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The Approach of Solutions for the Nonlocal Diffusion Equation to Traveling Fronts

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Abstract The paper is concerned with the asymptotic behavior as $t \to \pm\infty$ of an entire solution $u(x,t)$ for the nonlocal diffusion equation. With bistable assumption, it is well known that the model has three different types of traveling fronts. Under certain conditions on the wave speeds, and by some auxiliary rational functions with certain properties to construct appropriate super- and sub solutions of the model, we establish two new types of entire solutions $u(x,t)$ which approach to three travelling fronts or the positive equilibrium as $t \to \pm\infty$.

Keywords Entire solution, Traveling front, Nonlocal evolution equation, Super-sub solutions.

MSC(2010) 34K05, 34A34, 34E05

1. Introduction

Consider the following nonlocal evolution problem for an entire solution $u(x,t)$ defined on $\mathbb{R} \times \mathbb{R}$:

$$u_t(x,t) = (J * u)(x,t) - u(x,t) + f(u(x,t)), \tag{1.1}$$

where the kernel J of the convolution $(J * u)(x,t) := \int_{\mathbb{R}} J(x-y)u(y,t)dy$ is nonnegative, even, with unit integral, and the function f is bistable. We see (1.1) is a nonlocal analog of the usual bistable reaction diffusion equation

$$\frac{\partial u(x,t)}{\partial t} = \Delta u(x,t) + f(u(x,t)). \tag{1.2}$$

As such, (1.1) as well as this equation (1.2) may model a variety of physical and biological phenomena involving media with properties varying in space. The possible advantages of (1.1) lie in the fact that much more general types of interactions between states at nearby locations in the medium can be accounted for. Lee et al. also [14] argue that, for processes where the spatial scale for movement is large in comparison with its temporal scale, nonlocal models may allow for better estimation of parameters from data and provide more insight into the biological system.

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Convolution equations are related to the classic Laplacian diffusion equations, i.e., letting $J(x) = \delta(x) + \delta''(x)$, where δ is the Dirac delta (see, Medlock et al. [16]), (1.2) reduces to (1.1).

It is well known that travelling fronts are special examples of the so-called entire solutions which are defined in the whole space and for all time $t \in \mathbb{R}$. Travelling fronts are known to exist for (1.1) with reaction terms f of bistable, monostable, and ignition type. For (1.1) with the bistable or ignition nonlinearities, the existence results of travelling fronts can be studied by Bates et al. [1], Chen [4], Coville [7] and Yagisita [24]. In the case where f is monostable, this kind of equation (1.1) was originally introduced in 1937 by Kolmogorov, Petrovskii and Piskunov [17] as a way to derive the Fisher equation (1.2) with $f(u) = u(1-u)$. We refer to some works [2,3,8,9,16,23] for (1.1) with the monostable nonlinearity. Notice that the entire solutions can help us for the mathematical understanding of transient dynamics. In recent years, there have been lots of works devoted to the interaction of traveling fronts and entire solutions for various diffusion equations; see, e.g., [11–13,15,18–20,22,25] and the references cited therein.

We point out that authors [18, 19] investigated the entire solution behaving as two traveling fronts coming from both sides of the x-axis for (1.1) with the monostable and bistable nonlinearities, respectively. Recently, new types of entire solutions merging three fronts are concerned for some evolution equations with the bistable nonlinearity (see [5, 6, 21]). Motivated by these works, it is natural and interesting to study new entire solutions merging three fronts of (1.1) with the bistable nonlinearity.

Before to state our main results, we first give some assumptions for the functions $J(\cdot)$ and $f(\cdot)$, definitions of the traveling fronts and entire solutions for (1.1).

(J) $J(\cdot) \in C^1(\mathbb{R})$ has compact support, $J(x) = J(-x) \geq 0$,

$$\int_{\mathbb{R}} J(x)dx = 1 \text{ and } \int_{\mathbb{R}} J(x)e^{-\lambda x}dx < +\infty \text{ for all } \lambda > 0.$$

(A1) **[Bistable condition]** $f(u) \in C^2(\mathbb{R})$, $f(0) = f(a) = f(1) = 0$, $f'(0) < 0$, $f'(1) < 0$, $f'(a) > 0$, $f(u) < 0$ for any $u \in (0, a)$ and $f(u) > 0$ for any $u \in (a, 1)$.

Definition 1.1 (traveling fronts and entire solutions)**.**

(1) A solution $u(x,t)$ is called a traveling front of (1.1) connecting $\{e_1, e_2\} \subset \{0, a, 1\}$ with the wave speed k, if

$$u(x,t) = \phi(x+kt), \text{ or } u(x,t) = \phi(-x+kt),$$

for all $x \in \mathbb{R},\ t \in \mathbb{R}$ and some function $\phi(\cdot) = \{\phi(\cdot)\}_{x\in\mathbb{R},t\in\mathbb{R}}$, which satisfies

$$\phi(-\infty) = e_1 \text{ and } \phi(+\infty) = e_2, \quad \forall x \in \mathbb{R}.$$

(2) A function $u(x,t) = \{u(x,t)\}_{x\in\mathbb{R},t\in\mathbb{R}}$ is called an entire solution of (1.1) if for any $x \in \mathbb{R}$, $u(x,t)$ is differentiable for all $t \in \mathbb{R}$ and satisfies (1.1) for $x \in \mathbb{R}$ and $t \in \mathbb{R}$.

Now we recall the result in [1, 4, 24] as follows.

Proposition 1.1. *Assume that (J) and (A1) hold. Then* (1.1) *can admit a decreasing traveling front* $\hat{\varphi}_1(x+\hat{v}_1 t)$ *with* $\hat{v}_1<0$, *where* $\hat{\varphi}_1$ *satisfies*

$$\begin{cases} \hat{v}_1\hat{\varphi}_1'(\xi)=J*\hat{\varphi}_1(\xi)-\hat{\varphi}_1(\xi)+f(\hat{\varphi}_1(\xi)), \\ \hat{\varphi}_1(-\infty)=1,\ \hat{\varphi}_1(+\infty)=0. \end{cases} \tag{1.3}$$

In order to obtain monostable traveling fronts, we assume that $f(u)$ satifies

(A2): $f(u)\begin{cases} \geq f'(a)(u-a), \text{ if } u\in[0,a], \\ \leq f'(a)(u-a), \text{ if } u\in[a,1], \end{cases}$ for all $x\in\mathbb{R}$.

Similar to the arguments in [9,19,23], we easily obtain the following results.

Proposition 1.2. *Assume that (J), (A1) and (A2) hold. Then the following assertions hold.*

(i) There exists $v^*<0$ *such that for any* $\tilde{v}_2<v^*$, (1.1) *can admit an increasing traveling front* $\tilde{\varphi}_2(\pm x+\tilde{v}_2 t)$, *where* $\tilde{\varphi}_2$ *satisfies*

$$\begin{cases} \tilde{v}_2\tilde{\varphi}_2'(\xi)=J*\tilde{\varphi}_2(\xi)-\tilde{\varphi}_2(\xi)+f(\tilde{\varphi}_2(\xi)), \\ \tilde{\varphi}_2(-\infty)=0,\ \tilde{\varphi}_2(+\infty)=a. \end{cases} \tag{1.4}$$

(ii) There exists a ${v_3}^*>0$ *such that for every* $\tilde{v}_3>{v_3}^*$, (1.1) *admits an increasing traveling front* $\tilde{\varphi}_3(x+\tilde{v}_3 t)$, *where* $\tilde{\varphi}_3$ *satisfies*

$$\begin{cases} \tilde{v}_3\tilde{\varphi}_3'(\xi)=J*\tilde{\varphi}_3(\xi)-\tilde{\varphi}_3(\xi)+f(\tilde{\varphi}_3(\xi)), \\ \tilde{\varphi}_3(-\infty)=a,\ \tilde{\varphi}_3(+\infty)=1. \end{cases} \tag{1.5}$$

Remark 1.1. Taking $\hat{v}_2=-\tilde{v}_2$ and $\hat{\varphi}_2(x+\hat{v}_2 t)=a-\tilde{\varphi}_2(-x+\tilde{v}_2 t)=a-\tilde{\varphi}_2(-(x-\tilde{v}_2 t))=a-\tilde{\varphi}_2(-(x+\hat{v}_2 t))$, where $\hat{\varphi}_2$ satisfies

$$\begin{cases} \hat{v}_2\hat{\varphi}_2'(\xi)=J*\hat{\varphi}_2(\xi)-\hat{\varphi}_2(\xi)+f(\hat{\varphi}_2(\xi)), \\ \hat{\varphi}_2(-\infty)=a,\ \hat{\varphi}_2(+\infty)=0. \end{cases} \tag{1.6}$$

Remark 1.2. Let $u(x,t)$ be an entire solution of (1.1) and $\varphi_k(x+v_k t)$ $(k=1,2,3)$ be the traveling front of (1.1) with the wave speed v_k and connecting two different constant states. If $u(x,t)$ satisfies

$$\lim_{t\to-\infty}\Big\{\sum_{1\leq k\leq 3}\sup_{p_{k-1}(t)\leq x\leq p_k(t)}|u(x,t)-\varphi_k(x+v_k t+\theta_k)|\Big\}=0$$

then it is called the entire solution originating from three fronts of (1.1), where $v_1<v_2<v_3$ and θ_k $(k=1,2)$ is some constant, $p_0(t)=-\infty$, $p_k(t):=-(v_k+v_{k+1})t/2$ $(k=1,2)$ and $p_3(t)=+\infty$.

Next we will state our main results.

Theorem 1.1. *Assume that (J), (A1) and (A2) hold. Let* $(\hat{v}_1, \hat{\varphi}_1)$, $(\tilde{v}_2, \tilde{\varphi}_2)$ *and* $(\tilde{v}_3, \tilde{\varphi}_3)$ *be traveling fronts of* (1.1) *with* $\hat{v}_1 < \tilde{v}_2 < \tilde{v}_3$. *Then there exists an entire solution* $u(x,t)$ *originating from three fronts of* (1.1) *and* $\omega \in \mathbb{R}$ *which satisfies*

$$\lim_{t\to+\infty} \sup_{x\in\mathbb{R}} |u(x,t) - 1| = 0$$

and

$$\lim_{t\to-\infty} \Big\{ \sup_{x\le m_1} |u(x,t) - \hat{\varphi}_1(x + \hat{v}_1 t - \omega)| + \sup_{m_1\le x\le m_2} |u(x,t) - \tilde{\varphi}_2(x + \tilde{v}_2 t + \omega)| + \sup_{x\ge m_2} |u(x,t) - \tilde{\varphi}_3(x + \tilde{v}_3 t + \omega)| \Big\} = 0,$$

where $m_1 = -\frac{(\hat{v}_1+\tilde{v}_2)t}{2}$ *and* $m_2 = -\frac{(\tilde{v}_2+\tilde{v}_3)t}{2}$.

Theorem 1.2. *Assume that (J), (A1) and (A2) hold. Let* $(\hat{v}_1, \hat{\varphi}_1)$, $(\tilde{v}_2, \tilde{\varphi}_2)$ *and* $(\hat{v}_2, \hat{\varphi}_2)$ *be traveling fronts with* $\hat{v}_1 < \tilde{v}_2 < \hat{v}_2$. *Then there exists an entire solution* $v(x,t)$ *originating from three fronts of* (1.1) *and* $\omega_1, \omega_2 \in \mathbb{R}$ *which satisfies*

$$\lim_{t\to+\infty} \sup_{x\in\mathbb{R}} |v(x,t) - \hat{\varphi}_1(x + \hat{v}_1 t + \omega_1)| = 0$$

and

$$\lim_{t\to-\infty} \Big\{ \sup_{x\le n_1} |v(x,t) - \hat{\varphi}_1(x + \hat{v}_1 t + \omega_1)| + \sup_{n_1\le x\le n_2} |v(x,t) - \tilde{\varphi}_2(x + \tilde{v}_2 t + \omega_1)| + \sup_{x\ge n_2} |v(x,t) - \hat{\varphi}_2(x + \hat{v}_2 t - \omega_2)| \Big\} = 0,$$

where $n_1 = -\frac{(\hat{v}_1+\tilde{v}_2)t}{2}$ *and* $n_2 = -\frac{(\tilde{v}_2+\hat{v}_2)t}{2}$.

2. Proof of Theorem 1.1

Set $c_1 = \hat{v}_1$, $c_2 = \tilde{v}_2$ and $c_3 = \tilde{v}_3$. Let $\phi_1 = \hat{\varphi}_1$, $\phi_2 = \tilde{\varphi}_2$, $\phi_3 = \tilde{\varphi}_3$, $\xi = x + c_i t$, $x \in \mathbb{R}, i = 1,2,3$, be traveling fronts of (1.1) that satisfy

$$\begin{cases} c_i \phi_i'(\xi) = J * \phi_i(\xi) - \phi_i(\xi) + f(\phi_i(\xi)), \\ \phi_i(-\infty) = \alpha_i, \phi_i(\infty) = \beta_i, \end{cases} \tag{2.1}$$

where $(\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3) = (1, 0, 0, a, a, 1)$.

Without loss of generality, we assume that

$$\phi_1(0) = \frac{a}{2}, \phi_2(0) = \frac{a}{2}, \phi_3(0) = \frac{1+a}{2}. \tag{2.2}$$

By using the similar method as in [18,22,25], we can obtain the following asymptotic result.

