}{3}}y^j\,dy=\frac{\oint_{\Gamma_h} x^{i-\frac{5}{3}}dy^{j+1}}{j+1}=\frac{\frac{5}{3}-i}{j+1}\oint_{\Gamma_h} x^{i-\frac{5}{3}-1}y^{j+1}dx=\frac{\frac{5}{3}-i}{j+1}I_{i-1,j+1}(h).$$

Thus, the $A(h)$ can be written as

$$A(h)=\sum_{\substack{0\le i\le n,\\ 0\le j\le n,\\ 0\le i+j\le n}} b_{i,j}I_{i,j}(h)+\sum_{\substack{0\le i\le n,\\ 0\le j\le n,\\ 0\le i+j\le n}} a_{i,j}\frac{i-\frac{5}{3}}{j+1}I_{i-1,j+1}(h)=\sum_{\substack{-1\le i\le n,\\ 0\le j\le n+1,\\ 0\le i+j\le n}} \tilde{b}_{i,j}I_{i,j}(h). \tag{2.1}$$

where $\tilde{b}_{i,j}=b_{i,j}+(3i-2)/(3j)a_{i+1,j-1}$.

The following proposition gives a simple expression of Abelian integrals $A(h)$.

Proposition 2.1. *Abelian integrals $A(h)$ can be expressed as*

$$A(h)=\begin{cases}\alpha(h)J_{-\frac{1}{3}}(h)+\beta(h)J_0(h)+\gamma(h)J_{\frac{1}{3}}(h), & (n\ge 3),\\ \alpha(h)J_{-\frac{1}{3}}(h)+\beta(h)J_0(h), & (n=1,2),\\ \delta(h)J_{-1}(h), & (n=0),\end{cases} \tag{2.2}$$

where $\alpha(h)$, $\beta(h)$, $\gamma(h)$, and $\delta(h)$ are polynomials of h with $\deg(\alpha(h))\le 3n-5$, $\deg(\beta(h))\le 3n-6$, $\deg(\gamma(h))\le 3n-7$, for $n\ge 3$; $\alpha(h)=\bar{\alpha}h$, $\beta(h)=\bar{\beta}$, $\deg(\bar{\alpha})=0$, $\deg(\alpha(h))=1$, $\deg(\beta(h))=\deg(\bar{\beta})=0$, for $n=1,2$; $\deg(\delta(h))=0$, for $n=0$.

Proof. Since the periodic orbits Γ_h are symmetric about x-axis, thus, $I_{i,j}(h)=0$ as j is even, so, we only need to consider the case of j is odd.

From (1.4), we obtain

$$-\frac{1}{3}x^{-\frac{5}{3}}y^2+x^{-\frac{2}{3}}y\frac{\partial y}{\partial x}+\frac{2C}{3}x^{-\frac{2}{3}}-\frac{2C}{3}x^{-\frac{5}{3}}=0, \tag{2.3}$$

where $C=1/(3\cdot 2^5)$.

Multiplied (2.3) by $x^sy^{j-2}dx$ and integrated it over Γ_h, we have

$$\frac{3s+j-2}{j}I_{s,j}(h)=2C\left[I_{s+1,j-2}(h)-I_{s,j-2}(h)\right], \tag{2.4}$$

where $j=1,3,5,\cdots,2[n/2]+1$. We restrict $s=t/3$ $(t=-3,-2,-1,0,\cdots,3n-3)$, and $0\le s+j\le n$.

