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Bogdanov-Takens Bifurcation in a Host-parasitoid Model

Chuang Xiang[1], Ming Lu[1] and Jicai Huang[1,†]

Abstract In this paper, we study a host-parasitoid model with Holling II Functional response, where we focus on a special case: the carrying capacity K_2 for parasitoids is equal to a critical value $\frac{r_1}{\eta}$. It is shown that the model can undergo Bogdanov-Takens bifurcation. The approximate expressions for saddle-node, Homoclinic and Hopf bifurcation curves are calculated. Numerical simulations, including bifurcation diagrams and corresponding phase portraits, are also given to illustrate the theoretical results.

Keywords Host-parasitoid model, Holling II functional response, Bogdanov-Takens bifurcation.

MSC(2010) 34C23, 34C25.

1. Introduction

Following the pioneering work of Fisher [2], many mathematical models have been proposed to describe and stop or reverse biological invasions ([5,7,8,11]). Predators are able to stop or reverse invasions, as shown first by Owen and Lewis [9], and a slowdown of invasions can be obtained if the functional response is linear (Type I).

In order to study the invasion and biological control of leaf-mining microlepidopteron, which attacks horse chestnut trees in Europe. Magal *et al.* [6] developed a host-parasitoid model with Holling II Functional response as follows

$$\begin{aligned} \dot{u} &= r_1 u(1-\frac{u}{K_1}) - \frac{\eta u v}{1+\eta h u}, \\ \dot{v} &= r_2 v(1-\frac{v}{K_2}) + \frac{\gamma \eta u v}{1+\eta h u}, \end{aligned} \tag{1.1}$$

where $u(t)$ and $v(t)$ denote densities of the hosts (leafminers *microlepidopteron*) and generalist parasitoids (*Minotetrastichus frontalis*) at time t, respectively. r_1 and r_2 represent the intrinsic growth rate of the hosts and parasitoids, respectively, K_1 and K_2 represent the carrying capacity of the hosts population and parasitoids population, respectively. η is the encounter rate of hosts and parasitoids, γ is the conversion rate of parasitoids, h describes the harvesting time. r_i, $K_i (i=1,2)$, γ, η, h are all positive constants. Magal *et al.* [6] analyzed the number and stability of equilibria in system (1.1), and showed some complex dynamical behaviors, such as

[†]the corresponding author.
Email address: hjc@mail.ccnu.edu.cn(J. Huang)
[1]School of Mathematics and Statistics, Central China Normal University, Wuhan, Hubei 430079, China
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the existence of a cusp, an unstable limit cycle and a homoclinic loop, by numerical simulations. However, the complex nonlinear dynamics and bifurcation phenomena still remain unknown, which is the subject of this paper.

For simplicity, we first nondimensionalize system (1.1) with the following scaling

$$\bar{t} = r_1 t, \quad \bar{x} = \frac{u}{K_1}, \quad \bar{y} = \frac{r_2 v}{r_1 K_2},$$

dropping the bars, model (1.1) becomes

$$\begin{aligned} \dot{u} = x &= x(1 - x - \frac{by}{a+x}), \\ \dot{v} &= y(\delta - y + \frac{cx}{a+x}), \end{aligned} \tag{1.2}$$

where

$$a = \frac{1}{K_1 \eta h}, \quad b = \frac{K_2}{K_1 r_2 h}, c = \frac{\gamma}{r_1 h}, \quad \delta = \frac{r_2}{r_1},$$

and a, b, c, δ are all positive. In this paper, we focus on a special case of system (1.2): $\delta = \frac{a}{b}$, i.e., the carrying capacity K_2 for parasitoids is equal to a critical value $\frac{r_1}{\eta}$. It is shown that the model can undergo Bogdanov-Takens bifurcation. The approximate expressions about saddle-node, Homoclinic and Hopf bifurcation curves are calculated. Numerical simulations, including bifurcation diagrams and corresponding phase portraits, are also given to illustrate the results.

This paper is organized as follows. In section 2, we analyse the existence and types of equilibria in model (1.2) when $\delta = \frac{a}{b}$. In section 3, we show the existence of Bogdanov-Takens bifurcation, and some numerical simulations are also given to illustrate the theoretical results. The paper ends with a brief discussion.

2. Equilibria and their types

By the biological implication, we only consider system (1.2) in $\mathbb{R}_+^2 = \{(x, y)|x \geqslant 0, y \geqslant 0\}$. It is easy to see that the positive invariant and bounded region of system (1.2) is

$$\Omega = \{(x, y)|0 \leq x \leq 1, 0 \leq y \leq \delta + \frac{c}{a+1}\}.$$

It is easy to see that system (1.2) always has three boundary equilibria $(0, 0)$, $(1, 0)$ and $(0, \delta)$ for all permissible parameters. The Jacobian matrix of system (1.2) at any equilibrium $E(x, y)$ of system (1.2) takes the following form

$$J(E) = \begin{pmatrix} 1 - 2x - \frac{aby}{(a+x)^2} & -\frac{bx}{a+x} \\ \frac{acy}{(a+x)^2} & \frac{a}{b} - 2y + \frac{cx}{a+x} \end{pmatrix},$$

and

$$\mathrm{Det}(J(E)) = (1 - 2x - \frac{aby}{(a+x)^2})(\frac{a}{b} - 2y + \frac{cx}{a+x}) + \frac{abcxy}{(a+x)^3},$$

$$\mathrm{Tr}(J(E)) = 1 + \frac{a}{b} - 2(x+y) + \frac{cx}{a+x} - \frac{aby}{(a+x)^2}.$$

It implies that $E(x, y)$ is an elementary equilibrium if $\mathrm{Det}(J(E)) \neq 0$, a hyperbolic saddle if $\mathrm{Det}(J(E)) < 0$, or a degenerate equilibrium if $\mathrm{Det}(J(E)) = 0$, respectively.