Lemma 2.1. *Assume that (J), (A1) and (A2) hold. There exist positive numbers* C_0, C_1, C_2, η_1, η_2 *and* ρ, *which depend on* $\hat{v}_1$, $\tilde{v}_2$, $\hat{v}_2$, $\tilde{v}_3$, *for* $\xi \le 0$, *it holds*

$$0 \le \phi_1'(\xi), \phi_2'(\xi), \phi_3'(\xi) \le C_0 e^{\eta_1 \xi}, \tag{2.3}$$

$$C_1 e^{\eta_1 \xi} \le 1 - \phi_1(\xi), \phi_2(\xi), \phi_3(\xi) - a \le C_2 e^{\eta_1 \xi}, \tag{2.4}$$

$$\frac{\phi_1'(\xi)}{1-\phi_1(\xi)}, \ \frac{\phi_2'(\xi)}{\phi_2(\xi)}, \ \frac{\phi_3'(\xi)}{\phi_3(\xi)-a} \ge \rho. \tag{2.5}$$

For $\xi \ge 0$, it holds

$$0 \le \phi_1'(\xi), \phi_2'(\xi), \phi_3'(\xi) \le C_0 e^{-\eta_2 \xi}, \tag{2.6}$$

$$C_1 e^{-\eta_2 \xi} \le \phi_1(\xi), a - \phi_2(\xi), 1 - \phi_3(\xi) \le C_2 e^{-\eta_2 \xi}, \tag{2.7}$$

$$\frac{\phi_1'(\xi)}{\phi_1(\xi)}, \ \frac{\phi_2'(\xi)}{a-\phi_2(\xi)}, \ \frac{\phi_3'(\xi)}{1-\phi_3(\xi)} \ge \rho. \tag{2.8}$$

Now we consider the Cauchy problem of (1.1) with the initial data $u(x,t_0) = u_0(x)$, that is,

$$\begin{cases} u_t(x,t) = (J*u)(x,t) - u(x,t) + f(u(x,t)), \ x \in \mathbb{R}, \ t > t_0, \\ u(x,t_0) = u_0(x), x \in \mathbb{R}. \end{cases} \tag{2.9}$$

Definition 2.1 (Super- and subsolutions)**.**
Let $t_0 < T$, where t_0 and T are any two real constants. A sequence of continuous function $u(x,t) \in C(\mathbb{R}, [t_0,T))$ is called a supersolution (or subsolution) of (1.1) ,if

$$u_t(x,t) \ge (or \le)(J*u)(x,t) - u(x,t) + f(u(x,t)).$$

The results on the existence and comparison principle for the Cauchy problem of (1.1) are standard and common (see [15]).

Lemma 2.2. *Assume that (J),(A1) and (A2) hold. Then the following assertions hold.*

(1) *For any $u_0(x) \in C(\mathbb{R},[0,1])$, equation (2.9) admits a unique solution $u(x,t_0;u_0) \in C^{0,1}(\mathbb{R}\times[t_0,\infty),[0,1])$.*

(2) *Suppose $u^+(x,t)$ and $u^-(x,t)$ are nonnegative bounded supersolution and subsolution of (1.1) on $[t_0,+\infty)$, respectively, with the initial condition $u^+(x,t_0) \ge u^-(x,t_0)$ for $x \in \mathbb{R}, t \in [t_0,+\infty)$, then $u^+(x,t) \ge u^-(x,t) \ge 0$ for all $x \in \mathbb{R}, t \in [t_0,+\infty)$.*

Similar to the arguments in [25], we can also obtain the following estimate.

Lemma 2.3. *Assume that (J), (A1) and (A2) hold. Let $u(x,t)$ be a solution of (2.9) with the initial value $u_0(x) \in C(\mathbb{R},[0,1])$. There exists a positive constant M, such that*

$$\Big|u_t(x,t)\Big| \le M \ \ and \ \Big|u_{tt}(x,t)\Big| \le M, \ for \ x \in \mathbb{R}, \ t > t_0 \ .$$

In addition, assume $\max_{u\in[0,1]} f'(u) < 1$ and there exists a positive constant number K_1 such that for any $\eta > 0$,

$$\int_{\mathbb{R}} |J(x+\eta) - J(x)| dx \le K_1 \eta \ \ and \ \ |u_0(x+\eta) - u_0(x)| \le K_1 \eta,$$

then for any $x \in \mathbb{R}$, $t > 0$ and $\eta > 0$,one has

$$|u(x+\eta,t) - u(x,t)| \le K_2 \eta, \ \ |u_t(x+\eta,t) - u_t(x,t)| \le K_2 \eta,$$

where K_2 is some positive constant independent of u_0 and η.

2.1. Construction of super- and subsolutions

To construct a pair of super- and subsolutions of (1.1), we introduce the following auxiliary function linking three fronts(see [6])

$$Q(y,z,w) := z + (1-z)\frac{(1-y)z(w-a)+y(a-z)(1-w)}{(1-y)z(1-a)+(a-z)(1-w)} \quad \text{for any } (y,z,w)\in D_1, \tag{2.10}$$

where $D_1 = \{[0,1]\times[0,a]\times[a,1]\}\backslash(\{(1,a,w)|a\le w\le 1\}\cup\{(1,z,1)|0\le z\le a\}\cup\{(y,0,1)|0\le y\le 1\})$.

Lemma 2.4. *[6, Lemma 3.1] The function $Q(y,z,w)$ has the following properties.*

(i)

$$Q(y,z,w) = \begin{cases} y + (1-y)z\dfrac{(1-a)(w-y)}{(1-y)z(1-a)+(a-z)(1-w)}, \\ w + (a-z)(1-w)\dfrac{y-w}{(1-y)z(1-a)+(a-z)(1-w)}. \end{cases} \tag{2.11}$$

(ii) There exist functions $Q_i, i = 1,2,3$, such that

$$\begin{aligned} Q_y(y,z,w) &= \frac{a(1-z)(a-z)(1-w)^2}{[(1-y)z(1-a)+(a-z)(1-w)]^2} = (a-z)(1-w)Q_1(y,z,w), \\ Q_z(y,z,w) &= \frac{(1-a)a(1-y)(1-w)(w-y)}{[(1-y)z(1-a)+(a-z)(1-w)]^2} = (1-y)(1-w)Q_2(y,z,w), \\ Q_w(y,z,w) &= \frac{a(1-a)(1-y)^2z(1-z)}{[(1-y)z(1-a)+(a-z)(1-w)]^2} = (1-y)zQ_3(y,z,w) \ge 0. \end{aligned}$$

(iii) There exist functions $R_j, j = 1,\cdots,16$, such that

$$\begin{aligned} Q_{yy}(y,z,w) =& zR_1(y,z,w) = (a-z)R_2(y,z,w) = (1-w)R_3(y,z,w), \\ Q_{zz}(y,z,w) =& (1-y)R_4(y,z,w) = (1-w)R_5(y,z,w) \\ =& yR_6(y,z,w) + (w-a)R_7(y,z,w), \\ Q_{ww}(y,z,w) =& (1-y)R_8(y,z,w) = zR_9(y,z,w) = (a-z)R_{10}(y,z,w), \\ Q_{yz}(y,z,w) =& (1-w)R_{11}(y,z,w), Q_{zw}(y,z,w) = (1-y)R_{12}(y,z,w), \\ Q_{yw}(y,z,w) =& (1-y)R_{13}(y,z,w) = zR_{14}(y,z,w) \\ =& (a-z)R_{15}(y,z,w) + (1-w)R_{16}(y,z,w). \end{aligned}$$

By using this auxiliary function Q, we can construct a suitable pair of super-sub-solutions. Now let $u(x,t) = U(\xi,t)$ with $\xi = x+\bar{c}t$ and $\bar{c} = (c_1+c_2)/2 = (\hat{v}_1+\tilde{v}_2)/2$. Then (1.1) becomes

$$U_t = J * U - U - \bar{c}U_\xi - f(U). \tag{2.12}$$

Taking $s_1 = (c_2-c_1)/2$, $s_2 = c_3-\bar{c}$, it is obvious to see that $\phi_1(\xi-s_1t)$, $\phi_2(\xi+s_1t)$ and $\phi_3(\xi+s_2t)$ are traveling fronts of (2.12). We denote

$$F[U(\xi,t)] := U_t + \bar{c}U_\xi - (J*U-U) - f(U).$$

Consider $U(\xi,t) = Q\big(\phi_1(\xi - p_1(t)), \phi_2(\xi + p_2(t)), \phi_3(\xi + p_3(t))\big)$, where $p_1(t)$, $p_2(t)$, $p_3(t) < 0$ and $-p_2(t) < -p_3(t)$. Then we can caculate

$$
\begin{aligned}
&F[U(\xi,t)]\\
=&-p_1'Q_y\phi_1' + p_2'Q_z\phi_2' + p_3'Q_w\phi_3' + \bar{c}(Q_y\phi_1' + Q_z\phi_2' + Q_w\phi_3') - (J*U-U) - f(U(\xi,t))\\
=&-p_1'Q_y\phi_1' + p_2'Q_z\phi_2' + p_3'Q_w\phi_3' + (\bar{c}-c_1)Q_y\phi_1' + (\bar{c}-c_2)Q_z\phi_2' + (\bar{c}-c_3)Q_w\phi_3'\\
&-(J*U-U) - f(U(\xi,t)) + c_1Q_y\phi_1' + c_2Q_z\phi_2' + c_3Q_w\phi_3'\\
=&-(p_1'-s_1)Q_y\phi_1' + (p_2'-s_1)Q_z\phi_2' + (p_3'-s_2)Q_w\phi_3' - (J*U-U) - f(U(\xi,t))\\
&+Q_y[(J*\phi_1-\phi_1) - f(\phi_1)] + Q_z[(J*\phi_2-\phi_2) - f(\phi_2)] + Q_w[(J*\phi_3-\phi_3) - f(\phi_3))]\\
=&-(p_1'-s_1)Q_y\phi_1' + (p_2'-s_1)Q_z\phi_2' + (p_3'-s_2)Q_w\phi_3' + Q_y(J*\phi_1-\phi_1) + Q_z(J*\phi_2-\phi_2)\\
&+Q_w(J*\phi_3-\phi_3) - (J*U-U) + Q_yf(\phi_1) + Q_zf(\phi_2) + Q_wf(\phi_3) - f(U(\xi,t))\\
=&-(p_1'-s_1)Q_y\phi_1' + (p_2'-s_1)Q_z\phi_2' + (p_3'-s_2)Q_w\phi_3' - \mathcal{G}(\phi_1,\phi_2,\phi_3) - \mathcal{F}(\phi_1,\phi_2,\phi_3),
\end{aligned}
$$

where

$$
\begin{aligned}
&\mathcal{F}(\phi_1,\phi_2,\phi_3) := f(U(\xi,t)) - Q_yf(\phi_1) - Q_zf(\phi_2) - Q_wf(\phi_3)\\
&\mathcal{G}(\phi_1,\phi_2,\phi_3) := (J*U-U) - Q_y(J*\phi_1-\phi_1) - Q_z(J*\phi_2-\phi_2) - Q_w(J*\phi_3-\phi_3).
\end{aligned}
$$

Lemma 2.5. *[6, Lemma 3.2] Assume that (J), (A1) and (A2) hold. If $p_1(t), p_2(t), p_3(t) \le -\sigma$ for some positive constant σ, then there exist positive constants ϵ_1, ϵ_2 and ϵ_3 such that*

$$
\begin{aligned}
&Q_y(\phi_1(\xi-p_1(t)), \phi_2(\xi+p_2(t)), \phi_3(\xi+p_3(t))) \ge \epsilon_1, \quad \textit{for } \xi \le -p_2(t),\\
&Q_z(\phi_1(\xi-p_1(t)), \phi_2(\xi+p_2(t)), \phi_3(\xi+p_3(t))) \ge \epsilon_2, \quad \textit{for } p_1(t) \le \xi \le -p_3(t),\\
&Q_w(\phi_1(\xi-p_1(t)), \phi_2(\xi+p_2(t)), \phi_3(\xi+p_3(t))) \ge \epsilon_3, \quad \textit{for } \xi \ge -p_3(t).
\end{aligned}
$$

According to the above arguments, we can easily obtain that there exists a positive constant C such that

$$
\begin{aligned}
&|R_l(\phi_1,\phi_2,\phi_3)|, |Q_{yy}(\phi_1,\phi_2,\phi_3)|, |Q_{zz}(\phi_1,\phi_2,\phi_3)||Q_{ww}(\phi_1,\phi_2,\phi_3)|\\
&|Q_{yz}(\phi_1,\phi_2,\phi_3)|, |Q_{zw}(\phi_1,\phi_2,\phi_3)|, |Q_{yw}(\phi_1,\phi_2,\phi_3)| \le C
\end{aligned}
$$

for $\xi \in \mathbb{R}$, $p_1(t)$, $p_2(t)$, $p_3(t) < -\sigma$, $l = 1, \cdots, 16$.

Denote

$$
\begin{aligned}
&\mathcal{A}(\phi_1(\xi-p_1(t)), \phi_2(\xi+p_2(t)), \phi_3(\xi+p_3(t)))\\
=&-Q_y(\phi_1,\phi_2,\phi_3)\phi_1'(\xi-p_1(t) + Q_z(\phi_1,\phi_2,\phi_3)\phi_2'(\xi+p_2(t)) + Q_w(\phi_1,\phi_2,\phi_3)\phi_3'(\xi+p_3(t))).
\end{aligned}
$$

The next lemma shows some properties of the function $\mathcal{A}$.