(i) For $3s+j-2\ne 0$, i.e., $(s,j)\ne(1/3,1),(-1/3,3),(-1,5)$, from (2.4), we have

$$I_{s,j}(h)=\frac{2jC}{3s+j-2}\left[I_{s+1,j-2}(h)-I_{s,j-2}(h)\right], \tag{2.5}$$

which indicates that $I_{s,j}(h)$ can be expressed in terms of $I_{s+1,j-2}(h)$ and $I_{s,j-2}(h)$. Then step by step, we obtain that $I_{s,j}(h)$ can be written as a linear combination of $J_i(h)(i=-1,0,\cdots)$ and $I_{-1,5}(h)$ with the form

$$I_{s,j}(h)=\begin{cases}J_s(h), & (j=1),\\ \sum\limits_{k=0}^{\frac{j-1}{2}}c_{(s,j),\,s+k}J_{s+k}(h), & (s\ge 0, j\ge 3, or\, s=-1\, and\, j=3),\\ I_{-1,5}(h), & (s=-1\, and\, j=5),\\ \sum\limits_{k=0}^{\frac{j-3}{2}}c_{(-1,j),\,k}J_k(h)+c_{-1,j}I_{-1,5}(h), & (s=-1\, and\, j\ge 7),\end{cases} \tag{2.6}$$

where $c_{(s,j),\,s+k}$ represents the coefficient obtained when $I_{s,j}(h)$ generates $J_{s+k}(h)$, $c_{-1,j}$ represents the coefficient obtained when $I_{-1,j}(h)$ generates $I_{-1,5}(h)$, and they are all real number.
(ii) For $3s+j-2=0$, i.e., $(s,j)=(1/3,1)\,,(-1/3,3)\,,(-1,5)$, from (2.4), when $(s,j)=(-1,5),(-1/3,3)$ respectively, we obtain

$$I_{-1,3}(h)=I_{0,3}(h), \tag{2.7}$$

$$J_{\frac{2}{3}}(h)=J_{-\frac{1}{3}}(h). \tag{2.8}$$

From (2.1) and (2.6), the $A(h)$ can be decomposed several parts as follows:

$$A(h)=A_0(h)+A_1(h)+A_2(h)+A_3(h)+A_4(h), \tag{2.9}$$

where

$A_0(h)=p_{-1,5}I_{-1,5}(h)$,

$A_1(h)=\sum_{s=-1}^{n-1}\tilde{b}_{s,1}J_s(h)$,

$A_2(h)=\sum\limits_{\substack{0\le s\le n,\\ 3\le j\le n,\\ 3\le s+j\le n}}\tilde{b}_{s,j}\sum_{k=0}^{(j-1)/2}c_{(s,j),s+k}J_{s+k}(h)$,

$A_3(h)=\tilde{b}_{-1,3}\sum\limits_{k=0}^{1}c_{(-1,3),k}J_{k-1}(h)=\tilde{b}_{-1,3}c_{(-1,3),0}J_{-1}(h)+\tilde{b}_{-1,3}c_{(-1,3),1}J_0(h)$,

$A_4(h)=\sum_{j=7}^{2[n/2]+1}\tilde{b}_{-1,j}\sum_{k=0}^{(j-3)/2}c_{(-1,j),k}J_k(h)$.

For $A_0(h)$, $p_{-1,5}=\tilde{b}_{-1,5}+\sum_{j=7}^{2[n/2]+1}\tilde{b}_{-1,j}c_{-1,j}$.

For $A_2(h)$, the maximum number of $s+k$ for $k=0,1,\cdots,(j-1)/2$ is $n-3+(3-1)/2=n-2$, and the minimum number is $0+0=0$, i.e., $\{s+k\}=\{0,1,\cdots,n-3,n-2\}$.

For $A_4(h)$, the maximum number of k for $k=0,1,\cdots,(j-3)/2$ is $[n/2]-1$, and the minimum number is 0, i.e., $\{k\}=\{0,1,\cdots,[n/2]-2,[n/2]-1\}$, and $[n/2]-1\le n/2-1\le n-1$ $(n\ge 0)$.

Suppose that $A_5(h):=A_1(h)+A_2(h)+A_3(h)+A_4(h)$, so, $A_5(h)$ is a linear combination of $J_{-1}(h)$, $J_0(h)$, $J_1(h)$, $\cdots$, $J_{n-1}(h)$. By substituting these formulae into $A(h)$, we obtain

$$A(h)=\begin{cases}\sum\limits_{k=-1}^{n-1}p_kJ_k(h)+p_{-1,5}I_{-1,5}(h), & (n\ge 4),\\ \sum\limits_{k=-1}^{n-1}p_kJ_k(h), & (0\le n\le 3),\end{cases} \tag{2.10}$$

where $p_{-1,5}$ and $p_k(k=-1,0,\cdots,n-1)$ are all real number.