We can easily get the following results about the types of the boundary equilibria.

Lemma 2.1. *System* (1.2) *always has three boundary equilibria* $(0,0)$, $(1,0)$ *and* $(0,\delta)$. $(0,0)$ *is always a hyperbolic unstable node,* $(1,0)$ *is always a hyperbolic saddle.* $(0,\delta)$ *is a hyperbolic saddle if* $\delta < \frac{a}{b}$, *a hyperbolic stable node if* $\delta > \frac{a}{b}$, *a degenerate equilibrium if* $\delta = \frac{a}{b}$.

Next we consider the positive equilibria of system (1.2). The positive equilibrium $E(x,y)$ satisfies the following equation:

$$\begin{aligned} 1 - x - \frac{by}{a+x} &= 0, \\ \delta - y + \frac{cx}{a+x} &= 0, \end{aligned} \tag{2.1}$$

from which we can see that x is a root of the following equation:

$$x^3 + (2a-1)x^2 + (a^2 - 2a + bc + b\delta)x + ab\delta - a^2 = 0, \tag{2.2}$$

in the interval $(0,1)$. By the root formula of three-order algebraic equation, we can see that equation (2.2) may have three different real roots, which makes the qualitative and bifurcation analysis very difficult, thus, throughout this paper, we just consider a special case as follows:

$$\delta = \frac{a}{b} \;\; (i.e., K_2 = \frac{r_1}{\eta}). \tag{2.3}$$

We firstly define

$$\begin{aligned} F(x) &= x^2 + (2a-1)x + a^2 - a + bc, \\ f(x) &= \frac{dF(x)}{dx} = 2x + 2a - 1. \end{aligned} \tag{2.4}$$

It is easy to see that $F(x) = 0$ has at most two positive roots in the interval $(0,1)$, correspondingly, system (1.2) has at most two positive equilibria when (2.3) is satisfied.

From $F(x) = 0$, we can get

$$c = \frac{(a+x)(1-a-x)}{b}, \tag{2.5}$$

substituting (2.5) into $\mathrm{Det}(J(E))$, we can rewrite $\mathrm{Det}(J(E))$ as follows

$$\mathrm{Det}(J(E)) = \frac{x(1-x)}{b(a+x)} f(x). \tag{2.6}$$

We have the following results about the positive equilibria.

Lemma 2.2. *If* $\delta = \frac{a}{b}$, *then system* (1.2) *has at most two positive equilibria. Moreover,*

(I) system (1.2) *has no positive equilibrium if one of the following conditions is satisfied:*

(i) $c > \frac{1}{4b}$;

(ii) $c = \frac{1}{4b}$ *and* $a \geq \frac{1}{2}$;

(iii) $\frac{a(1-a)}{b} \le c < \frac{1}{4b}$, $a > \frac{1}{2}$.

(II) *if* $\frac{a(1-a)}{b} < c < \frac{1}{4b}$ *and* $0 < a < \frac{1}{2}$, *then system* (1.2) *has two positive equilibria* $E_1(x_1, y_1)$ *and* $E_2(x_2, y_2)$, *which are all elementary equilibria and* E_1 *is a hyperbolic saddle, where* $0 < x_1 < x_2 < 1$.

(III) *if* $c = \frac{a(1-a)}{b}$ *and* $0 < a < \frac{1}{2}$, *or* $0 < c \le \frac{a(1-a)}{b}$ *and* $0 < a < 1$, *then system* (1.2) *has a unique positive equilibrium* $E_2(x_2, y_2)$ *which is an elementary and anti-saddle equilibrium, where* $0 < x_2 < 1$.

(IV) *if* $c = \frac{1}{4b}$ *and* $0 < a < \frac{1}{2}$, *then system* (1.2) *has a unique positive equilibrium* E^* *which is degenerate.*

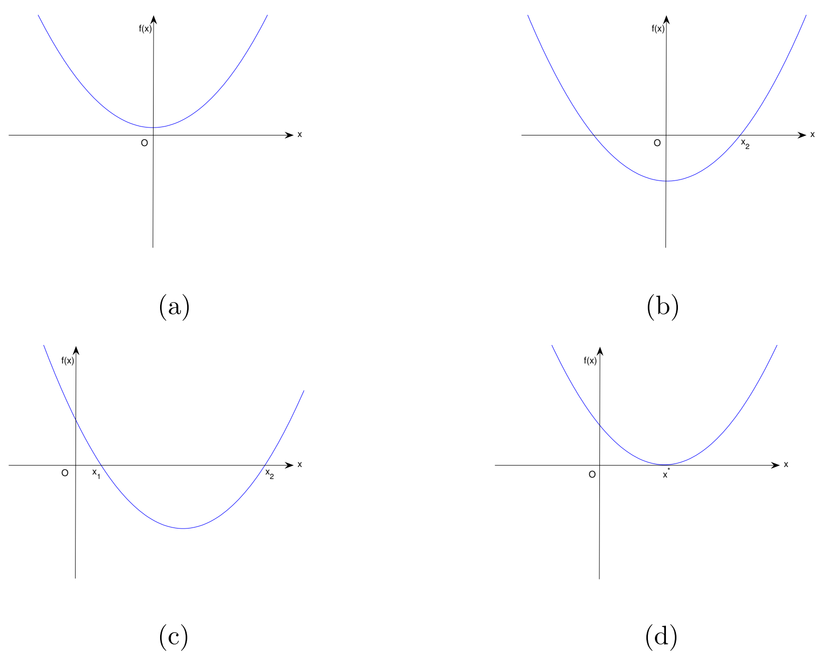

Figure 1. The positive roots of $F(x) = 0$. (a) No positive root; (b) A unique positive single root x_2; (c) Two different positive single roots x_1 and x_2; (d) A double positive root x^*.