Lemma 2.6. *[6, Lemma 3.3]Assume that (J), (A1) and (A2) hold. Then there exists a large number $\sigma > 0$ such that*

$$
\mathcal{A}(\phi_1(\xi-p_1(t)), \phi_2(\xi+p_2(t)), \phi_3(\xi+p_3(t))) > 0 \quad \textit{for any } \xi \in \mathbb{R},\ p_1, p_2, p_3 \le -\sigma. \tag{2.13}
$$

More precisely, for $p_1, p_2, p_3 \le -\sigma$, the following statements hold:

$$
\mathcal{A}(\phi_1,\phi_2,\phi_3) \ge \frac{1}{2}Q_y|\phi_1'(\xi-p_1(t))| \quad \textit{for } \xi \le p_1(t), \tag{2.14}
$$

$$\mathcal{A}(\phi_1,\phi_2,\phi_3) \geq \frac{1}{2}\Big[Q_1|\phi_1'(\xi-p_1(t))| + Q_2|\phi_2'(\xi+p_2(t))|\Big] \ \textit{for}\ p_1(t) \leq \xi \leq -p_2(t), \tag{2.15}$$

$$\mathcal{A}(\phi_1,\phi_2,\phi_3) \geq \frac{1}{2}\Big[Q_2|\phi_2'(\xi+p_2(t))| + Q_3|\phi_3'(\xi+p_3(t))|\Big] \ \textit{for}\ -p_2(t) \leq \xi \leq -p_3(t), \tag{2.16}$$

$$\mathcal{A}(\phi_1,\phi_2,\phi_3) \geq \frac{1}{2}Q_3|\phi_3'(\xi+p_3(t))| \ \ \textit{for}\ \xi \geq -p_3(t). \tag{2.17}$$

Notice that the expression of $\mathcal{F}$ is the same as that in [6], thus we can obtain the following lemma.

Lemma 2.7. *Assume that (J), (A1) and (A2) hold. Then*

$$\Big|\frac{\mathcal{F}(\phi_1,\phi_2,\phi_3)}{\mathcal{A}(\phi_1,\phi_2,\phi_3)}\Big| \leq \begin{cases} L_1(e^{\eta_1 p_2(t)} + e^{\eta_1 p_3(t)}), & \textit{for}\ \xi \leq 0, \\ L_2(e^{\eta_2 p_1(t)} + e^{\eta_1 (p_3(t)-p_2(t))/2}), & \textit{for}\ 0 \leq \xi \leq (-p_3(t)-p_2(t))/2, \\ L_3(e^{\eta_2 p_1(t)} + e^{\eta_2 (p_3(t)-p_2(t))/2}), & \textit{for}\ \xi \geq (-p_3(t)-p_2(t))/2, \end{cases}$$

where L_1, L_2, L_3 are positive constants.

Lemma 2.8. *Assume that (J), (A1) and (A2) hold. Then*

$$\Big|\frac{\mathcal{G}(\phi_1,\phi_2,\phi_3)}{\mathcal{A}(\phi_1,\phi_2,\phi_3)}\Big| \leq \begin{cases} L_4(e^{\eta_1 p_2(t)} + e^{\eta_1 p_3(t)}), & \textit{for}\ \xi \leq 0, \\ L_5\left(e^{\eta_2 p_1(t)} + e^{\eta_1 (p_3(t)-p_2(t))/2}\right), & \textit{for}\ 0 \leq \xi \leq (-p_3(t)-p_2(t))/2, \\ L_6\left(e^{\eta_2 p_1(t)} + e^{\eta_2 (p_3(t)-p_2(t))/2}\right), & \textit{for}\ \xi \geq (-p_3(t)-p_2(t))/2, \end{cases}$$

where L_4, L_5, L_6 are positive constants.

Proof. *By using the similar method as in [21], here we give the details for convenience. We denote*

$$\hat{\phi}_1(\theta) := \phi_1(\xi-p_1(t)-\theta s),\ \hat{\phi}_2(\theta) := \phi_2(\xi+p_2(t)-\theta s)\ \textit{and}\ \hat{\phi}_3(\theta) := \phi_3(\xi+p_3(t)-\theta s)$$

for $\theta \in [0,1]$, $s \in \mathbb{R}$. Then we have

$$\begin{aligned}
\mathcal{G}(\phi_1,\phi_2,\phi_3) = & \int_{\mathbb{R}} J(s)[Q(\hat{\phi}_1(1),\hat{\phi}_2(1),\hat{\phi}_3(1)) - Q(\hat{\phi}_1(0),\hat{\phi}_2(0),\hat{\phi}_3(0))]ds \\
& - Q_y\int_{\mathbb{R}} J(s)[\hat{\phi}_1(1)-\hat{\phi}_1(0)]ds - Q_z\int_{\mathbb{R}} J(s)[\hat{\phi}_2(1)-\hat{\phi}_2(0)]ds \\
& - Q_w\int_{\mathbb{R}} J(s)[\hat{\phi}_3(1)-\hat{\phi}_3(0)]ds \\
= & \int_{\mathbb{R}} J(s)Q_y\big(\theta_1\hat{\phi}_1(1)+(1-\theta_1)\hat{\phi}_1(0),\hat{\phi}_2(1),\hat{\phi}_3(1)\big)[\hat{\phi}_1(1)-\hat{\phi}_1(0)]ds \\
& + \int_{\mathbb{R}} J(s)Q_z\big(\hat{\phi}_1(0),\theta_2\hat{\phi}_2(1)+(1-\theta_2)\hat{\phi}_2(0),\hat{\phi}_3(1)\big)[\hat{\phi}_2(1)-\hat{\phi}_2(0)]ds \\
& + \int_{\mathbb{R}} J(s)Q_w\big(\hat{\phi}_1(0),\hat{\phi}_2(0),\theta_3\hat{\phi}_3(1)+(1-\theta_3)\hat{\phi}_3(0)\big)[\hat{\phi}_3(1)-\hat{\phi}_3(0)]ds \\
& - \int_{\mathbb{R}} J(s)Q_y(\hat{\phi}_1(0),\hat{\phi}_2(0),\hat{\phi}_3(0))[\hat{\phi}_1(1)-\hat{\phi}_1(0)]ds
\end{aligned}$$

$$
\begin{aligned}
&-\int_{\mathbb{R}} J(s)Q_z(\hat{\phi}_1(0),\hat{\phi}_2(0),\hat{\phi}_3(0))[\hat{\phi}_2(1)-\hat{\phi}_2(0)]ds\\
&-\int_{\mathbb{R}} J(s)Q_w(\hat{\phi}_1(0),\hat{\phi}_2(0),\hat{\phi}_3(0))[\hat{\phi}_3(1)-\hat{\phi}_3(0)]ds\\
=&\int_{\mathbb{R}} J(s)\mathcal{G}_1[\hat{\phi}_1(1)-\hat{\phi}_1(0)]ds+\int_{\mathbb{R}} J(s)\mathcal{G}_2[\hat{\phi}_2(1)-\hat{\phi}_2(0)]ds\\
&+\int_{\mathbb{R}} J(s)\mathcal{G}_3[\hat{\phi}_3(1)-\hat{\phi}_3(0)]ds,
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{G}_1 :=&Q_y\big(\theta_1\hat{\phi}_1(1)+(1-\theta_1)\hat{\phi}_1(0),\hat{\phi}_2(1),\hat{\phi}_3(1)\big)-Q_y(\hat{\phi}_1(0),\hat{\phi}_2(0),\hat{\phi}_3(0)),\\
\mathcal{G}_2 :=&Q_z\big(\hat{\phi}_1(0),\theta_2\hat{\phi}_2(1)+(1-\theta_2)\hat{\phi}_2(0),\hat{\phi}_3(1)\big)-Q_y(\hat{\phi}_1(0),\hat{\phi}_2(0),\hat{\phi}_3(0)),\\
\mathcal{G}_3 :=&Q_w\big(\hat{\phi}_1(0),\hat{\phi}_2(0),\theta_3\hat{\phi}_3(1)+(1-\theta_3)\hat{\phi}_3(0)\big)-Q_w(\hat{\phi}_1(0),\hat{\phi}_2(0),\hat{\phi}_3(0)).
\end{aligned}
$$

By a direct computation, we have

$$
\begin{aligned}
\mathcal{G}_1 =&Q_{yy}(\theta_4\hat{\phi}_1(1)+(1-\theta_4)\hat{\phi}_1(0),\hat{\phi}_2(1),\hat{\phi}_3(1))\theta_1[\hat{\phi}_1(1)-\hat{\phi}_1(0)]\\
&+Q_{yz}(\hat{\phi}_1(0),\theta_5\hat{\phi}_2(1)+(1-\theta_5)\hat{\phi}_2(0),\hat{\phi}_3(1))[\hat{\phi}_2(1)-\hat{\phi}_2(0)]\\
&+Q_{yw}(\hat{\phi}_1(0),\hat{\phi}_2(0),\theta_6\hat{\phi}_3(1)+(1-\theta_6)\hat{\phi}_3(0))[\hat{\phi}_3(1)-\hat{\phi}_3(0)],\\
\mathcal{G}_2 =&Q_{zz}(\hat{\phi}_1(0),\theta_7\hat{\phi}_2(1)+(1-\theta_7)\hat{\phi}_2(0),\hat{\phi}_3(1))\theta_2[\hat{\phi}_2(1)-\hat{\phi}_2(0)]\\
&+Q_{zw}(\hat{\phi}_1(0),\hat{\phi}_2(0),\theta_8\hat{\phi}_3(1)+(1-\theta_8)\hat{\phi}_3(0))[\hat{\phi}_3(1)-\hat{\phi}_3(0)],\\
\mathcal{G}_3 =&Q_{ww}(\hat{\phi}_1(0),\hat{\phi}_2(0),\theta_9\hat{\phi}_3(1)+(1-\theta_9)\hat{\phi}_3(0))\theta_3[\hat{\phi}_3(1)-\hat{\phi}_3(0)].
\end{aligned}
$$

Let

$$
\begin{aligned}
\mathcal{G}_{11} &= Q_{yy}\big(\theta_4\hat{\phi}_1(1)+(1-\theta_4)\hat{\phi}_1(0),\hat{\phi}_2(1),\hat{\phi}_3(1)\big)\theta_1[\hat{\phi}_1(1)-\hat{\phi}_1(0)]^2,\\
\mathcal{G}_{22} &= Q_{yz}\big(\hat{\phi}_1(0),\theta_5\hat{\phi}_2(1)+(1-\theta_5)\hat{\phi}_2(0),\hat{\phi}_3(1)\big)[\hat{\phi}_1(1)-\hat{\phi}_1(0)][\hat{\phi}_2(1)-\hat{\phi}_2(0)],\\
\mathcal{G}_{33} &= Q_{yw}\big(\hat{\phi}_1(0),\hat{\phi}_2(0),\theta_6\hat{\phi}_3(1)+(1-\theta_6)\hat{\phi}_3(0)\big)[\hat{\phi}_1(1)-\hat{\phi}_1(0)][\hat{\phi}_3(1)-\hat{\phi}_3(0)],\\
\mathcal{G}_{44} &= Q_{zz}\big(\hat{\phi}_1(0),\theta_7\hat{\phi}_2(1)+(1-\theta_7)\hat{\phi}_2(0),\hat{\phi}_3(1)\big)\theta_2[\hat{\phi}_2(1)-\hat{\phi}_2(0)]^2,\\
\mathcal{G}_{55} &= Q_{zw}\big(\hat{\phi}_1(0),\hat{\phi}_2(0),\theta_8\hat{\phi}_3(1)+(1-\theta_8)\hat{\phi}_3(0)\big)[\hat{\phi}_2(1)-\hat{\phi}_2(0)][\hat{\phi}_3(1)-\hat{\phi}_3(0)],\\
\mathcal{G}_{66} &= Q_{ww}\big(\hat{\phi}_1(0),\hat{\phi}_2(0),\theta_9\hat{\phi}_3(1)+(1-\theta_9)\hat{\phi}_3(0)\big)\theta_3[\hat{\phi}_3(1)-\hat{\phi}_3(0)]^2,
\end{aligned}
$$

where $\theta_i\in(0,1)$, $i=1,\cdots,9$. *Then by mean theorem,* $\mathcal{G}_{11}$, $\mathcal{G}_{22}$, $\mathcal{G}_{33}$, $\mathcal{G}_{44}$, $\mathcal{G}_{55}$ *and* $\mathcal{G}_{66}$ *can be rewritten as*

$$
\begin{aligned}
\mathcal{G}_{11} &= Q_{yy}\big(\theta_4\hat{\phi}_1(1)+(1-\theta_4)\hat{\phi}_1(0),\hat{\phi}_2(1),\hat{\phi}_3(1)\big)\theta_1 s^2[\phi_1'(\xi-p_1(t)-\theta_{10}s)]^2,\\
\mathcal{G}_{22} &= Q_{yz}\big(\hat{\phi}_1(0),\theta_5\hat{\phi}_2(1)+(1-\theta_5)\hat{\phi}_2(0),\hat{\phi}_3(1)\big)s^2\phi_1'(\xi-p_1(t)-\theta_{11}s)\phi_2'(\xi+p_2(t)-\theta_{12}s),\\
\mathcal{G}_{33} &= Q_{yw}\big(\hat{\phi}_1(0),\hat{\phi}_2(0),\theta_6\hat{\phi}_3(1)+(1-\theta_6)\hat{\phi}_3(0)\big)s^2\phi_1'(\xi-p_1(t)-\theta_{13}s)\phi_3'(\xi+p_3(t)-\theta_{14}s),\\
\mathcal{G}_{44} &= Q_{zz}\big(\hat{\phi}_1(0),\theta_7\hat{\phi}_2(1)+(1-\theta_7)\hat{\phi}_2(0),\hat{\phi}_3(1)\big)\theta_2 s^2[\phi_2'(\xi+p_2(t)-\theta_{15}s)]^2,\\
\mathcal{G}_{55} &= Q_{zw}\big(\hat{\phi}_1(0),\hat{\phi}_2(0),\theta_8\hat{\phi}_3(1)+(1-\theta_8)\hat{\phi}_3(0)\big)s^2\phi_2'(\xi+p_2(t)-\theta_{16}s)\phi_3'(\xi+p_3(t)-\theta_{17}s),\\
\mathcal{G}_{66} &= Q_{ww}\big(\hat{\phi}_1(0),\hat{\phi}_2(0),\theta_9\hat{\phi}_3(1)+(1-\theta_9)\hat{\phi}_3(0)\big)\theta_3 s^2[\phi_3'(\xi+p_3(t)-\theta_{18}s)]^2,
\end{aligned}
$$

where $\theta_i\in(0,1)$, $i=4,\ldots,18$, $s\in[-M,M]$.