Again, it follows from (1.4) that

$$\frac{1}{2}x^{-\frac{2}{3}}y^2+2Cx^{\frac{1}{3}}+Cx^{-\frac{2}{3}}=h. \tag{2.11}$$

Multiplied (2.11) by $x^{s-1}y^{j-2}dx$ and integrated it over Γ_h, we obtain

$$\frac{1}{2}I_{s,j}(h)=hI_{s+\frac{2}{3},j-2}(h)-2CI_{s+1,j-2}(h)-CI_{s,j-2}(h). \tag{2.12}$$

(i) For $3s+j-2\neq 0$, i.e., $(s,j)\neq(1/3,1),(-1/3,3),(-1,5)$, when $j=3$ $(s\neq -1/3)$, by (2.5), the equality (2.12) becomes

$$(6s+5)CJ_{s+1}(h)=(3s+1)hJ_{s+\frac{2}{3}}(h)-(3s-2)CJ_s(h),\ \ \left(s\neq -\tfrac{1}{3}\right). \tag{2.13}$$

A) For $s\geq 1$, we rewrite (2.13) as

$$J_s(h)=\frac{3s-2}{(6s-1)C}hJ_{s-\frac{1}{3}}(h)-\frac{3s-5}{6s-1}J_{s-1}(h), \tag{2.14}$$

which indicates that $J_s(h)$ can be expressed in terms of $hJ_{s-1/3}(h)$ and $J_{s-1}(h)$. Then step by step, we obtain that $J_s(h)$ can be written as a linear combination of $J_{-1/3}(h)$, $J_0(h)$, and $J_{1/3}(h)$ with polynomial coefficients of h.

$$J_s(h)=\alpha_s(h)J_{-\frac{1}{3}}(h)+\beta_s(h)J_0(h)+\gamma_s(h)J_{\frac{1}{3}}(h), \tag{2.15}$$

where $\alpha_s(h)$, $\beta_s(h)$, and $\gamma_s(h)$ are polynomials of h with $\deg(\alpha_s(h))\leq 3s-2$, $\deg(\beta_s(h))\leq 3s-3$, $\deg(\gamma_s(h))\leq 3s-4$, for $s\geq 2$; and $\deg(\alpha_s(h))\leq 1$, $\deg(\beta_s(h))=0$, $\gamma_s(h)=0$, for $s=1$.

B) For $s=0$, $J_s(h)$ can also be a linear combination of $J_{-1/3}(h)$, $J_0(h)$ and $J_{1/3}(h)$ as $J_0(h)=J_0(h)$.

C) For $s<0$, from (2.13), we get

$$J_s(h)=\frac{3s+1}{(3s-2)C}hJ_{s+\frac{2}{3}}(h)-\frac{6s+5}{3s-2}J_{s+1}(h),\left(s\neq -\frac{1}{3}\right). \tag{2.16}$$

From (2.16), we obtain

$$J_{-1}(h)=\frac{2}{5C}hJ_{-\frac{1}{3}}(h)-\frac{1}{5}J_0(h). \tag{2.17}$$

As a consequence, all $J_i(h)$ $(i=-1,0,1,\cdots,n-1)$ can be expressed in terms of $J_{-1/3}(h)$, $J_0(h)$ and $J_{1/3}(h)$.