Proof. From equation (2.6), it is easy to see that $\text{Det}(J(E_2)) > 0$, $\text{Det}(J(E_1)) < 0$, $\text{Det}(J(E^*)) = 0$, then E_1, E_2 are all elementary equilibria and E_1 is a hyperbolic saddle, E^* is a degenerate equilibrium. □

Next we consider the case (IV) in Lemma 2.2, and look for some parameter values such that the unique degenerate equilibrium $E^*(x^*, y^*)$ satisfied $\text{Tr}(J(E^*)) = 0$. From $F(x^*) = f(x^*) = 0$, we can express c, x^* and y^* by a and b as follows:

$$c = \frac{1}{4b}, \quad x^* = \frac{1-2a}{2}, \quad y^* = \frac{1+2a}{4b}. \tag{2.7}$$

Moreover, from $\text{Tr}(J(E^*)) = 0$ and (2.7), we have

$$b = \frac{1+2a}{4a(1-2a)}. \tag{2.8}$$

Defining the unique positive root of $8a^3 + 4a^2 - 1 = 0$ as follows

$$a_0 \triangleq \frac{(100 + 12\sqrt{69})^{\frac{1}{3}} + (100 - 12\sqrt{69})^{\frac{1}{3}} - 2}{12} \approx 0.377439, \tag{2.9}$$

we have the following results.

Theorem 2.1. *If $\delta = \frac{a}{b}$, $0 < a < \frac{1}{2}$ and the conditions in (2.7) are satisfied, then system (1.2) has a unique positive equilibrium $E^*(\frac{1-2a}{2}, \frac{1+2a}{4b})$. Moreover,*

(I) if $b \neq \frac{1+2a}{4a(1-2a)}$, then $E^*(\frac{1-2a}{2}, \frac{1+2a}{4b})$ is a saddle-node, which includes a stable parabolic sector (or an unstable parabolic sector) if $0 < b < \frac{1+2a}{4a(1-2a)}$ (or $b > \frac{1+2a}{4a(1-2a)}$);

(II) if $b = \frac{1+2a}{4a(1-2a)}$, then $E^*(\frac{1-2a}{2}, a(1-2a))$ is a cusp of codimension 2 if $a \neq a_0$. The phase portrait is given in Figure 2.

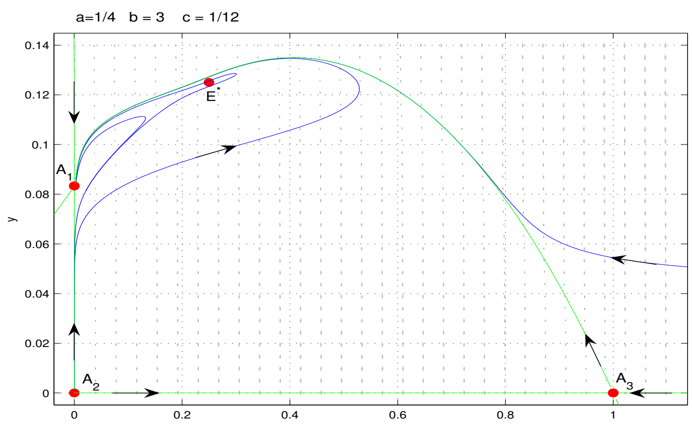

Figure 2. A unique positive equilibrium E^* if $c = \frac{1}{4b}$ and $\delta = \frac{a}{b}$, which is a cusp of codimension 2 if $b = \frac{1+2a}{4a(1-2a)}$ and $a \neq a_0$.

Proof. (**I**) If $b \neq \frac{1+2a}{4a(1-2a)}$, then the Jacobian matrix of equilibrium E^* has only one zero eigenvalue. We first let $u = x - \frac{1-2a}{2}, v = y - \frac{1+2a}{4b}$, then system (1.2) becomes

$$\begin{aligned}\dot{u} &= (a - 2a^2)u + (2a-1)v + (4a^2 + 2a - 1)u^2 - 4abuv + o(|u, v|^2),\\ \dot{v} &= \frac{a(1+2a)}{4b^2}u - \frac{1+2a}{4b}v - \frac{a(1+2a)}{2b^2}u^2 + \frac{a}{b}uv - v^2 + o(|u, v|^2).\end{aligned} \tag{2.10}$$

Let $u = \frac{b}{a}X - \frac{4(2a-1)b^2}{1+2a}Y$, $v = X+Y$ and make a time variation $\tau = -\frac{4b}{1+2a-4ab+8a^2b}t$, then system (2.10) can be rewritten as

$$\begin{aligned}\dot{X} &= AX^2 + BXY + CY^2 + o(|X, Y|^2),\\ \dot{Y} &= Y + DX^2 + \overline{E}XY + FY^2 + o(|X, Y|^2),\end{aligned} \tag{2.11}$$

where

$$\begin{aligned}
A &= \frac{4b^2(1-2a)^2(1+2a)}{a(1+a(2-4b)+8a^2b)^2},\\
B &= \frac{32b^2(2a^2+b-4ab+a^3(4+8b))}{(1+2a)(1+2a-4ab+8a^2b)^2},\\
C &= -\frac{16ab^2(2a-1)(-1+4b^2+32a^2b^2+a(-2+8b-24b^2))}{(1+2a)(1+2a-4ab+8a^2b)^2},\\
D &= \frac{2b(1+2a)(1-2b+a(2+4b))}{a(1+a(2-4b)+8a^2b)^2},\\
\overline{E} &= -\frac{4b(-1-4b+8b^2+32a^3b^2+16a^2b(1+b)+a(-2+8b-32b^2))}{(1+2a-4ab+8a^2b)^2},\\
F &= \frac{F_1+F_2}{(1+2a)(1+2a-4ab+8a^2b)^2},\\
F_1 &= 256a^5b^3+128a^4b^2(1-b)+16a^3b(1-2b-8b^2),\\
F_2 &= 4a^2(1-8b^2+24b^3)+4a(1-b+2b^2-4b^3)+1,
\end{aligned}$$

we can see that $A\neq 0$, since $b\neq\frac{1+2a}{4a(1-2a)}$ and $0<a<\frac{1}{2}$, then according to Theorem 7.1 in Zhang *et al.* [13], $E^*(\frac{1-2a}{a},\frac{1+2a}{4b})$ is a saddle-node which includes a stable (or an unstable) parabolic sector if $0<b<\frac{1+2a}{4a(1-2a)}$ (or $b>\frac{1+2a}{4a(1-2a)}$).