We now divide the proof into the following six cases.

Case 1. $\xi \le p_1(t)$. *From Lemma 2.4*(iii), *one can show that there exists a general positive constant* C' *such that*

$$
\begin{aligned}
&|Q_{yy}\big(\theta_4\hat{\phi}_1(1)+(1-\theta_4)\hat{\phi}_1(0),\hat{\phi}_2(1),\hat{\phi}_3(1)\big)| \le C'|\hat{\phi}_2(1)|\\
&|Q_{zz}\big(\hat{\phi}_1(0),\theta_7\hat{\phi}_2(1)+(1-\theta_7)\hat{\phi}_2(0),\hat{\phi}_3(1)\big)| \le C'|1-\hat{\phi}_1(0)|,\\
&|Q_{ww}\big(\hat{\phi}_1(0),\hat{\phi}_2(0),\theta_9\hat{\phi}_3(1)+(1-\theta_9)\hat{\phi}_3(0)\big)| \le C'|1-\hat{\phi}_1(0)|,\\
&|Q_{yz}\big(\hat{\phi}_1(0),\theta_5\hat{\phi}_2(1)+(1-\theta_5)\hat{\phi}_2(0),\hat{\phi}_3(1)\big)| \le C',\\
&|Q_{yw}\big(\hat{\phi}_1(0),\hat{\phi}_2(0),\theta_6\hat{\phi}_3(1)+(1-\theta_6)\hat{\phi}_3(0)\big)| \le C',\\
&|Q_{zw}\big(\hat{\phi}_1(0),\hat{\phi}_2(0),\theta_8\hat{\phi}_3(1)+(1-\theta_8)\hat{\phi}_3(0)\big)| \le C'|1-\hat{\phi}_1(0)|,
\end{aligned}
$$

and

$$
\begin{aligned}
&|\mathcal{G}_{11}| \le C'\theta_1|\phi_2(1)|s^2[\phi_1'(\xi-p_1(t)-\theta_{10}s)]^2,\\
&|\mathcal{G}_{44}| \le C'\theta_2 s^2|1-\hat{\phi}_1(0)|[\phi_2'(\xi+p_2(t)-\theta_{15}s)]^2,\\
&|\mathcal{G}_{66}| \le C'\theta_3 s^2|1-\hat{\phi}_1(0)|[\phi_3'(\xi+p_3(t)-\theta_{18}s)]^2,\\
&|\mathcal{G}_{22}| \le C's^2|\phi_1'(\xi-p_1(t)-\theta_{11}s)||\phi_2'(\xi+p_2(t)-\theta_{21}s)|,\\
&|\mathcal{G}_{33}| \le C's^2|\phi_1'(\xi-p_1(t)-\theta_{31}s)||\phi_3'(\xi+p_3(t)-\theta_{14}s)|,\\
&|\mathcal{G}_{55}| \le C'|1-\hat{\phi}_1(0)||s^2\phi_2'(\xi+p_2(t)-\theta_{16}s)||\phi_3'(\xi+p_3(t)-\theta_{17}s)|.
\end{aligned}
$$

Since $\xi \le p_1(t)$, *we have* $\xi-p_1-\theta_i s \le M$, $\xi+p_2-\theta_i s \le M$ *and* $\xi+p_3-\theta_i s \le M$. *Then, by Lemma 2.1, Lemma 2.5 and* (2.14), *there exists some positive constant* $\tilde{L}_1$ *such that*

$$
\begin{aligned}
\Big|\frac{\mathcal{G}}{\mathcal{A}}\Big| \le& \frac{2|\mathcal{G}_{11}|}{Q_y|\phi_1'|}+\frac{2|\mathcal{G}_{44}|}{Q_y|\phi_1'|}+\frac{2|\mathcal{G}_{66}|}{Q_y|\phi_1'|}+\frac{2|\mathcal{G}_{22}|}{Q_y|\phi_1'|}+\frac{2|\mathcal{G}_{33}|}{Q_y|\phi_1'|}+\frac{2|\mathcal{G}_{55}|}{Q_y|\phi_1'|}\\
\le& 2C's^2\Big[\frac{|\hat{\phi}_2(1)|[\phi_1'(\xi-p_1(t)-\theta_{10}s)]^2}{\epsilon_1|\phi_1'|}+\frac{|1-\hat{\phi}_1(0)|[\phi_2'(\xi+p_2(t)-\theta_{15}s)]^2}{\epsilon_1|\phi_1'|}\\
&+\frac{|1-\hat{\phi}_1(0)|[\phi_3'(\xi+p_3-\theta_{18}s)]^2}{\epsilon_1|\phi_1'|}+\frac{|\phi_1'(\xi-p_1-\theta_{11}s)||\phi_2'(\xi+p_2-\theta_{21}s)|}{\epsilon_1|\phi_1'|}\\
&+\frac{|\phi_1'(\xi-p_1-\theta_{31}s)\phi_3'(\xi+p_3-\theta_{14}s)|}{\epsilon_1|\phi_1'|}\\
&+\frac{|1-\hat{\phi}_1(0)\phi_2'(\xi+p_2-\theta_{16}s)\phi_3'(\xi+p_3-\theta_{17}s)|}{\epsilon_1|\phi_1'|}\Big]\\
\le& \frac{2C's^2C_0^2C_2e^{\eta_1(\xi+p_2-s)}e^{2\eta_1(\xi-p_1-\theta_{10}s)}}{\epsilon_1 C_1 e^{\eta_1(\xi-p_1)}/\rho}+\frac{2C's^2C_0^2C_2e^{\eta_1(\xi-p_1)}e^{2\eta_1(\xi+p_2-\theta_{15}s)}}{\epsilon_1 C_1 e^{\eta_1(\xi-p_1)}/\rho}\\
&+\frac{2C's^2C_0^2C_2e^{\eta_1(\xi-p_1)}e^{2\eta_1(\xi+p_3-\theta_{18}s)}}{\epsilon_1 C_1 e^{\eta_1(\xi-p_1)}/\rho}+\frac{2C's^2C_0^2e^{\eta_1(\xi-p_1)-\theta_{11}s)}e^{\eta_1(\xi+p_2-\theta_{21}s)}}{\epsilon_1 C_1 e^{\eta_1(\xi-p_1)}/\rho}\\
&+\frac{2C's^2C_0^2e^{\eta_1(\xi-p_1-\theta_{31}s)}e^{\eta_1(\xi+p_3-\theta_{14}s)}}{\epsilon_1 C_1 e^{\eta_1(\xi-p_1)}/\rho}+\frac{2C's^2C_0^2C_2e^{\eta_1(\xi-p_1)}e^{\eta_1(\xi+p_2-\theta_{16}s)}e^{\eta_1(\xi+p_3-\theta_{17}s)}}{\epsilon_1 C_1 e^{\eta_1(\xi-p_1)}/\rho}\\
\le& \tilde{L}_1\big[e^{\eta_1(\xi+p_2(t))}+e^{\eta_1(\xi+p_3(t))}\big] \le \tilde{L}_1\big[e^{\eta_1 p_2(t)}+e^{\eta_1 p_3(t)}\big].
\end{aligned}
$$

Case 2. $p_1(t) \le \xi \le 0$. *From Lemma 2.4*(iii), *one can show that there exists a* *general positive constant* C' *, it follows that*

$$\begin{aligned}
&|Q_{yy}\big(\theta_4\hat{\phi}_1(1)+(1-\theta_4)\hat{\phi}_1(0),\hat{\phi}_2(1),\hat{\phi}_3(1)\big)|\le C'|\hat{\phi}_2(1)|,\\
&|Q_{zz}\big(\hat{\phi}_1(0),\theta_7\hat{\phi}_2(1)+(1-\theta_7)\hat{\phi}_2(0),\hat{\phi}_3(1)\big)|\le C',\\
&|Q_{ww}\big(\hat{\phi}_1(0),\hat{\phi}_2(0),\theta_9\hat{\phi}_3(1)+(1-\theta_9)\hat{\phi}_3(0)\big)|\le C'|\hat{\phi}_2(0)|,\\
&|Q_{yz}\big(\hat{\phi}_1(0),\theta_5\hat{\phi}_2(1)+(1-\theta_5)\hat{\phi}_2(0),\hat{\phi}_3(1)\big)|\le C',\\
&|Q_{yw}\big(\hat{\phi}_1(0),\hat{\phi}_2(0),\theta_8\hat{\phi}_3(1)+(1-\theta_8)\hat{\phi}_3(0)\big)|\le C',\\
&|Q_{zw}\big(\hat{\phi}_1(0),\hat{\phi}_2(0),\theta_6\hat{\phi}_3(1)+(1-\theta_6)\hat{\phi}_3(0)\big)|\le C',
\end{aligned}$$

and

$$\begin{aligned}
&|\mathcal{G}_{11}|\le C'|\hat{\phi}_2(1)|\theta_1 s^2[\phi_1'(\xi-p_1(t)-\theta_{10}s)]^2,\\
&|\mathcal{G}_{44}|\le C'\theta_2 s^2[\phi_2'(\xi+p_2(t)-\theta_{15}s)]^2,\\
&|\mathcal{G}_{66}|\le C'|\hat{\phi}_2(0)|\theta_3 s^2|\hat{\phi}_2(0)|[\phi_3'(\xi+p_3(t)-\theta_{18}s)]^2,\\
&|\mathcal{G}_{22}|\le C's^2|\phi_1'(\xi-p_1(t)-\theta_{11}s)||\phi_2'(\xi+p_2(t)-\theta_{21}s)|,\\
&|\mathcal{G}_{33}|\le C's^2|\phi_2'(\xi+p_2(t)-\theta_{16}s)||\phi_3'(\xi+p_3(t)-\theta_{17}s)|,\\
&|\mathcal{G}_{55}|\le C's^2|\phi_1'(\xi-p_1(t)-\theta_{31}s)||\phi_3'(\xi+p_3(t)-\theta_{14}s)|.
\end{aligned}$$