(ii) For $3s+j-2=0$, i.e., $(s,j)=(1/3,1),(-1/3,3),(-1,5)$, moreover $I_{1/3,1}(h)=J_{1/3}(h)$, when $(s,j)=(-1/3,3)$, $(-1,5)$ respectively, from (2.12), we obtain

$$I_{-\frac{1}{3},3}(h)=2hJ_{\frac{1}{3}}(h)-4CJ_{\frac{2}{3}}(h)-2CJ_{-\frac{1}{3}}(h), \tag{2.18}$$

$$I_{-1,5}(h)=2hI_{-\frac{1}{3},3}(h)-4CI_{0,3}(h)-2CI_{-1,3}(h). \tag{2.19}$$

From (2.7), (2.8), (2.12), (2.18), (2.19), we obtain

$$I_{-\frac{1}{3},3}(h)=2hJ_{\frac{1}{3}}(h)-6CJ_{-\frac{1}{3}}(h),$$

$$I_{0,3}(h)=\frac{6}{5}hJ_{-\frac{1}{3}}(h)-\frac{18C}{5}J_0(h),$$

$$I_{-1,5}(h)=-\frac{1}{5}hJ_{-\frac{1}{3}}(h)+\frac{9C}{40}J_0(h)+4h^2J_{\frac{1}{3}}(h). \tag{2.20}$$

For $n\geq 3$, substituting formulae (2.15), (2.17), (2.20) into (2.10), we obtain

$$A(h)=\alpha(h)J_{-\frac{1}{3}}(h)+\beta(h)J_0(h)+\gamma(h)J_{\frac{1}{3}}(h),$$

where $\alpha(h)$, $\beta(h)$, $\gamma(h)$ are polynomials of h with $\deg(\alpha(h)) \le 3n-5$, $\deg(\beta(h)) \le 3n-6$, $\deg(\gamma(h)) \le 3n-7$.

For $n = 1, 2$, from (2.10), we obtain

$$A(h) = p_{-1}J_{-1}(h) + p_0 J_0(h) + p_1 J_1(h). \tag{2.21}$$

Substituting formulae (2.15) and (2.17) into (2.21), we obtain

$$A(h) = \alpha(h)J_{-\frac{1}{3}}(h) + \beta(h)J_0(h),$$

where $\alpha(h) = \bar{\alpha}h$, $\bar{\alpha} = (2p_{-1} + p_1)/(5C)$, $\beta(h) = \bar{\beta} = p_0 - p_{-1}/5 + 2p_1/5$, with $\deg(\alpha(h)) = 1$, $\deg(\bar{\alpha}) = 0$, $\deg(\beta(h)) = \deg(\bar{\beta}) = 0$.

For $n = 0$, from (2.10), we obtain $A(h) = \delta(h)J_{-1}(h)$, where $\delta(h) = p_{-1}$ with $\deg(\delta(h)) = 0$. □

3. Picard-Fuchs Equations and Riccati Equation

In this section, we give a relation among functions $J_s(h)$ and their derivatives $J_s'(h)$ for $s = -1/3, 0, 1/3$; a relation among functions $J_s'(h)$ and the second derivatives $J_s''(h)$ for $s = -1/3, 0, 1/3$; a relation between $J_{-1/3}'(h)$ and $J_{1/3}'(h)$, obtaining two lemmas and a corollary.

For the relation among functions $J_s(h)$ and their derivatives $J_s'(h)$ for $s = -1/3$, 0, 1/3, we obtain the following lemma.

Lemma 3.1. *The functions $J_s(h)$ for $s = -1/3, 0, 1/3$ satisfy the following Picard-Fuchs equation*

$$\begin{pmatrix} J_{-\frac{1}{3}}(h) \\ J_0(h) \\ J_{\frac{1}{3}}(h) \end{pmatrix} = \begin{pmatrix} \frac{2}{3}h & -2C & 0 \\ 0 & h & -3C \\ -6C & 0 & 2h \end{pmatrix} \begin{pmatrix} J'_{-\frac{1}{3}}(h) \\ J'_0(h) \\ J'_{\frac{1}{3}}(h) \end{pmatrix}. \tag{3.1}$$