(**II**) Next we prove that the unique positive equilibrium is a cusp if $b=\frac{1+2a}{4a(1-2a)}$. Substituting (2.8) into system (2.10), then we have

$$\begin{aligned}
\dot u &= (a-2a^2)u-\frac{1+2a}{4a}v+(-1+2a+4a^2)u^2+\frac{1+2a}{2a-1}uv+o(|u,v|^2)),\\
\dot v &= \frac{4(1-2a)^2a^3}{1+2a}u+a(2a-1)v-\frac{8a^3(1-2a)^2}{1+2a}u^2+\frac{4a^2(1-2a)}{1+2a}uv-v^2\\
&\quad +o(|u,v|^2).
\end{aligned}\tag{2.12}$$

Let $u=\frac{1+2a}{4a^2-8a^3}x+\frac{1+2a}{4a^3(1-2a)^2}y$ and $v=x$, system (2.12) becomes

$$\begin{aligned}
\dot x &= y-\frac{2a+1}{2a}x^2+\frac{1+a}{a^2(2a-1)}xy-\frac{1+2a}{2a^3(1-2a)^2}y^2+o(|x,y|^2),\\
\dot y &= -\frac{(1-2a)^2(1+2a)}{4a}x^2+\frac{1-2a-4a^2}{4a^3-2a^2}xy+\frac{-1+2a+8a^2}{4a^3(1-2a)^2}y^2\\
&\quad +o(|x,y|^2).
\end{aligned}\tag{2.13}$$

Making a C^∞-change of variables $x=X+\frac{8a^3+12a^2-2a-1}{8a^3(1-2a)^2}X^2-\frac{1+2a}{2a^3(1-2a)^2}XY$, $y=Y+\frac{1+2a}{2a}X^2+\frac{8a^2+2a-1}{4a^3(1-2a)^2}XY$, we have

$$\begin{aligned}
\dot X &= Y+o(|X,Y|^2),\\
\dot Y &= -\frac{(1-2a)^2(1+2a)}{4a}X^2+\frac{1-4a^2-8a^3}{4a^3-2a^2}XY+o(|X,Y|^2),
\end{aligned}\tag{2.14}$$

where $\frac{(1-2a)^2(1+2a)}{4a}\neq 0$ since $0<a<\frac{1}{2}$. If $a\neq a_0$, then $\frac{1-4a^2-8a^3}{4a^3-2a^2}\neq 0$, and $E^*(\frac{1-2a}{2},a(1-2a))$ is a cusp of codimension 2 by Perko [10]; if $a=a_0\approx 0.377439$, then $E^*(\frac{1-2a}{2},a(1-2a))$ is a cusp of codimension at least 3 ([4]). □

3. Bogdanov-Takens bifurcation

From Theorem 2.1 we know that system (1.2) has a cusp E^* of codimension 2 when $c = \frac{1}{4b}$, $\delta = \frac{a}{b}$, $b = \frac{1+2a}{4a(1-2a)}$ and $a \neq a_0$, then it may exhibit Bogdanov-Takens bifurcation around E^*. Choosing b and c as bifurcation parameters, then the unfolding system of system (1.2) is as follows

$$\begin{aligned}\dot{x} &= x(1 - x - \frac{(\frac{1+2a}{4a(1-2a)} + \lambda_1)y}{a+x}),\\ \dot{y} &= y(\frac{a}{\frac{1+2a}{4a(1-2a)} + \lambda_1} - y + \frac{(\frac{a(1-2a)}{1+2a} + \lambda_2)x}{a+x}),\end{aligned} \tag{3.1}$$

where $\lambda = (\lambda_1, \lambda_2)$ are small parameters vector in a small neighborhood of $(0,0)$ and $0 < a < \frac{1}{2}$.

In the following we study if system (3.1) can undergo a Bogdanov-Takens bifurcation in a small neighborhood of equilibrium E^* as parameters (b, c) varies in a small neighborhood of $(\frac{1+2a}{4a(1-2a)}, \frac{a(1-2a)}{1+2a})$ and the result as follows.

Theorem 3.1. *When $0 < a < \frac{1}{2}$ and $a \neq a_0$, parameters (λ_1, λ_2) vary in a small neighborhood of the origin, system (3.1) undergoes a Bogdanov-Takens bifurcation in a small neighborhood of E^*. Moreover,*

(I) if $0 < a < a_0$, then there exist a subcritical Bogdanov-Takens bifurcation. Hence, there exist some various parameter values such that system (3.1) has an unstable limit cycle or an unstable homoclinic loop around E^*;

(II) if $a_0 < a < \frac{1}{2}$, then there exist an supercritical g Bogdanov-Takens bifurcation. Hence, there exist some various parameter values such that system ((3.1) has a stable limit cycle or a stable homoclinic loop around E^*.