Since $p_1(t)\le\xi\le 0$*, we have* $\xi-p_1-\theta_i\ge -M$*,* $\xi+p_2-\theta_i\le M$ *and* $\xi+p_3-\theta_i\le M$*, then it follows from Lemma 2.1, Lemma 2.5 and* (2.15) *that there exists a* $\tilde{L}_2$ *such that*

$$\begin{aligned}
\Big|\frac{\mathcal{G}}{\mathcal{A}}\Big|\le&\frac{2|\mathcal{G}_{11}|}{Q_y|\phi_1'|}+\frac{2|\mathcal{G}_{44}|}{Q_z|\phi_2'|}+\frac{2|\mathcal{G}_{66}|}{Q_z|\phi_2'|}+\frac{2|\mathcal{G}_{22}|}{Q_y|\phi_1'|}+\frac{2|\mathcal{G}_{33}|}{Q_y|\phi_1'|}+\frac{2|\mathcal{G}_{55}|}{Q_z|\phi_2'|}\\
\le&2C's^2\Big[\frac{|\hat{\phi}_2(1)|[\phi_1'(\xi-p_1-\theta_{10}s)]^2}{\epsilon_1|\phi_1'|}+\frac{[\phi_2'(\xi+p_2-\theta_{15}s)]^2}{\epsilon_2|\phi_2'|}\\
&+\frac{|\hat{\phi}_2(0)|[\phi_3'(\xi+p_3-\theta_{18}s)]^2}{\epsilon_2|\phi_2'|}+\frac{|\phi_1'(\xi-p_1-\theta_{11}s)||\phi_2'(\xi+p_2-\theta_{21}s)|}{\epsilon_1|\phi_1'|}\\
&+\frac{|\phi_1'(\xi-p_1-\theta_{31}s)\phi_3'(\xi+p_3-\theta_{14}s)|}{\epsilon_1|\phi_1'|}+\frac{|\phi_2'(\xi+p_2-\theta_{16}s)\phi_3'(\xi+p_3-\theta_{17}s)|}{\epsilon_2|\phi_2'|}\Big]\\
\le&2C's^2C_0^2\Big[\frac{C_2e^{\eta_1(\xi+p_2(t)-s)}e^{-2\eta_2(\xi-p_1(t)-\theta_{10}s)}}{\epsilon_1C_1e^{-\eta_2(\xi-p_1(t))}/\rho}+\frac{e^{2\eta_1(\xi+p_2(t)-\theta_{15}s)}}{\epsilon_2C_1e^{\eta_1(\xi+p_2(t))}/\rho}\\
&+\frac{C_2e^{\eta_1(\xi+p_2(t))}e^{2\eta_1(\xi+p_3(t)-\theta_{18}s)}}{\epsilon_2C_1e^{\eta_1(\xi+p_2(t))}/\rho}+\frac{e^{-\eta_2(\xi-p_1(t)-\theta_{11}s)}e^{\eta_1(\xi+p_2(t)-\theta_{21}s)}}{\epsilon_1C_1e^{-\eta_2(\xi-p_1(t))}/\rho}\\
&+\frac{e^{-\eta_2(\xi-p_1(t)-\theta_{31}s)}e^{\eta_1(\xi+p_3(t)-\theta_{14}s)}}{\epsilon_1C_1e^{-\eta_2(\xi-p_1(t))}/\rho}+\frac{e^{\eta_1(\xi+p_2(t)-\theta_{16}s)}e^{\eta_1(\xi+p_3(t)-\theta_{17}s)}}{\epsilon_2C_1e^{\eta_1(\xi+p_2(t))}/\rho}\Big]\\
\le&\tilde{L}_2\big[e^{\eta_1(\xi+p_2(t))}+e^{\eta_1(\xi+p_3(t))}\big]\le\tilde{L}_2\big[e^{\eta_1p_2(t)}+e^{\eta_1p_3(t)}\big].
\end{aligned}$$

By **Case 1** *and* **Case 2***, we can obtain*

$$\Big|\frac{\mathcal{G}(\phi_1,\phi_2,\phi_3)}{\mathcal{A}\phi_1,\phi_2,\phi_3)}\Big|\le L_4(e^{\eta_1p_2(t)}+e^{\eta_1p_3(t)}),\quad for\ \xi\le 0,$$

where $L_4=\max\{\tilde{L}_1,\tilde{L}_2\}$.

Case 3. $0 \le \xi \le -p_2(t)$. *From Lemma 2.4*(iii)*, one can show that there exists a general positive constant* C' *such that*

$$
\begin{aligned}
&|Q_{yy}\big(\theta_4\hat{\phi}_1(1)+(1-\theta_4)\hat{\phi}_1(0),\hat{\phi}_2(1),\hat{\phi}_3(1)\big)| \le C',\\
&|Q_{ww}\big(\hat{\phi}_1(0),\theta_7\hat{\phi}_2(1)+(1-\theta_7)\hat{\phi}_2(0),\hat{\phi}_3(1)\big)| \le C'(|\hat{\phi}_1(0)|+|\hat{\phi}_3(1)-a|),\\
&|Q_{ww}\big(\hat{\phi}_1(0),\hat{\phi}_2(0),\theta_9\hat{\phi}_3(1)+(1-\theta_9)\hat{\phi}_3(0)\big)| \le C'|\hat{\phi}_2(0)|,\\
&|Q_{yz}\big(\hat{\phi}_1(0),\theta_5\hat{\phi}_2(1)+(1-\theta_5)\hat{\phi}_2(0),\hat{\phi}_3(1)\big)| \le C',\\
&|Q_{zw}\big(\hat{\phi}_1(0),\hat{\phi}_2(0),\theta_8\hat{\phi}_3(1)+(1-\theta_8)\hat{\phi}_3(0)\big)| \le C',\\
&|Q_{yw}\big(\hat{\phi}_1(0),\hat{\phi}_2(0),\theta_6\hat{\phi}_3(1)+(1-\theta_6)\hat{\phi}_3(0)\big)| \le C',
\end{aligned}
$$

and

$$
\begin{aligned}
&|\mathcal{G}_{11}| \le C's^2\theta_1[\phi_1'(\xi-p_1(t)-\theta_{10}s)]^2,\\
&|\mathcal{G}_{44}| \le C'(|\hat{\phi}_1(0)|+|\hat{\phi}_3(1)-a|)\theta_2 s^2[\phi_2'(\xi+p_2(t)-\theta_{15}s)]^2,\\
&|\mathcal{G}_{66}| \le C'|\hat{\phi}_2(0)|\theta_3 s^2[\phi_3'(\xi+p_3(t)-\theta_{18}s)]^2,\\
&|\mathcal{G}_{22}| \le C's^2|\phi_1'(\xi-p_1(t)-\theta_{11}s)||\phi_2'(\xi+p_2(t)-\theta_{21}s)|,\\
&|\mathcal{G}_{33}| \le C's^2|\phi_1'(\xi-p_1(t)-\theta_{31}s)||\phi_3'(\xi+p_3(t)-\theta_{14}s)|,\\
&|\mathcal{G}_{55}| \le C'|s^2\phi_2'(\xi+p_2(t)-\theta_{16}s)||\phi_3'(\xi+p_3(t)-\theta_{17}s)|.
\end{aligned}
$$

By Lemma 2.1, Lemma 2.5 and (2.15)*, there exists a positive constant* $\tilde{L}_3$ *such that*

$$
\begin{aligned}
\Big|\frac{\mathcal{G}}{\mathcal{A}}\Big| \le& \frac{2|\mathcal{G}_{11}|}{Q_y|\phi_1'|}+\frac{2|\mathcal{G}_{44}|}{Q_z|\phi_2'|}+\frac{2|\mathcal{G}_{66}|}{Q_z|\phi_2'|}+\frac{2|\mathcal{G}_{22}|}{Q_y|\phi_1'|}+\frac{2|\mathcal{G}_{33}|}{Q_y|\phi_1'|}+\frac{2|\mathcal{G}_{55}|}{Q_z|\phi_2'|}\\
\le& 2C's^2\Big[\frac{[\phi_1'(\xi-p_1-\theta_{10}s)]^2}{\epsilon_1|\phi_1'|}+\frac{(|\hat{\phi}_1(0)|+|\hat{\phi}_3(1)-a|)[\phi_2'(\xi+p_2-\theta_{15}s)]^2}{\epsilon_2|\phi_2'|}\\
&+\frac{|\hat{\phi}_2(0)|[\phi_3'(\xi+p_3-\theta_{18}s)]^2}{\epsilon_2|\phi_2'|}+\frac{|\phi_1'(\xi-p_1-\theta_{11}s)||\phi_2'(\xi+p_2-\theta_{21}s)|}{\epsilon_2|\phi_2'|}\\
&+\frac{|\phi_1'(\xi-p_1-\theta_{31}s)||\phi_3'(\xi+p_3-\theta_{14}s)|}{\epsilon_1|\phi_1'|}+\frac{|\phi_2'(\xi+p_2-\theta_{16}s)||\phi_3'(\xi+p_3-\theta_{17}s)|}{\epsilon_2|\phi_2'|}\Big]\\
\le& 2C's^2\Big[\frac{C_0^2e^{-2\eta_2(\xi-p_1(t)-\theta_{10}s)}}{\epsilon_1C_1e^{-\eta_2(\xi-p_1(t))}/\rho}+\frac{C_2(e^{-\eta_2(\xi-p_1(t))}+e^{\eta_1(\xi+p_3(t)-s)})C_0^2e^{2\eta_1(\xi+p_2(t)-\theta_{15}s)}}{\epsilon_2C_1e^{\eta_1(\xi+p_2(t))}/\rho}\\
&+\frac{C_0^2C_2e^{\eta_1(\xi+p_2(t))}e^{2\eta_1(\xi+p_3(t)-\theta_{18}s)}}{\mu_2C_1e^{\eta_1(\xi+p_2(t))}/\rho}+\frac{C_0^2e^{-\eta_2(\xi-p_1(t)-\theta_{11}s)}e^{\eta_1(\xi+p_2(t)-\theta_{21}s)}}{\epsilon_2C_1e^{\eta_1(\xi+p_2(t))}/\rho}\\
&+\frac{C_0^2e^{-\eta_2(\xi-p_1(t)-\theta_{31}s)}e^{\eta_1(\xi+p_3(t)-\theta_{14}s)}}{\epsilon_1C_1e^{-\eta_2(\xi-p_1(t))}/\rho}+\frac{C_0^2e^{\eta_1(\xi+p_2(t)-\theta_{16}s)}e^{\eta_1(\xi+p_3(t)-\theta_{17}s)}}{\epsilon_2C_1e^{\eta_1(\xi+p_2(t))}/\rho}\Big]\\
\le& \tilde{L}_3\big[e^{\eta_2p_1(t)}+e^{\eta_1(p_3(t)-p_2(t))/2}\big].
\end{aligned}
$$

Case 4. $-p_2(t) \le \xi \le (-p_2(t)-p_3(t))/2$. *From Lemma 2.4*(iii)*, one can show that there exists a general positive constant* C' *such that*

$$
\begin{aligned}
&|Q_{yy}\big(\theta_4\hat{\phi}_1(1)+(1-\theta_4)\hat{\phi}_1(0),\hat{\phi}_2(1),\hat{\phi}_3(1)\big)| \le C'|a-\hat{\phi}_2(1)|,\\
&|Q_{zz}\big(\hat{\phi}_1(0),\theta_7\hat{\phi}_2(1)+(1-\theta_7)\hat{\phi}_2(0),\hat{\phi}_3(1)\big)| \le C'(|\hat{\phi}_1(0)|+|\hat{\phi}_3(1)-a|),\\
&|Q_{ww}\big(\hat{\phi}_1(0),\hat{\phi}_2(0),\theta_9\hat{\phi}_3(1)+(1-\theta_9)\hat{\phi}_3(0)\big)| \le C',
\end{aligned}
$$

$$|Q_{yz}\big(\hat\phi_1(0),\theta_5\hat\phi_2(1)+(1-\theta_5)\hat\phi_2(0),\hat\phi_3(1)\big)|\le C',$$
$$|Q_{zw}\big(\hat\phi_1(0),\hat\phi_2(0),\theta_8\hat\phi_3(1)+(1-\theta_8)\hat\phi_3(0)\big)|\le C',$$
$$|Q_{yw}\big(\hat\phi_1(0),\hat\phi_2(0),\theta_6\hat\phi_3(1)+(1-\theta_6)\hat\phi_3(0)\big)|\le C',$$

and

$$|\mathcal{G}_{11}|\le C'|a-\hat\phi_2(1)|\theta_1 s^2[\phi_1'(\xi-p_1(t)-\theta_{10}s)]^2,$$
$$|\mathcal{G}_{44}|\le C'(|\hat\phi_1(0)|+|\hat\phi_3(1)-a|)\theta_2 s^2[\phi_2'(\xi+p_2(t)-\theta_{15}s)]^2,$$
$$|\mathcal{G}_{66}|\le C'\theta_3 s^2[\phi_3'(\xi+p_3(t)-\theta_{18}s)]^2,$$
$$|\mathcal{G}_{22}|\le C's^2|\phi_1'(\xi-p_1(t)-\theta_{11}s)||\phi_2'(\xi+p_2(t)-\theta_{21}s)|,$$
$$|\mathcal{G}_{33}|\le C's^2|\phi_2'(\xi+p_2(t)-\theta_{16}s)||\phi_3'(\xi+p_3(t)-\theta_{17}s)|,$$
$$|\mathcal{G}_{55}|\le C's^2|\phi_1'(\xi-p_1(t)-\theta_{31}s)||\phi_3'(\xi+p_3(t)-\theta_{14}s)|.$$