Proof. By (1.4), we have $y^2 = 2hx^{2/3} - 4Cx - 2C$, $\partial y/\partial h = x^{2/3}/y$, $ydy = \left(2/3hx^{-1/3} - 2C\right)dx$. Since $J_s(h) = \oint_{\Gamma_h} x^{s-5/3}ydx$, $J_s'(h) = \oint_{\Gamma_h} x^{s-1}/ydx$. Thus

$$\begin{aligned} J_s(h) &= \oint_{\Gamma_h} \frac{x^{s-\frac{5}{3}}y^2}{y}dx = \oint_{\Gamma_h} \frac{x^{s-\frac{5}{3}}\left(2hx^{\frac{2}{3}} - 4Cx - 2C\right)}{y}dx \\ &= 2hJ_s'(h) - 4CJ'_{s+\frac{1}{3}}(h) - 2CJ'_{s-\frac{2}{3}}(h), \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} \left(s - \tfrac{2}{3}\right)J_s(h) &= \oint_{\Gamma_h}\left(s - \tfrac{2}{3}\right)x^{s-\frac{5}{3}}ydx = \oint_{\Gamma_h} ydx^{s-\frac{2}{3}} \\ &= -\oint_{\Gamma_h} \frac{x^{s-\frac{2}{3}}\left(\frac{2}{3}hx^{-\frac{1}{3}} - 2C\right)}{y}dx = -\tfrac{2}{3}hJ_s'(h) + 2CJ'_{s+\frac{1}{3}}(h). \end{aligned} \tag{3.3}$$

From (3.2) and (3.3), we have

$$(3s-2)J_s(h) = -2hJ_s'(h) + 6CJ'_{s+\frac{1}{3}}(h), \tag{3.4}$$

$$(6s-1)J_s(h) = 2hJ_s'(h) - 6CJ'_{s-\frac{2}{3}}(h). \tag{3.5}$$

From (3.4) and (3.5), let $s=-1/3,0,1/3$ respectively, we obtain

$$3J_{-\frac{1}{3}}(h) = 2hJ'_{-\frac{1}{3}}(h) - 6CJ'_0(h), \tag{3.6}$$

$$J_0(h) = hJ'_0(h) - 3CJ'_{\frac{1}{3}}(h), \tag{3.7}$$

$$J_{\frac{1}{3}}(h) = 2hJ'_{\frac{1}{3}}(h) - 6CJ'_{-\frac{1}{3}}(h). \tag{3.8}$$

From (3.6)–(3.8), we obtain (3.1). □

For the relation among functions $J'_s(h)$ and the second derivatives $J''_s(h)$ for $s=-1/3,0,1/3$, we get the following lemma.

Lemma 3.2. *The functions $J'_s(h)$ for $s=-1/3,\ 0,\ 1/3$ satisfy the following Picard-Fuchs equation*

$$\begin{pmatrix} J''_{-\frac{1}{3}}(h) \\ J''_0(h) \\ J''_{\frac{1}{3}}(h) \end{pmatrix} = \frac{1}{B(h)} \begin{pmatrix} h^2 & -9C^2 \\ 9C^2 & -3Ch \\ 3Ch & -h^2 \end{pmatrix} \begin{pmatrix} J'_{-\frac{1}{3}}(h) \\ J'_{\frac{1}{3}}(h) \end{pmatrix}, \tag{3.9}$$

where $B(h)=2\left(h-1/2^5\right)\left(h^2+1/2^5h+1/2^{10}\right)$.

Proof. By differentiating both sides of (3.1) with respect to h, we have

$$J'_{-\frac{1}{3}}(h) = 2hJ''_{-\frac{1}{3}}(h) - 6CJ''_0(h), \tag{3.10}$$

$$0\cdot J'_0(h) = -hJ''_0(h) + 3CJ''_{\frac{1}{3}}(h), \tag{3.11}$$

$$J'_{\frac{1}{3}}(h) = 6CJ''_{-\frac{1}{3}}(h) - 2hJ''_{\frac{1}{3}}(h). \tag{3.12}$$