Proof. Firstly, we translate E^* to the origin by letting $u = x - \frac{1-2a}{2}$, $v = y - a(1-2a)$, and the Taylor expansion of system (3.1) around the origin takes the form

$$\begin{aligned}\dot{u} &= a_{00} + a_{10}u + a_{01}v + a_{20}u^2 + a_{11}uv + P_1(u, v, \lambda),\\ \dot{v} &= b_{00} + b_{10}u + b_{01}v + b_{20}u^2 + b_{11}uv + b_{02}v^2 + P_2(u, v, \lambda),\end{aligned} \tag{3.2}$$

where P_1, P_2 are C^∞ function at least of the third order with respect to (u, v), whose coefficients depend smoothly on λ_1, λ_2 and

$$\begin{aligned}
a_{00} &= -(1-2a)^2 a\lambda_1, \quad a_{10} = a(1-2a)(1-4a\lambda_1),\\
a_{01} &= -\frac{1+2a}{4a} + (2a-1)\lambda_1, \quad a_{20} = (-1 + 2a + 4a^2 + 8(1-2a)a^2\lambda_1),\\
a_{11} &= \frac{1+2a+4(1-2a)a\lambda_1}{2a-1},\\
b_{00} &= \frac{a(1-2a)^2(a(1-4a^2)\lambda_1\lambda_2 - 16a^3(1-2a)\lambda_1 + (1+2a)^2\lambda_2)}{(1+2a)(1+2a+4a(1-2a)\lambda_1)},\\
b_{10} &= \frac{4a^2(a(1-2a)^2 + (1-4a^2)\lambda_2)}{1+2a}, \quad b_{02} = -1,\\
b_{01} &= -\frac{(1-2a)((1+2a)^2(a-\lambda_2) + 4a(1-2a)(a(1+6a) - (1+2a)\lambda_2)\lambda_1)}{(1+2a)(1+2a+4a(1-2a)\lambda_1)},
\end{aligned}$$

$$b_{20} = -\frac{8a^2(a(1-2a)^2 + (1-4a^2)\lambda_2)}{1+2a}, \quad b_{11} = \frac{4a(a(1-2a) + (1+2a)\lambda_2)}{1+2a}.$$

Secondly, let

$$x = u, \quad y = a_{00} + a_{10}u + a_{01}v + a_{20}u^2 + a_{11}uv + P_1(u, v, \lambda),$$

then system (3.2) can be written as

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= c_{00} + c_{10}x + c_{01}y + c_{20}x^2 + c_{11}xy + c_{02}y^2 + Q_1(x, y, \lambda), \end{aligned} \tag{3.3}$$

where $Q_1(x, y, \lambda)$ is a C^∞ function at least third order with respect to (x, y) and

$$\begin{aligned} c_{00} &= a_{01}b_{00} - a_{00}b_{01} + \frac{a_{00}^2 b_{02}}{a_{01}}, \; c_{01} = a_{10} + b_{01} - \frac{2a_{00}b_{02} + a_{00}a_{11}}{a_{01}}, \\ c_{10} &= a_{01}b_{10} + a_{11}b_{00} - a_{10}b_{01} - a_{00}b_{11} + \frac{a_{00}^2 a_{11}^2 + 2a_{00}a_{10}a_{01}b_{02}}{a_{01}^2}, \\ c_{20} &= a_{01}b_{20} + a_{11}b_{10} - a_{10}b_{11} - a_{20}b_{01} + \frac{a_{00}a_{10}a_{11}^2 - a_{00}a_{01}a_{20}a_{11} + a_{10}^2 a_{01}b_{02}}{a_{01}^2}, \\ c_{11} &= 2a_{20} + b_{11} - \frac{2a_{10}a_{01}b_{02} + a_{10}a_{01}a_{11} + a_{00}a_{11}^2}{a_{01}^2}, \; c_{02} = \frac{a_{11} + b_{02}}{a_{01}}. \end{aligned}$$

Thirdly, we introduce a new time variable τ by $dt = (1 - c_{02}x)d\tau$, and still denote τ by t, then system (3.3) can be written as

$$\begin{aligned} \dot{x} &= y(1 - c_{02}x), \\ \dot{y} &= (1 - c_{02}x)(c_{00} + c_{10}x + c_{01}y + c_{20}x^2 + c_{11}xy + c_{02}y^2 + Q_1(x, y, \lambda)), \end{aligned} \tag{3.4}$$

next let $X = x, Y = y(1 - c_{02}x)$, then system (3.4) takes

$$\begin{aligned} \dot{X} &= Y, \\ \dot{Y} &= d_{00} + d_{10}X + d_{01}Y + d_{20}X^2 + d_{11}XY + Q_2(X, Y, \lambda), \end{aligned} \tag{3.5}$$

where Q_2 has the same property as $P_i(u, v, \lambda)$ $(i = 1, 2)$ and

$$\begin{aligned} d_{00} &= c_{00}, \;\; d_{10} = c_{10} - 2c_{00}c_{02}, \;\; d_{01} = c_{01}, \\ d_{20} &= c_{20} - 2c_{10}c_{02} + c_{00}c_{02}^2, \;\; d_{11} = c_{11} - c_{01}c_{02}. \end{aligned}$$

Notice that when $0 < a < \frac{1}{2}$ and λ_i are small, $d_{20} = -a(1-2a)^3 < 0$, we make a change of variables as follows

$$x = X, \quad y = -\frac{Y}{\sqrt{-d_{20}}}, \quad \tau = \sqrt{-d_{20}}\,t,$$

then system (3.5) can be rewritten as

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= e_1 + e_2x + e_3y - x^2 + e_4xy + Q_3(x, y, \lambda), \end{aligned} \tag{3.6}$$

where $Q_3(X,Y,\lambda)$ has the same property as $P_i(u,v,\lambda)$ $(i=1,2)$ and

$$e_1=-\frac{d_{00}}{d_{20}},\ \ e_3=-\frac{d_{10}}{d_{20}},\ \ e_3=\frac{d_{01}}{\sqrt{-d_{20}}},\ \ e_4=\frac{d_{11}}{\sqrt{-d_{20}}}.$$