By using Lemma 2.1, Lemma 2.5 and (2.16)*, we can get a positive constant* $\tilde L_4$*, such that*

$$\begin{aligned}
\Big|\frac{\mathcal{G}}{\mathcal{A}}\Big| \le&\frac{2|\mathcal{G}_{11}|}{Q_y|\phi_1'|}+\frac{2|\mathcal{G}_{44}|}{Q_z|\phi_2'|}+\frac{2|\mathcal{G}_{66}|}{Q_z|\phi_2'|}+\frac{2|\mathcal{G}_{22}|}{Q_y|\phi_1'|}+\frac{2|\mathcal{G}_{33}|}{Q_y|\phi_1'|}+\frac{2|\mathcal{G}_{55}|}{Q_z|\phi_2'|}\\
\le&2C's^2\Big[\frac{|a-\hat\phi_2(1)|[\phi_1'(\xi-p_1-\theta_{10}s)]^2}{\epsilon_2|\phi_2'|}+\frac{(|\hat\phi_1(0)|+|\hat\phi_3(1)-a|)[\phi_2'(\xi+p_2-\theta_{15}s)]^2}{\epsilon_2|\phi_2'|}\\
&+\frac{[\phi_3'(\xi+p_3-\theta_{18}s)]^2}{\epsilon_3|\phi_3'|}+\frac{|\phi_1'(\xi-p_1-\theta_{11}s)||\phi_2'(\xi+p_2-\theta_{21}s)|}{\epsilon_2|\phi_2'|}\\
&+\frac{|\phi_1'(\xi-p_1-\theta_{31}s)||\phi_3'(\xi+p_3-\theta_{14}s)|}{\epsilon_3|\phi_3'|}+\frac{|\phi_2'(\xi+p_2-\theta_{16}s)||\phi_3'(\xi+p_3-\theta_{17}s)|}{\epsilon_2|\phi_2'|}\Big]\\
\le&2C's^2C_0^2\Big[\frac{C_2e^{-\eta_2(\xi+p_1-s)}e^{-2\eta_2(\xi-p_1-\theta_{10}s)}}{\epsilon_2C_1e^{-\eta_2(\xi+p_2)}/\rho}+\frac{C_2(e^{-\eta_2(\xi-p_1)}+e^{\eta_1(\xi+p_3-s)})e^{-2\eta_2(\xi+p_2-\theta_{15}s)}}{\epsilon_2C_1e^{-\eta_2(\xi+p_1)}/\rho}\\
&+\frac{e^{2\eta_1(\xi+p_3-\theta_{18}s)}}{\epsilon_3C_1e^{\eta_1(\xi+p_3)}/\rho}+\frac{e^{-\eta_2(\xi-p_1-\theta_{11}s)}e^{-\eta_2(\xi+p_2-\theta_{21}s)}}{\epsilon_2C_1e^{-\eta_2(\xi+p_2)}/\rho}+\frac{e^{-\eta_2(\xi-p_1-\theta_{31}s)}e^{\eta_1(\xi+p_3-\theta_{14}s)}}{\epsilon_3C_1e^{\eta_1(\xi+p_3)}/\rho}\\
&+\frac{e^{-\eta_2(\xi+p_2-\theta_{16}s)}e^{\eta_1(\xi+p_3-\theta_{17}s)}}{\epsilon_2C_1e^{-\eta_2(\xi+p_2)}/\rho}\Big]\\
\le&\tilde L_4\big[e^{\eta_2p_1(t)}+e^{\eta_1(p_3(t)-p_2(t))/2}\big].
\end{aligned}$$

By **Case 3** *and* **Case 4***, we can obtain*

$$\Big|\frac{\mathcal{G}(\phi_1,\phi_2,\phi_3)}{\mathcal{A}\phi_1,\phi_2,\phi_3)}\Big|\le L_5(e^{\eta_2p_1(t)}+e^{\eta_1(p_3(t)-p_2(t))/2}),\quad \textit{for } 0\le\xi\le(-p_3(t)-p_2(t))/2,$$

where $L_5=\max\{\tilde L_3,\tilde L_4\}$*.*

Case 5. $(-p_3(t)-p_2(t))/2\le\xi\le-p_3(t)$.

From Lemma 2.4(iii)*, one can show that there exists a general positive constant* C' *such that*

$$|Q_{yy}\big(\theta_4\hat\phi_1(1)+(1-\theta_4)\hat\phi_1(0),\hat\phi_2(1),\hat\phi_3(1)\big)|\le C'|a-\hat\phi_2(1)|,$$
$$|Q_{zz}\big(\hat\phi_1(0),\theta_7\hat\phi_2(1)+(1-\theta_7)\hat\phi_2(0),\hat\phi_3(1)\big)|\le C',$$

$$
\begin{aligned}
&|Q_{ww}\big(\hat{\phi}_1(0), \hat{\phi}_2(0), \theta_9\hat{\phi}_3(1) + (1-\theta_9)\hat{\phi}_3(0)\big)| \le C'|a - \hat{\phi}_2(0)|,\\
&|Q_{yz}\big(\hat{\phi}_1(0), \theta_5\hat{\phi}_2(1) + (1-\theta_5)\hat{\phi}_2(0), \hat{\phi}_3(1)\big)| \le C',\\
&|Q_{zw}\big(\hat{\phi}_1(0), \hat{\phi}_2(0), \theta_8\hat{\phi}_3(1) + (1-\theta_8)\hat{\phi}_3(0)\big)| \le C',\\
&|Q_{yw}\big(\hat{\phi}_1(0), \hat{\phi}_2(0), \theta_6\hat{\phi}_3(1) + (1-\theta_6)\hat{\phi}_3(0)\big)| \le C',
\end{aligned}
$$

and

$$
\begin{aligned}
&|\mathcal{G}_{11}| \le C'|a - \hat{\phi}_2(1)|\theta_1 s^2[\phi_1'(\xi - p_1(t) - \theta_{10}s)]^2,\\
&|\mathcal{G}_{44}| \le C'\theta_2 s^2[\phi_2'(\xi + p_2(t) - \theta_{15}s)]^2,\\
&|\mathcal{G}_{66}| \le C'|a - \hat{\phi}_2(0)|\theta_3 s^2|a - \hat{\phi}_1(0)|[\phi_3'(\xi + p_3(t) - \theta_{18}s)]^2,\\
&|\mathcal{G}_{22}| \le C's^2|\phi_1'(\xi - p_1(t) - \theta_{11}s)||\phi_2'(\xi + p_2(t) - \theta_{21}s)|,\\
&|\mathcal{G}_{55}| \le C'|s^2\phi_2'(\xi + p_2(t) - \theta_{16}s)||\phi_3'(\xi + p_3(t) - \theta_{17}s)|,\\
&|\mathcal{G}_{33}| \le C's^2|\phi_1'(\xi - p_1(t) - \theta_{31}s)||\phi_3'(\xi + p_3(t) - \theta_{14}s)|.
\end{aligned}
$$

By applying Lemma 2.1, Lemma 2.5 and (2.16)*, we have a positive constant* $\tilde{L}_5$*, it can be obtained*

$$
\begin{aligned}
\Big|\frac{\mathcal{G}}{\mathcal{A}}\Big| \le& \frac{2|\mathcal{G}_{11}|}{Q_z|\phi_2'|} + \frac{2|\mathcal{G}_{44}|}{Q_z|\phi_2'|} + \frac{2|\mathcal{G}_{66}|}{Q_w|\phi_3'|} + \frac{2|\mathcal{G}_{22}|}{Q_z|\phi_2'|} + \frac{2|\mathcal{G}_{33}|}{Q_w|\phi_3'|} + \frac{2|\mathcal{G}_{55}|}{Q_w|\phi_3'|}\\
\le& 2C's^2\Big[\frac{|a - \hat{\phi}_2(1)|[\phi_1'(\xi - p_1 - \theta_{10}s)]^2}{\rho_2|\phi_2'|} + \frac{[\phi_2'(\xi + p_2 - \theta_{15}s)]^2}{\rho_2|\phi_2'|}\\
&+ \frac{|a - \hat{\phi}_2(0)|[\phi_3'(\xi + p_3 - \theta_{18}s)]^2}{\epsilon_3|\phi_3'|} + \frac{|\phi_1'(\xi - p_1 - \theta_{11}s)||\phi_2'(\xi + p_2(t) - \theta_{21}s)|}{\epsilon_2|\phi_2'|}\\
&+ \frac{|\phi_1'(\xi - p_1 - \theta_{31}s)\phi_3'(\xi + p_3 - \theta_{14}s)|}{\epsilon_3|\phi_3'|} + \frac{|\phi_2'(\xi + p_2 - \theta_{16}s)\phi_3'(\xi + p_3 - \theta_{17}s)|}{\epsilon_3|\phi_3'|}\Big]\\
\le& 2C's^2C_0^2\Big[\frac{C_2e^{-\eta_2(\xi+p_2(t)-s)}e^{-2\eta_2(\xi-p_1(t)-\theta_{10}s)}}{\epsilon_2C_1e^{-\eta_2(\xi+p_2(t))}/\rho} + \frac{e^{-2\eta_2(\xi+p_2(t)-\theta_{15}s)}}{\epsilon_2C_1e^{-\eta_2(\xi+p_2(t))}/\rho}\\
&+ \frac{C_2e^{-\eta_2(\xi+p_2(t))}e^{2\eta_1(\xi+p_3(t)-\theta_{18}s)}}{\epsilon_3C_1e^{\eta_1(\xi+p_3(t))}/\rho} + \frac{e^{-\eta_2(\xi-p_1(t)-\theta_{11}s)}e^{-\eta_2(\xi+p_2(t)-\theta_{21}s)}}{\epsilon_2C_1e^{-\eta_2(\xi+p_2(t))}/\rho}\\
&+ \frac{e^{-\eta_2(\xi-p_1(t)-\theta_{31}s)}e^{\eta_1(\xi+p_3(t)-\theta_{14}s)}}{\epsilon_3C_1e^{\eta_1(\xi+p_3(t))}/\rho} + \frac{e^{-\eta_2(\xi+p_2(t)-\theta_{16}s)}e^{\eta_1(\xi+p_3(t)-\theta_{17}s)}}{\epsilon_3C_1e^{\eta_1(\xi+p_3(t))}/\rho}\Big]\\
\le& \tilde{L}_5\big[e^{\eta_2p_1(t)} + e^{\eta_2(p_2(t)-p_1(t))/2}\big].
\end{aligned}
$$

Case 6. $\xi \ge -p_3(t)$.

From Lemma 2.4(iii)*, one can show that there exists a general positive constant* C' *such that*

$$
\begin{aligned}
&|Q_{yy}\big(\theta_4\hat{\phi}_1(1) + (1-\theta_4)\hat{\phi}_1(0), \hat{\phi}_2(1), \hat{\phi}_3(1)\big)| \le C'|1 - \hat{\phi}_3(1)|,\\
&|Q_{zz}\big(\hat{\phi}_1(0), \theta_7\hat{\phi}_2(1) + (1-\theta_7)\hat{\phi}_2(0), \hat{\phi}_3(1)\big)| \le C'|1 - \hat{\phi}_3(1)|,\\
&|Q_{ww}\big(\hat{\phi}_1(0), \hat{\phi}_2(0), \theta_9\hat{\phi}_3(1) + (1-\theta_9)\hat{\phi}_3(0)\big)| \le C'|a - \hat{\phi}_2(0)|,\\
&|Q_{yz}\big(\hat{\phi}_1(0), \theta_5\hat{\phi}_2(1) + (1-\theta_5)\hat{\phi}_2(0), \hat{\phi}_3(1)\big)| \le C'|1 - \hat{\phi}_3(1)|,\\
&|Q_{zw}\big(\hat{\phi}_1(0), \hat{\phi}_2(0), \theta_8\hat{\phi}_3(1) + (1-\theta_8)\hat{\phi}_3(0)\big)| \le C',
\end{aligned}
$$

$$|Q_{yw}\big(\hat\phi_1(0),\hat\phi_2(0),\theta_6\hat\phi_3(1)+(1-\theta_6)\hat\phi_3(0)\big)|\le C',$$

and

$$\begin{aligned}
&|\mathcal{G}_{11}|\le C'|1-\hat\phi_3(1)|\theta_1 s^2[\phi_1'(\xi-p_1(t)-\theta_{10}s)]^2,\\
&|\mathcal{G}_{44}|\le C'|1-\hat\phi_3(1)|\theta_2 s^2[\phi_2'(\xi+p_2(t)-\theta_{15}s)]^2,\\
&|G_{66}|\le C'|a-\hat\phi_2(0)|\theta_3 s^2[\phi_3'(\xi+p_3(t)-\theta_{18}s)]^2,\\
&|G_{22}|\le C'|1-\hat\phi_3(1)|s^2|\phi_1'(\xi-p_1(t)-\theta_{11}s)||\phi_2'(\xi+p_2(t)-\theta_{21}s)|,\\
&|G_{55}|\le C'|s^2\phi_2'(\xi+p_2(t)-\theta_{16}s)||\phi_3'(\xi+p_3(t)-\theta_{17}s)|,\\
&|G_{33}|\le C's^2|\phi_1'(\xi-p_1(t)-\theta_{31}s)||\phi_3'(\xi+p_3(t)-\theta_{14}s)|.
\end{aligned}$$