From (3.10)−(3.12) , we obtain

$$\begin{pmatrix} J''_{-\frac{1}{3}}(h) \\ J''_0(h) \\ J''_{\frac{1}{3}}(h) \end{pmatrix} = \begin{pmatrix} 2h & -6C & 0 \\ 0 & -h & 3C \\ 6C & 0 & -2h \end{pmatrix}^{-1} \begin{pmatrix} J'_{-\frac{1}{3}}(h) \\ 0\cdot J'_0(h) \\ J'_{\frac{1}{3}}(h) \end{pmatrix}. \tag{3.13}$$

From (3.13), we obtain (3.9). □

Since $J_s(h)=\oint_{\Gamma_h} x^{s-5/3}ydx$, and $J'_s(h)=\oint_{\Gamma_h} x^{s-1}/ydx$, after some simple calculations, we obtain the following lemma.

Lemma 3.3. $J_s\left(1/2^5\right)=0$ $(s=-1/3,0,1/3)$, $J'_s(h)>0$ $(s=-1/3,0,1/3)$ *and* $J_{-1}(h)<0$, *when* $h\in\left(1/2^5,+\infty\right)$.

For the relation of $J'_{-1/3}(h)$ and $J'_{1/3}(h)$, assume that $U(h):=J'_{-1/3}(h)/J'_{1/3}(h)$, we have the following corollary.

Corollary 3.1. *The function $U(h)$ satisfies the following Riccati equation*

$$B(h)U'(h) = -3ChU^2(h) + 2h^2U(h) - 9C^2, \tag{3.14}$$

where $B(h)=2\left(h-1/2^5\right)\left(h^2+1/2^5h+1/2^{10}\right)$.

Proof. Using (3.9), and differentiated both sides of $U(h)$ with respect to h, we obtain (3.14). □

4. The Upper Bound of the Number of Zeros

In this section, we give a relation between $A(h)$ and its derivative $A'(h)$. Finally, we prove Theorem 1.1.

For $n \geq 1$, by (2.2) and (3.1), we have

$$A(h) = \alpha_1(h)J'_{-\frac{1}{3}}(h) + \beta_1(h)J'_0(h) + \gamma_1(h)J'_{\frac{1}{3}}(h), \tag{4.1}$$

where $\alpha_1(h) = 2h\alpha(h)/3 - 6C\gamma(h)$, $\beta_1(h) = h\beta(h) - 2C\alpha(h)$, $\gamma_1(h) = 2h\gamma(h) - 3C\beta(h)$. Thus, $\deg(\alpha_1(h)) \leq 3n-4$, $\deg(\beta_1(h)) \leq 3n-5$, $\deg(\gamma_1(h)) \leq 3n-6$, for $n \geq 3$; $\alpha_1(h) = 2\bar{\alpha}h^2/3$, $\beta_1(h) = Qh$, $\gamma_1(h) = -3C\bar{\beta}$, $Q = \bar{\beta} - 2C\bar{\alpha}$, for $n = 1, 2$.

Differentiated both sides of (4.1) with respect to h and using (3.9), we obtain

$$B(h)A'(h) = \alpha_2(h)J'_{-\frac{1}{3}}(h) + B(h)\beta'_1(h)J'_0(h) + \gamma_2(h)J'_{\frac{1}{3}}(h), \tag{4.2}$$

where $\alpha_2(h) = B(h)\alpha'_1(h) + h^2\alpha_1(h) + 9C^2\beta_1(h) + 3Ch\gamma_1(h)$, $\gamma_2(h) = B(h)\gamma'_1(h) - h^2\gamma_1(h) - 3Ch\beta_1(h) - 9C^2\alpha_1(h)$. So, $\deg(\alpha_2(h)) \leq 3n-2$, $\deg(\gamma_2(h)) \leq 3n-4$, for $n \geq 3$; $\alpha_2(h) = 5\bar{\alpha}hB(h)/3$, $\gamma_2(h) = 0$, for $n = 1, 2$.