Next, let $X=x+\frac{e_2}{2}, Y=y$, then system (3.6) can be transformed as

$$\begin{aligned}\dot{X}&=Y,\\ \dot{Y}&=f_1+f_2Y-X^2+f_3XY+Q_4(X,Y,\lambda),\end{aligned} \tag{3.7}$$

where $Q_4(X,Y,\lambda)$ has the same property as $P_i(u,v,\lambda)$ $(i=1,2)$ and

$$f_1=e_1+\frac{e_2^2}{4},\ \ f_2=e_3+\frac{e_2e_4}{2},\ \ f_3=e_4.$$

When $0<a<\frac{1}{2}$, $8a^3+4a^2-1\neq 0$ and λ_i are small, then $d_{11}\neq 0$ and $f_3\neq 0$. Make the last change of variables

$$x=-f_3^2X,\ \ y=f_3^3Y,\ \ \tau=-\frac{t}{f_3},$$

then system (3.7) takes the following form

$$\begin{aligned}\dot{x}&=y,\\ \dot{y}&=\mu_1+\mu_2y+x^2+xy+Q_5(x,y,\lambda),\end{aligned} \tag{3.8}$$

where Q_5 has the same property as $P_i(u,v,\lambda)$ $(i=1,2)$ and

$$\mu_1=-f_1f_3^4,\ \ \mu_2=-f_2f_3.$$

By lengthy calculation, we can express μ_i by $\lambda_i(i=1,2)$ as follows

$$\begin{aligned}\mu_1&=g_1\lambda_1+g_2\lambda_2+g_3{\lambda_1}^2+g_4\lambda_1\lambda_2+g_5{\lambda_2}^2+o(|\lambda_1,\lambda_2|^2),\\ \mu_2&=k_1\lambda_1+k_2\lambda_2+k_3{\lambda_1}^2+k_4\lambda_1\lambda_2+k_5{\lambda_2}^2+o(|\lambda_1,\lambda_2|^2),\end{aligned} \tag{3.9}$$

where

$$\begin{aligned}
g_1&=\frac{16(8a^3+4a^2-1)^4}{a(1-4a^2)^5},\ \ g_2=\frac{4(8a^3+4a^2-1)^4}{a^3(1-2a)^7(1+2a)^3},\\
g_3&=\frac{64(8a^3+4a^2-1)^3g_{31}}{(1-2a)^6(1+2a)^8},\ \ g_4=\frac{32(8a^3+4a^2-1)^3g_{41}}{a(1-2a)^8(1+2a)^6},\\
g_5&=-\frac{4(8a^3+4a^2-1)^3g_{51}}{a^4(1-2a)^{10}(1+2a)^4},\ \ k_1=\frac{8a(1-8a^2+16a^4-64a^6)}{(1-2a)^2(1+2a)^4},\\
k_2&=\frac{4(1-8a^2)(8a^3+4a^2-1)}{(1-2a)^4(1+2a)^2},\ \ k_3=-\frac{32a^2(k_{31}+k_{32})}{(1-2a)^4(1+2a)^7},\\
k_4&=-\frac{8(k_{41}+k_{42})}{(1-4a^2)^5},\ \ k_5=\frac{2(k_{51}+k_{52})}{a^2(1-2a)^7(1+2a)^3},
\end{aligned}$$

and

$$g_{31}=224a^7+1360a^6+488a^5-112a^4-100a^3-47a^2+26a+4,$$

$$\begin{aligned}
g_{41} &= 32a^6 + 1104a^5 + 488a^4 + 8a^3 - 36a^2 - 45 + 13,\\
g_{51} &= 288a^7 - 784a^6 - 552a^5 - 176a^4 - 20a^3 + 55a^2 + 3,\\
k_{31} &= 2048a^{11} - 9216a^{10} + 2560a^9 + 9984a^8 + 6784a^7 + 1344a^6,\\
k_{32} &= -1600a^5 - 640a^4 - 80a^3 + 48a^2 + 18a + 1,\\
k_{41} &= 12800a^9 + 9728a^8 + 1536a^7 - 1472a^6,\\
k_{42} &= -1344a^5 + 48a^4 + 144a^3 + 36a^2 - 10a - 3,\\
k_{51} &= 1536a^{10} - 5120a^9 - 5632a^8 - 1920a^7 + 256a^6,\\
k_{52} &= 736a^5 + 144a^4 - 8a^3 - 14a^2 - 2a - 1.
\end{aligned}$$

Notice that

$$\left|\frac{\partial(\mu_1,\mu_2)}{\partial(\lambda_1,\lambda_2)}\right|_{\lambda=0} = \frac{32(8a^3+4a^2-1)^5}{a^2(1-2a)^8(1+2a)^5} \neq 0$$

since $0 < a < \frac{1}{2}$ and $a \neq a_0$, then the parameters transformation (3.9) is a homeomorphism in a small neighborhood of the origin, and μ_1, μ_2 are independent parameters. By the results in [1] and [12] (see also [3, 10]), we obtain the following local representations of the bifurcation curves up to second-order approximations:

(1) The saddle-node bifurcation curve $SN = \{(\mu_1,\mu_2) : \mu_1 = 0, \mu_2 \neq 0\}$, i.e.,

$$\begin{aligned}
SN = \{(\lambda_1,\lambda_2) :\ & \frac{16(8a^3+4a^2-1)^4}{a(1-4a^2)^5}\lambda_1 + \frac{4(8a^3+4a^2-1)^4}{a^3(1-2a)^7(1+2a)^3}\lambda_2\\
&+ \frac{64(8a^3+4a^2-1)^3N_1}{(1-2a)^6(1+2a)^8}\lambda_1^2 + \frac{32(8a^3+4a^2-1)^3N_2}{a(1-2a)^8(1+2a)^6}\lambda_1\lambda_2\\
&- \frac{4(8a^3+4a^2-1)^3N_3}{a^4(1-2a)^{10}(1+2a)^4}\lambda_2^2 + O(|\lambda_1,\lambda_2|^3) = 0, \mu_1(\lambda_1,\lambda_2) = 0\},
\end{aligned}$$

where

$$\begin{aligned}
N_1 &= 224a^7 + 1360a^6 + 488a^5 - 112a^4 - 100a^3 - 47a^2 + 26a + 4,\\
N_2 &= 32a^6 + 1104a^5 + 488a^4 + 8a^3 - 36a^2 - 45 + 13,\\
N_3 &= 288a^7 - 784a^6 - 552a^5 - 176a^4 - 20a^3 + 55a^2 + 3.
\end{aligned}$$