According to Lemma 2.1, Lemma 2.5 and (2.17)*, there exist a positive constant* $\tilde L_6$ *such that*

$$\begin{aligned}
\Big|\frac{\mathcal{G}}{\mathcal{A}}\Big| \le&\frac{2|\mathcal{G}_{11}|}{Q_w|\phi_3'|}+\frac{2|\mathcal{G}_{44}|}{Q_w|\phi_3'|}+\frac{2|\mathcal{G}_{66}|}{Q_w|\phi_3'|}+\frac{2|\mathcal{G}_{22}|}{Q_w|\phi_3'|}+\frac{2|\mathcal{G}_{33}|}{Q_w|\phi_3'|}+\frac{2|\mathcal{G}_{55}|}{Q_w|\phi_3'|}\\
\le&2C's^2\Big[\frac{|1-\hat\phi_3(1)|[\phi_1'(\xi-p_1-\theta_{10}s)]^2}{\epsilon_3|\phi_3'|}+\frac{|1-\hat\phi_3(1)|[\phi_2'(\xi+p_2-\theta_{15}s)]^2}{\epsilon_3|\phi_3'|}\\
&+\frac{|a-\hat\phi_2(0)|[\phi_3'(\xi+p_3-\theta_{18}s)]^2}{\epsilon_3|\phi_3'|}+\frac{|1-\hat\phi_3(1)\phi_1'(\xi-p_1-\theta_{11}s)\phi_2'(\xi+p_2-\theta_{21}s)|}{\epsilon_3|\phi_3'|}\\
&+\frac{|\phi_1'(\xi-p_1-\theta_{31}s)||\phi_3'(\xi+p_3-\theta_{14}s)|}{\epsilon_3|\phi_3'|}+\frac{|\phi_2'(\xi+p_2-\theta_{16}s)||\phi_3'(\xi+p_3-\theta_{17}s)|}{\epsilon_3|\phi_3'|}\Big]\\
\le&2C's^2C_0^2\Big[\frac{C_2e^{-\eta_2(\xi+p_3(t)-s)}e^{-2\eta_2(\xi-p_1(t)-\theta_{10}s)}}{\epsilon_3C_1e^{-\eta_2(\xi+p_3(t))}/\rho}+\frac{C_2e^{-\eta_2(\xi+p_3(t)-s)}e^{-2\eta_2(\xi+p_2(t)-\theta_{15}s)}}{\epsilon_3C_1e^{-\eta_2(\xi+p_3(t))}/\rho}\\
&+\frac{C_2e^{-\eta_2(\xi+p_2(t))}e^{-2\eta_2(\xi+p_3(t)-\theta_{18}s)}}{\epsilon_3C_1e^{-\eta_2(\xi+p_3(t))}/\rho}+\frac{C_2e^{-\eta_2(\xi+p_3(t)-s)}e^{-\eta_2(\xi-p_1(t)-\theta_{11}s)}e^{-\eta_2(\xi+p_2(t)-\theta_{21}s)}}{\epsilon_2C_1e^{-\eta_2(\xi+p_3(t))}/\rho}\\
&+\frac{e^{-\eta_2(\xi-p_1(t)-\theta_{31}s)}e^{-\eta_2(\xi+p_3(t)-\theta_{14}s)}}{\epsilon_3C_1e^{-\eta_2(\xi+p_3(t))}/\rho}+\frac{e^{-\eta_2(\xi+p_2(t)-\theta_{16}s)}e^{-\eta_2(\xi+p_3(t)-\theta_{17}s)}}{\epsilon_3C_1e^{-\eta_2(\xi+p_3(t))}/\rho}\Big]\\
\le&\tilde L_6(e^{-\eta_2(\xi-p_1(t))}+e^{-\eta_2(\xi+p_1(t))})\le\tilde L_6\big[e^{\eta_2p_1(t)}+e^{\eta_2(p_3(t)-p_2(t))/2}\big].
\end{aligned}$$

By **Case 5** *and* **Case 6***, we can obtain*

$$\Big|\frac{\mathcal{G}(\phi_1,\phi_2,\phi_3)}{\mathcal{A}\phi_1,\phi_2,\phi_3)}\Big|\le L_6(e^{\eta_2p_1(t)}+e^{\eta_2(p_3(t)-p_2(t))/2}),\quad \textit{for}\ \xi\ge(-p_3(t)-p_2(t))/2,$$

where $L_6=\max\{\tilde L_5,\tilde L_6\}$. □

Next, we consider the following four ordinary differential equations with the initial conditions with $s_2>s_1>0$ (see [5,6,21]):

$$q_1'(t)=s_1+Le^{\kappa q_1(t)},\ -\infty<t<0,\ q_1(0)=q_0; \tag{2.18}$$

$$q_2'(t)=s_2+Le^{\kappa q_1(t)},\ -\infty<t<0,\ q_2(0)=q_0; \tag{2.19}$$

$$r_1'(t)=s_1-Le^{\kappa r_1(t)},\ -\infty<t<0,\ r_1(0)=r_0; \tag{2.20}$$

$$r_2'(t)=s_2-Le^{\kappa r_1(t)},\ -\infty<t<0,\ r_2(0)=r_0, \tag{2.21}$$

where L, s_1, s_2, q_0, r_0 and κ are positive constants. (2.18)-(2.21) have the following unique solutions for $t \le 0$, respectively,

$$q_1(t) = s_1 t - \frac{1}{\kappa}\ln\big[e^{-\kappa q_0} + \frac{L(1-e^{\kappa s_1 t})}{s_1}\big], \tag{2.22}$$

$$q_2(t) = s_2 t - \frac{1}{\kappa}\ln\big[e^{-\kappa q_0} + \frac{L(1-e^{\kappa s_1 t})}{s_1}\big], \tag{2.23}$$

$$r_1(t) = s_1 t - \frac{1}{\kappa}\ln\big[e^{-\kappa r_0} + \frac{L(1-e^{\kappa s_1 t})}{s_1}\big], \tag{2.24}$$

$$r_2(t) = s_2 t - \frac{1}{\kappa}\ln\big[e^{-\kappa r_0} + \frac{L(1-e^{\kappa s_1 t})}{s_1}\big]. \tag{2.25}$$

For a sufficient large constant $\sigma > 0$, we choose q_0 and r_0 satisfying

$$q_0 = -\frac{1}{\kappa}\ln(e^{-\kappa r_0} - \frac{2L}{s_1}) < -\sigma,\ \ r_0 < -\frac{1}{\kappa}\ln(\frac{2L}{s_1} + e^{\kappa\sigma}). \tag{2.26}$$

Moreover,

$$\lim_{t\to-\infty}(q_1(t) - r_1(t)) = \lim_{t\to-\infty}(q_2(t) - r_2(t)) = 0, \tag{2.27}$$

$$\lim_{t\to-\infty}(q_1(t) - s_1 t) = \lim_{t\to-\infty}(q_2(t) - s_2 t) = -\frac{1}{\kappa}\ln\big[e^{-\kappa q_0} + \frac{L}{s_1}\big], \tag{2.28}$$

$$\lim_{t\to-\infty}(r_1(t) - s_1 t) = \lim_{t\to-\infty}(r_2(t) - s_2 t) = -\frac{1}{\kappa}\ln\big[e^{-\kappa r_0} + \frac{L}{s_1}\big]. \tag{2.29}$$

Also, there exists a positive constant N such that for all $t \le 0$,

$$0 < q_1(t) - r_1(t) = q_2(t) - r_2(t) \le N e^{\kappa s_1 t}, \tag{2.30}$$

and $q_1(t), q_2(t), r_1(t), r_2(t) \le -\sigma$.

Moreover, we have

$$q_2(t) - q_1(t) = r_2(t) - r_1(t) = (s_2 - s_1)t \to -\infty \tag{2.31}$$

Therefore, by the choice of κ, there exists a $t_0 < 0$ such that

$$\max\big\{\frac{\eta_1(q_2(t) - q_1(t))}{2}, \frac{\eta_2(q_2(t) - q_1(t))}{2}\big\} < \kappa q_1(t) < 0,\ \ \forall t \le t_0. \tag{2.32}$$

$$\max\big\{\frac{\eta_1(r_2(t) - r_1(t))}{2}, \frac{\eta_2(r_2(t) - r_1(t))}{2}\big\} < \kappa r_1(t) < 0,\ \ \forall t \le t_0. \tag{2.33}$$

Lemma 2.9. *Assume that (J), (A1) and (A2) hold. Define the functions* $\overline{U}(\xi, t)$ *and* $\underline{U}(\xi, t)$ *by*

$$\overline{U}(\xi, t) = Q\big(\phi_1(\xi - q_1(t)), \phi_2(\xi + q_1(t)), \phi_3(\xi + q_2(t))\big),$$
$$\underline{U}(\xi, t) = Q\big(\phi_1(\xi - r_1(t)), \phi_2(\xi + r_1(t)), \phi_3(\xi + r_2(t))\big),$$

then $\overline{U}(\xi, t)$ *and* $\underline{U}(\xi, t)$ *are a pair of super- and subsolutions of* (2.12) *for* $t \le t_0$*, with some* $t_0 < 0$*. Moreover, there exists a positive constant* γ *such that*

$$\overline{U}(\xi, t) > \underline{U}(\xi, t) \quad \textit{and} \quad \sup_{\xi\in\mathbb{R}}(\overline{U}(\xi, t) - \underline{U}(\xi, t) \le \gamma e^{\kappa s_1 t}, \quad \textit{for } t \le t_0. \tag{2.34}$$

Proof. **Step 1.**We prove that $\overline{U}(\xi,t)$ is a super-solution for $t \le t_0$. By Lemma 2.7, Lemma 2.8, we can get

$$
\begin{aligned}
&\Big|\frac{\mathcal{G}(\phi_1,\phi_2,\phi_3)+\mathcal{F}(\phi_1,\phi_2,\phi_3)}{\mathcal{A}(\phi_1,\phi_2,\phi_3)}\Big| \\
\le& \begin{cases} (L_1+L_4)(e^{\eta_1 q_1(t)}+e^{\eta_1 q_2(t)}), & \text{for } \xi \le 0,\\ (L_2+L_5)(e^{\eta_2 q_1(t)}+e^{\eta_1 (q_2(t)-q_1(t))/2}), & \text{for } 0\le \xi \le (-q_2(t)-q_1(t))/2,\\ (L_3+L_6)(e^{\eta_2 q_1(t)}+e^{\eta_2 (q_2(t)-q_1(t))/2}), & \text{for } \xi \ge (-q_2(t)-q_1(t))/2,\end{cases}
\end{aligned}
$$

where $M=\max\{(L_1+L_4),(L_2+L_5),(L_3+L_6)\}$.

Thus, based on (2.13), (2.18), (2.19), (2.32), we have

$$
\begin{aligned}
&F(\overline{U}(\xi,t))\\
=&-(q_1'-s_1)Q_y\phi_1'+(q_1'-s_1)Q_z\phi_2'+(q_2'-s_2)Q_w\phi_3'-\mathcal{G}(\phi_1,\phi_2,\phi_3)-\mathcal{F}(\phi_1,\phi_2,\phi_3)\\
=&\mathcal{A}(\phi_1,\phi_2,\phi_3)\Big[Le^{\kappa q_1}-\frac{\mathcal{G}(\phi_1,\phi_2,\phi_3)+\mathcal{F}(\phi_1,\phi_2,\phi_3)}{\mathcal{A}(\phi_1,\phi_2,\phi_3)}\Big]\\
\ge&\mathcal{A}(\phi_1,\phi_2,\phi_3)(L-M)Le^{\kappa q_1}\ge 0,
\end{aligned}
$$

where $L>M$. Hence, $\overline{U}(\xi,t)$ is a super-solution of (2.12).

Step 2. We prove that $\underline{U}(\xi,t)$ is a sub-solution for $t\le t_0$. By Lemmas 2.7-2.8, it holds

$$
\begin{aligned}
&\Big|\frac{\mathcal{G}(\phi_1,\phi_2,\phi_3)+\mathcal{F}(\phi_1,\phi_2,\phi_3)}{\mathcal{A}(\phi_1,\phi_2,\phi_3)}\Big| \\
\le& \begin{cases} (L_1+L_4)(e^{\eta_1 r_1(t)}+e^{\eta_1 r_2(t)}), & \text{for } \xi \le 0,\\ (L_2+L_5)(e^{\eta_2 r_1(t)}+e^{\eta_1 (r_2(t)-r_1(t))/2}), & \text{for } 0\le \xi \le (-r_2(t)-r_1(t))/2,\\ (L_3+L_6)(e^{\eta_2 r_1(t)}+e^{\eta_2 (r_2(t)-r_1(t))/2}), & \text{for } \xi \ge (-r_2(t)-r_1(t))/2,\end{cases}
\end{aligned}
$$

where $M=\max\{(L_1+L_4),(L_2+L_5),(L_3+L_6)\}$. Thus, by (2.13), (2.20), (2.21) and (2.33), we have

$$
\begin{aligned}
&F(\underline{U}(\xi,t))\\
=&-(r_1'-s_1)Q_y\phi_1'+(r_1'-s_1)Q_z\phi_2'+(r_2'-s_2)Q_w\phi_3'-\mathcal{G}(\phi_1,\phi_2,\phi_3)-\mathcal{F}(\phi_1,\phi_2,\phi_3)\\
=&-\mathcal{A}(\phi_1,\phi_2,\phi_3)\Big[Le^{\kappa r_1}-\frac{\mathcal{G}(\phi_1,\phi_2,\phi_3)+\mathcal{F}(\phi_1,\phi_2,\phi_3)}{\mathcal{A}(\phi_1,\phi_2,\phi_3)}\Big]\\
\le&-\mathcal{A}(\phi_1,\phi_2,\phi_3)(L-M)Le^{\kappa r_1}\le 0.
\end{aligned}
$$

Hence, $\underline{U}(\xi,t)$ is a sub-solution of (2.12).