According to (4.1) and (4.2), we have

$$B(h)\beta_1(h)A'(h) = B(h)\beta'_1(h)A(h) + D(h), \tag{4.3}$$

$$D(h) = E(h)J'_{-\frac{1}{3}}(h) + F(h)J'_{\frac{1}{3}}(h), \tag{4.4}$$

where $E(h) = \beta_1(h)\alpha_2(h) - B(h)\alpha_1(h)\beta'_1(h)$, $F(h) = \beta_1(h)\gamma_2(h) - B(h)\beta'_1(h)\gamma_1(h)$. Thus, $\deg(E(h)) \leq 6n-7$, $\deg(F(h)) \leq 6n-9$, for $n \geq 3$; $E(h) = \bar{\alpha}Qh^2B(h)$, $F(h) = 3CQ\bar{\beta}B(h)$, for $n = 1, 2$.

Assume that $V(h) := D(h)/J'_{1/3}(h)$, from the equality (4.4) and Corollary 3.1, after some simple calculations, we obtain the following lemma.

Lemma 4.1. *For $n \geq 1$, the function $V(h)$ satisfies the following Riccati equation*

$$B(h)E(h)V'(h) = -3ChV^2(h) + I(h)V(h) + G(h), \tag{4.5}$$

where $G(h) = B(h)E(h)F'(h) - B(h)E'(h)F(h) - 2h^2E(h)F(h) - 9C^2E^2(h) - 3ChF^2(h)$, $I(h) = B(h)E'(h) + 2h^2E(h) + 6ChF(h)$. Thus, $\deg(I(h)) \leq 6n-5$, $\deg(G(h)) \leq 12n-14$, for $n \geq 3$; $G(h) = -9CQ^2hB^2(h)(Sh^3+T)$, $S = \bar{\alpha}(2\bar{\beta}+C\bar{\alpha})$, $T = 3C^2\bar{\beta}(\bar{\beta} - 36C\bar{\alpha})$, for $n = 1, 2$.

We use $\sharp A(h)$ to denote the number of zeros of Abelian integrals $A(h)$ in Δ, and we need the following lemma.

Lemma 4.2 (Lemma 5.3, [12]). *The smooth functions $W(h)$, $\phi(h)$, $\psi(h)$, $\xi(h)$, and $\eta(h)$ satisfy the following Riccati equation*

$$\eta(h)W'(h) = \phi(h)W^2(h) + \psi(h)W(h) + \xi(h),$$

then

$$\sharp W(h) \leq \sharp\eta(h) + \sharp\xi(h) + 1.$$

Lemma 4.2 is the Lemma 5.3 in [12], and the proof can be found in [12], so it is omitted.

Finally, we complete the proof of Theorem 1.1.

Proof. Using Lemma 4.2, from the equality (4.3), we have

$$\sharp A(h) \leq \sharp B(h) + \sharp\beta_1(h) + \sharp D(h) + 1.$$

Using Lemma 4.1, we obtain

$$\sharp D(h) = \sharp V(h) \leq \sharp B(h) + \sharp E(h) + \sharp G(h) + 1,$$

thus,

$$\sharp A(h) \leq 2\sharp B(h) + \sharp\beta_1(h) + \sharp E(h) + \sharp G(h) + 2.$$

For $n \geq 3$, since $\deg(\beta_1(h)) \leq 3n-5$, $\deg(E(h)) \leq 6n-7$, and $\deg(G(h)) \leq 12n-14$, noticing that $B(h) = 2\left(h - 1/2^5\right)\left(h^2 + 1/2^5 h + 1/2^{10}\right)$ and there is no zero when $h \in \left(1/2^5, +\infty\right)$, we obtain

$$\sharp A(h) \leq (3n-5) + (6n-7) + (12n-14) + 2 = 21n - 24.$$

For $n = 1, 2$, since $1/2^5 < h < +\infty$, so $\sharp\beta_1(h) = \sharp(Qh) = 0$, $\sharp E(h) = \sharp[\bar{\alpha} Q h^2 B(h)] = 0$, $\sharp G(h) = \sharp[-9CQ^2 h B^2(h)(Sh^3 + T)] = \sharp(Sh^3 + T) = 1$, noticing that $Sh^3 + T$ and there is at most one zero when $h \in \left(1/2^5, +\infty\right)$, we obtain

$$\sharp A(h) \leq 0 + 0 + 0 + 1 + 2 = 3.$$

For $n = 0$, using Proposition 2.1, since $A(h) = \delta(h) J_{-1}(h)$, and $\deg(\delta(h)) = 0$, $J_{-1}(h) < 0$, we have $\sharp A(h) = 0$. □