(2) The Hopf bifurcation curve $H = \{(\mu_1,\mu_2) : \mu_2 = -\sqrt{-\mu_1}, \mu_1 < 0\}$, i.e.,

$$\begin{aligned}
H = \{(\lambda_1,\lambda_2) :\ & \frac{16(8a^3+4a^2-1)^4}{a(1-4a^2)^5}\lambda_1 + \frac{4(8a^3+4a^2-1)^4}{a^3(1-2a)^7(1+2a)^3}\lambda_2\\
&+ \frac{128(8a^3+4a^2-1)^2H_1}{(1-2a)^6(1+2a)^8}\lambda_1^2 + \frac{32(8a^3+4a^2-1)^2H_2}{a(1-2a)^8(1+2a)^6}\lambda_1\lambda_2\\
&- \frac{4(8a^3+4a^2-1)^2H_3}{a^4(1-2a)^{10}(1+2a)^4}\lambda_2^2 + O(|\lambda_1,\lambda_2|^3) = 0, \mu_1(\lambda_1,\lambda_2) < 0\},
\end{aligned}$$

where

$$\begin{aligned}
H_1 &= 1024a^{10} + 5632a^9 + 4864a^8 + 384a^7 - 13446a^6 - 608a^5 + 64a^4\\
&\quad + 116a^3 + 32a^2 - 13a - 2,\\
H_2 &= 768a^9 + 8192a^8 + 8640a^7 + 2080a^6 - 1472a^5 - 976a^4 - 76a^3
\end{aligned}$$

$$+ 86a^2 + 45a - 13,$$
$$H_3 = 1280a^{10} - 4096a^9 - 7552a^8 - 4160a^7 - 32a^6 + 928a^5 + 392a^4 + 44a^3 - 43a^2 - 3.$$

(3) The homoclinic bifurcation curve $HL = \{(\mu_1, \mu_2) : \mu_2 = -\frac{5}{7}\sqrt{-\mu_1}, \mu_1 < 0\}$, i.e.,

$$\begin{aligned} HL = \{(\lambda_1, \lambda_2) : & \frac{16(8a^3 + 4a^2 - 1)^4}{a(1 - 4a^2)^5}\lambda_1 + \frac{4(8a^3 + 4a^2 - 1)^4}{a^3(1 - 2a)^7(1 + 2a)^3}\lambda_2 \\ & + \frac{128(8a^3 + 4a^2 - 1)^2 l_1}{49(1 - 2a)^6(1 + 2a)^8}\lambda_1^2 + \frac{32(8a^3 + 4a^2 - 1)^2 l_2}{49a(1 - 2a)^8(1 + 2a)^6}\lambda_1\lambda_2 \\ & - \frac{4(8a^3 + 4a^2 - 1)^2 l_3}{49a^4(1 - 2a)^{10}(1 + 2a)^4}\lambda_2^2 + O(|\lambda_1, \lambda_2|^3) = 0, \mu_1(\lambda_1, \lambda_2) < 0\}, \end{aligned}$$

where

$$\begin{aligned} l_1 &= 47104a^{10} + 282112a^9 + 233728a^8 + 19584a^7 - 64896a^6 - 30368a^5 \\ &\quad + 3184a^4 + 5732a^3 + 1556a^2 - 637a - 98, \\ l_2 &= 25344a^9 + 419840a^8 + 415680a^7 + 99616a^6 - 69440a^5 - 48208a^4 \\ &\quad - 3916a^3 + 4262a^2 + 2205a - 637, \\ l_3 &= 87296a^{10} - 225280a^9 - 370048a^8 - 197696a^7 - 2720a^6 + 45088a^5 \\ &\quad + 19304a^4 + 2156a^3 - 2107a^2 - 147. \end{aligned}$$

From the expression of f_3 and the transformation $\tau = \frac{-t}{f_3}$ to get system (3.8), we can get the following results ([4]):

(1) If $0 < a < a_0$, then $f_3 < 0$, therefore, there exists a subcritical Bogdanov-Takens bifurcation at $(\frac{1-2a}{2}, a(1-2a))$, and system (1.2) has an unstable limit cycle or an unstable homoclinic loop for some various parameters values;

(2) If $a_0 < a < \frac{1}{2}$, then $f_3 > 0$, therefore, there exists a supercritical Bogdanov-Takens bifurcation at $(\frac{1-2a}{2}, a(1-2a))$, and system (1.2) has a stable limit cycle or an stable homoclinic loop for some various parameters values. □

The supercritical Bogdanov-Takens bifurcation diagram of codimension 2 and phase portraits of system (1.2) with $a = \frac{2}{5}$, $b = \frac{48}{5}$ and $c = \frac{2}{45}$ are given in Figure 3. These bifurcation curves SN, H and HL divide the small neighborhood of the origin in the parameter plane (λ_1, λ_2) into four regions (see Figure 3(a)).

(i) There are no positive equilibria when the parameters lie in region I (see Figure 3(b)).

(ii) When the parameters lie on curve SN, there is a positive equilibrium, which is a saddle-node.

(iii) Two positive equilibria, which are an unstable focus and a saddle, will occur through the saddle-node bifurcation when the parameters cross SN into region II (see Figure 3(c)).