Finally, according to (2.30) and Lemma 2.6, the function $\mathcal{A}$ is bounded above and

$$
\begin{aligned}
&\overline{U}(\xi,t)-\underline{U}(\xi,t)\\
=&Q\big(\phi_1(\xi-p_1(t)),\phi_2(\xi+p_1(t)),\phi_3(\xi+p_2(t))\big)\\
&-Q\big(\phi_1(\xi-r_1(t)),\phi_2(\xi+r_1(t)),\phi_3(\xi+r_2(t))\big)
\end{aligned}
$$

$$= \int_0^1 \mathcal{A}(\phi_1(\xi - \theta p_1 - (1-\theta) r_1), \phi_2(\xi + \theta p_1 + (1-\theta) r_1),$$
$$\phi_3(\xi + \theta p_2 + (1-\theta) r_2)) d\theta \times (p_1 - r_1),$$

which implies that (2.34) holds. This completes the proof. □

3. Theorem 1.1

Consider (2.9) with initial $u(x,-n) = \underline{U}(x - \bar{c}n, -n)$ for any $x \in \mathbb{R}$. According to Lemma 2.2 (1), (2.9) admits a unique solution $P^n(x,t)$. Moreover, by Lemma 2.9 and Lemma 2.2 (2), we get

$$\underline{U}(x + \bar{c}t, t) \leq P^n(x,t) \leq P^{n+1}(x,t) \leq \overline{U}(x + \bar{c}t, t)$$

for large $n \in \mathbb{N}$ and $-n < t \leq t_0$. From Lemma 2.3 and taking the diagonal extraction process, there exists a subsequence $P^{n_k}(x,t)$ of $P^n(x,t)$ and a function $u(x,t)$ such that $P^{n_k}(x,t)$ and $\frac{d}{dt}P^{n_k}(x,t)$ converge locally uniformly to $u(x,t)$ and $\frac{d}{dt}u(x,t)$, respectively as $k \to \infty$. Since $P^n(x,t) \leq P^{n+1}(x,t)$ for any $t > -n$, we have

$$\lim_{n\to\infty} P^n(x,t) = u(x,t), \ \ x \in \mathbb{R}, \ t \leq t_0,$$

there exists a unique entire solution $u(x,t)$ of (1.1), such that

$$\underline{U}(x + \bar{c}t, t) \leq u(x,t) \leq \overline{U}(x + \bar{c}t, t)$$

for all $x \in \mathbb{R}$ and $t \leq t_0$.

Next, we study the asymptotic behavior of the entire solution in Theorem 1.1. Define

$$\omega := -\frac{1}{\kappa} log(e^{-\kappa r_0} - \frac{L}{s_1}), \tag{3.1}$$

then we can obtain there exists a constant $R > 0$, such that

$$-Re^{\kappa s_1 t} < r_1(t) - s_1 t - \omega = r_2(t) - s_2 t - \omega < 0. \tag{3.2}$$

When $t \to +\infty$, this part can be obtained as in [10]. Similar to the process in [6, Theorem 4.3] , we divided it into three cases for $t \to -\infty$.

Case 1. Recall that $\xi = x + \bar{c}t$, for $x \leq -(c_1 + c_2)t/2$ and $t \leq t_0$, we have $\xi \leq 0 \leq -r_1$. By (2.11), (2.24), (2.34), Lemma 2.1 and (3.2), we have

$$\begin{aligned}
&|u(x,t) - \phi_1(x + c_1 t - \omega)| = |U(\xi,t) - \phi_1(\xi - s_1 t - \omega)| \\
&\leq |U(\xi,t) - \underline{U}(\xi,t)| + |\underline{U}(\xi,t) - \phi_1(\xi - s_1 t - \omega)| \\
&\leq |U(\xi,t) - \underline{U}(\xi,t)| + |\phi_1(\xi - r_1(t)) - \phi_1(\xi - s_1 t - \omega)| + |\phi_2(\xi + r_1(t))| \times \\
&\Big| \frac{(1-a)(1 - \phi_1(\xi - r_1(t)))(\phi_3(\xi + r_2(t))) - \phi_1(\xi - r_1(t))}{(1 - \phi_1(\xi - r_1(t)))\phi_2(\xi + r_1(t))(1-a) + (a - (\phi_2(\xi + r_1(t))))(1 - \phi_3(\xi + r_2(t)))} \Big| \\
&\leq |\overline{U}(\xi,t) - \underline{U}(\xi,t)| + |\phi_1(\xi - r_1(t)) - \phi_1(\xi - s_1 t - \omega)| + K_1 |\phi_2(\xi + r_1(t))| \\
&\leq \gamma e^{\kappa s_1 t} + \sup_{\zeta \in \mathbb{R}} |\phi_1'(\zeta)| |r_1(t) - s_1 t - \omega| + K_1 C_2 e^{\eta_1 (\xi + r_1(t))}
\end{aligned}$$

$\leq \gamma e^{\kappa s_1 t} + C_0 R e^{\kappa s_1 t} + K_1 C_2 e^{\eta_1 r_1(t)}$.

Case 2. For $-(c_1+c_2)/2 \leq x \leq -(c_2+c_3)t/2$ and $t \leq t_0$, we have $0 \leq \xi \leq -(s_1+s_2)t/2$. By (2.24), (2.10), (2.34), Lemma 2.1 and (3.2), we have

$$\begin{aligned}
&|u(x,t) - \phi_2(x + c_2 t + \omega)| = |U(\xi,t) - \phi_2(\xi + s_1 t + \omega)| \\
&\leq |U(\xi,t) - \underline{U}(\xi,t)| + |\underline{U}(\xi,t) - \phi_2(\xi + s_1 t + \omega)| \\
&\leq |\overline{U}(\xi,t) - \underline{U}(\xi,t)| + |\phi_2(\xi + r_1(t)) - \phi_2(\xi + s_1 t + \omega)| + |\phi_3(\xi + r_2(t)) - a| \times \\
&\Big|\frac{(1-\phi_2(\xi + r_2(t)))(1-\phi_1(\xi - r_1(t)))\phi_2(\xi + r_2(t))}{(1-\phi_1(\xi - r_1(t)))\phi_2(\xi + r_1(t))(1-a) + (a - (\phi_2(\xi + r_1(t))))(1-\phi_3(\xi + r_2(t)))}\Big| \\
&+ |\phi_1(\xi - r_1(t))| \times \\
&\Big|\frac{(1-\phi_2(\xi + r_1(t)))(a-\phi_2(\xi + r_1(t)))(1-\phi_3(\xi + r_2(t)))}{(1-\phi_1(\xi - r_1(t)))\phi_2(\xi + r_1(t))(1-a) + (a - (\phi_2(\xi + r_1(t))))(1-\phi_3(\xi + r_2(t)))}\Big| \\
&\leq |\overline{U}(\xi,t) - \underline{U}(\xi,t)| + |\phi_2(\xi + r_1(t)) - \phi_2(\xi + s_1 t + \omega)| + K_2(|\phi_3(\xi + r_2(t)) - a| \\
&+ |\phi_1(\xi - r_1(t))|) \\
&\leq \gamma e^{\kappa s_1 t} + \sup_{\zeta \in \mathbb{R}} |\phi_2'(\zeta)||r_1(t) - s_1 t - \omega| + K_2(|\phi_3(\xi + r_2(t)) - a| + |\phi_1(\xi - r_1(t))|) \\
&\leq \gamma e^{\kappa s_1 t} + C_0 R e^{\kappa s_1 t} + K_2(C_0 e^{\eta_1(\xi + r_2(t))} + C_0 e^{-\eta_2(\xi - r_1(t))}) \\
&\leq \gamma e^{\kappa s_1 t} + C_0 R e^{\kappa s_1 t} + K_2 C_0 (e^{\eta_1(-(s_1+s_2)t/2 + r_2(t))} + e^{\eta_2 r_1(t)}).
\end{aligned}$$

Case 3. For $x \geq -(c_2+c_3)t/2$ and $t \leq t_0$, we have $\xi \geq -(s_1+s_2)t/2$. By (2.11), (2.25), (2.34), Lemma 2.1 and (3.2), we have

$$\begin{aligned}
&|u(x,t) - \phi_3(x + c_3 t + \omega)| = |U(\xi,t) - \phi_3(\xi + s_2 t + \omega)| \\
&\leq |U(\xi,t) - \underline{U}(\xi,t)| + |\underline{U}(\xi,t) - \phi_3(\xi + s_2 t + \omega)| \\
&\leq |\overline{U}(\xi,t) - \underline{U}(\xi,t)| + |\phi_3(\xi + r_2(t)) - \phi_3(\xi + s_2 t + \omega)| + |a - \phi_{i,j;2}(\xi + r_1(t))| \cdot \\
&\Big|\frac{(1-\phi_3(\xi + r_2(t)))(\phi_1(\xi - r_1(t)) - \phi_3(\xi + r_2(t)))}{(1-\phi_1(\xi - r_1(t)))\phi_2(\xi + r_1(t))(1-a) + (a - (\phi_2(\xi + r_1(t))))(1-\phi_3(\xi + r_2(t)))}\Big| \\
&\leq |\overline{U}(\xi,t) - \underline{U}(\xi,t)| + |\phi_3(\xi + r_2(t)) - \phi_3(\xi + s_2 t + \omega)| + K_3|a - \phi_2(\xi + r_1(t))| \\
&\leq \gamma e^{\kappa s_1 t} + \sup_{\zeta \in \mathbb{R}} |\phi_3'(\zeta)||r_2(t) - s_2 t - \omega| + K_3 C_0 e^{\eta_2(\xi + r_1(t))} \\
&\leq \gamma e^{\kappa s_1 t} + C_0 R e^{\kappa s_1 t} + K_3 C_0 e^{\eta_2((s_1+s_2)t/2 - r_1(t))}.
\end{aligned}$$

Therefore, as $t \to -\infty$, the asymptotic behavior of the entire solution can be obtained. This completes the proof of Theorem 1.1.

4. Proof of Theorem 1.2

This section is similar to the proof of Theorem 1.1, so we only give the outline of the proof.

Firstly, set $c_1 = \hat{v}_1$, $c_2 = \tilde{v}_2$ and $c_3 = \hat{v}_2$. Let $\phi_1 = \hat{\varphi}_1$, $\phi_2 = \tilde{\varphi}_2$, $\phi_3 = \hat{\varphi}_2$, $i = 1,2,3$, be traveling fronts of (1.1) that satisfy

$$\begin{cases} c_i \phi_i'(\xi) = J * \phi_i(\xi) - \phi_i(\xi) + f(\phi_i(\xi)), \\ \phi_i(-\infty) = \alpha_i, \phi_i(\infty) = \beta_i, \end{cases} \tag{4.1}$$

where $(\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3) = (1, 0, 0, a, a, 0)$. Then, we define the auxiliary rational function

$$\tilde{Q}(y, z, w) := z + \frac{(1-y)z(a-w)(-z) + y(a-z)w(1-z)}{(1-y)za + (a-z)w},$$

where $D_1 := \{[0,1] \times [0,a] \times [0,a]\} \backslash (\{(1, a, w) | 0 \le w \le a\} \cup \{(1, z, 0) | 0 \le z \le a\} \cup \{(y, 0, 0) | 0 \le y \le 1\})$.

Secondly, we take the functions $\tilde{q}_i(t)$ and $\tilde{r}_i(t)$, $i = 1, 2$, which are the solutions of the following initial value problems

$$\tilde{q}_1'(t) = s_1 + Le^{\kappa \tilde{q}_1(t)}, \ -\infty < t < 0, \ \tilde{q}_1(0) = \tilde{q}_0; \tag{4.2}$$

$$\tilde{r}_1'(t) = s_1 - Le^{\kappa \tilde{r}_1(t)}, \ -\infty < t < 0, \ \tilde{r}_1(0) = \tilde{r}_0; \tag{4.3}$$

$$\tilde{q}_2'(t) = s_2 - Le^{\kappa \tilde{q}_1(t)}, \ -\infty < t < 0, \ \tilde{q}_2(0) = \tilde{r}_0; \tag{4.4}$$

$$\tilde{r}_2'(t) = s_2 + Le^{\kappa \tilde{r}_1(t)}, \ -\infty < t < 0, \ \tilde{r}_2(0) = \tilde{q}_0, \tag{4.5}$$

where $L, s_1, s_2, \tilde{q}_0, \tilde{r}_0$ and κ are the same as in Section 2.

Finally, we define the functions $\overline{U}^*(\xi, t)$ and $\underline{U}^*(\xi, t)$ by

$$\overline{U}^*(\xi, t) = \tilde{Q}\big(\phi_1(\xi - \tilde{q}_1(t)), \phi_2(\xi + \tilde{q}_1(t)), \phi_3(\xi + \tilde{q}_2(t))\big),$$
$$\underline{U}^*(\xi, t) = \tilde{Q}\big(\phi_1(\xi - \tilde{r}_1(t)), \phi_2(\xi + \tilde{r}_1(t)), \phi_3(\xi + \tilde{r}_2(t))\big),$$

then the functions $\overline{U}^*(\xi, t)$ and $\underline{U}^*(\xi, t)$ are a pair of super- and subsolutions of (2.12) for $t \le t_0$ with some constant $t_0 < 0$ as in Section 2. Meanwhile, the existence of the unique entire solution $v(x, t)$ of (1.1) can be shown and $v(x, t)$ satisfies

$$\underline{U}^*(x + \bar{c}t, t) \le v(x, t) \le \overline{U}^*(x + \bar{c}t, t)$$

for all $x \in \mathbb{R}$ and $t \le t_0$. By choosing

$$\omega_1 := -\frac{1}{\kappa} log(e^{-\kappa \tilde{r}_0} - \frac{L}{s_1}), \ \omega_2 := -\frac{1}{\kappa} log(e^{-\kappa \tilde{r}_0} - \frac{L}{s_1}) + \tilde{p}_0 + \tilde{r}_0$$

and similar to the proof of Theorem 1.1, the entire solution and the asymptotic behavior can be obtained. We omit the detail.

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