5. Conclusion

In this paper, we research the upper bound of the number of zeros of Abelian integrals for the quadratic reversible system $(r10)$ under arbitrary polynomial perturbations of degree n, according to the method of Picard-Fuchs equation and Riccati equation. At the same time, we prove that the upper bound is $21n - 24$ ($n \geq 3$). Our result shows that the upper bound depends linearly on n.

Acknowledgements

The authors would like to thank Editor-in-Chief Professor Maoan Han for his very valuable modifications and suggestions.

References

[1] V.I. Arnold, *Ten problems*, Adv. Sov. Math., 1990, 1, 1–8.

[2] A. Boulfoul, A. Makhlouf and N. Mellahi, *On the limit cycles for a class of generalized Kukles differential systems*, J. Appl. Anal. Comput., 2019, 9(3), 864–883.

[3] C. Christopher and C. Li, *Limit cycles of differential equations*, Birkhäuser Verlag, Basel, 2007.

[4] S. Gautier, L. Gavrilov and I.D. Iliev, *Perturbations of quadratic centers of genus one*, Disc. Contin. Dyn. Syst., 2009, 25(2), 511–535.

[5] M. Han, *Bifurcation theory of limit cycles*, Science Press, Beijing, 2013.

[6] L. Hong, X. Hong and J. Lu, *A linear estimation for the number of zeros for Abelian integrals in a kind of quadratic reversible centers of genus one*, J. Appl. Anal. Comput., 2019, Online First.

[7] X. Hong, J. Lu and Y. Wang, *Upper bounds for the associated number of zeros of Abelian integrals for two classes of quadratic reversible centers of genus one*, J. Appl. Anal. Comput., 2018, 8(6), 1959–1970.

[8] X. Hong, S. Xie and L. Chen, *Estimating the number of zeros for Abelian integrals of quadratic reversible centers with orbits formed by higher-order curves*, Int. J. Bifurcation and Chaos, 2016, 26(2), 1650020-1–16.

[9] X. Hong, S. Xie and R. Ma, *On the Abelian integrals of quadratic reversible centers with orbits formed by genus one curves of higher degree*, J. Math. Anal. Appl., 2015, 429(2), 924–941.

[10] B. Huang and W. Niu, *Limit cycles for two classes of planar polynomial differential systems with uniform isochronous centers*, J. Appl. Anal. Comput., 2019, 9(3), 943–961.

[11] J. Li, *Hilbert's 16th problem and bifurcations of planar polynomial vector fields*, Int. J. Bifurcation and Chaos, 2003, 13(1), 47–106.

[12] W. Li, Y. Zhao, C. Li and Z. Zhang, *Abelian integrals for quadratic centres having almost all their orbits formed by quartics*, Nonlinearity, 2002, 15(3), 863–885.

[13] S. Sui and B. Li, *Bifurcation of limit cycles from the global center of a class of integrable non-Hamilton systems*, J. Appl. Anal. Comput., 2018, 8(5), 1441–1451.

[14] P. Yu, M. Han and Y. Bai, *Dynamics and bifurcation study on an extended Lorenz system*, Journal of Nonlinear Modeling and Analysis, 2019, 1(1), 107–128.

[15] L. Zhao, M. Qi and C. Liu, *The cyclicity of period annuli of a class of quintic Hamiltonian systems*, J. Math. Anal. Appl., 2013, 403(2), 391–407.

[16] Y. Zhao, W. Li, C. Li and Z. Zhang, *Linear estimate of the number of zeros of Abelian integrals for quadratic centers having almost all their orbits formed by cubics*, Sci. China Ser. A, 2002, 45(8), 964–974.