(iv) A stable homoclinic cycle enclosing an unstable hyperbolic focus will occur through the homoclinic bifurcation when the parameters pass region III and lie on curve HL (see Figure 3(d)).

(v) A stable limit cycle will appear through the subcritical Hopf bifurcation when the parameters cross H into region III (see Figure 3(e)).

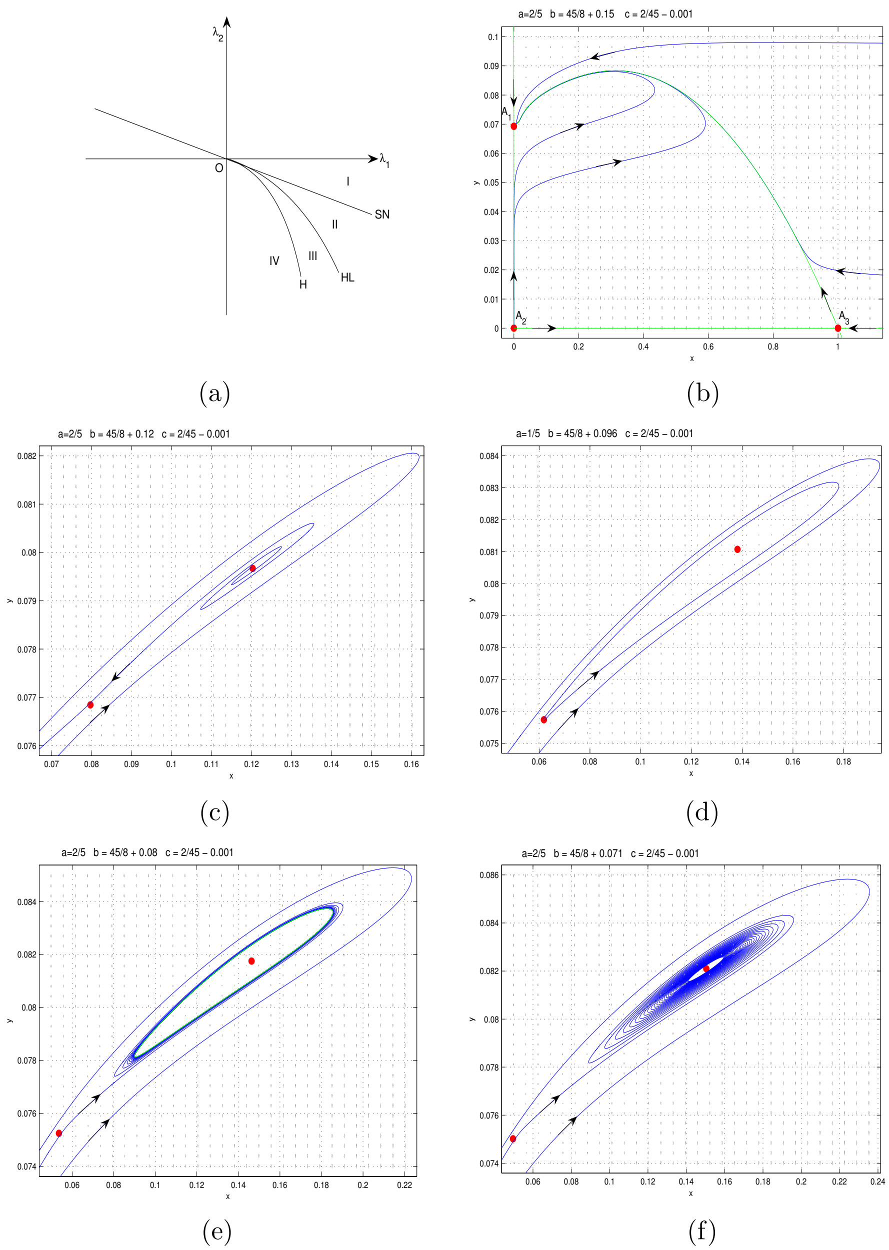

Figure 3. The supercritical Bogdanov-Takens bifurcation diagram and corresponding phase portraits of system ((1.2)) with $a = \frac{2}{5}$, $b = 45/8$ and $c = \frac{2}{45}$. (a) Bifurcation diagram; (b) No equilibria when $(\lambda_1, \lambda_2) = (0.15, -0.001)$ lies in region I; (c) An unstable focus when $(\lambda_1, \lambda_2) = (0.12, -0.001)$ lies in region II; (d) A stable homoclinic cycle when $(\lambda_1, \lambda_2) = (0.08, -0.001)$ lies in region III; (e) A stable limit cycle when $(\lambda_1, \lambda_2) = (0.071, -0.001)$ lies in region IV; (f) A stable focus when $(\lambda_1, \lambda_2) = (0.1, -0.00124)$ lies in region IV.

4. Discussion

In this paper, we revisited a host-parasitoid model with Holling II functional response, which was proposed by Magal *et al.* [6], we focus on a special case: $K_2 = \frac{r_1}{\eta}$, i.e., the carrying capacity K_2 for parasitoids is equal to a critical value $\frac{r_1}{\eta}$. After performing a qualitative and bifurcation analysis, our results reveal that model (1.2) exhibits Bogdanov-Takens bifurcation. The approximate expressions for saddle-node, Homoclinic and Hopf bifurcation curves are calculated. Numerical simulations, including bifurcation diagrams and corresponding phase portraits, are also given to illustrate theoretical results. Moreover, from Lemma 2.2 and Theorems 2.1 and 3.1, we can see that the invasion hosts can be stopped or reversed when model (1.2) has no positive equilibrium, i.e., the generalist parasitoids can control the invasion of the invading hosts; the invading hosts can persist in the form of multiple positive steady states or one periodic oscillation when model (1.2) has one or two positive equilibria. In both cases, the parasitoids always persist.

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