Existence and Blowup of Solutions for Neutral Partial Integro-differential Equations with State-dependent Delay

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Abstract In this paper, we study the existence and blowup of solutions for a neutral partial functional integro-differential equation with state-dependent delay in Banach space. The mild solutions are obtained by Sadovskii fixed point theorem under compactness condition for the resolvent operator, the theory of fractional power and α-norm are also used in the discussion since the nonlinear terms of the system involve spacial derivatives. The strong solutions are obtained under the lipschitz condition. In addition, based on the local existence result and a piecewise extended method, we achieve a blowup alternative result as well for the considered equation. Finally, an example is provided to illustrate the application of the obtained results.

Keywords Neutral partial integro-differential equation, Analytic semigroup, Resolvent operator, Fractional power operator, State-dependent delay.

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1. Introduction

In this paper, we study the existence and blowup of solutions for the semilinear neutral partial integro-differential equation with state-dependent delay of the form

$$
\begin{cases}
\dfrac{d}{dt}\left[x(t) + F(t, x_t)\right] = -A[x(t) + F(t, x_t)] + \displaystyle\int_0^t \Upsilon(t-s)x(s)ds + G(t, x_{\rho(t,x_t)}), \\
t \in [0, T], \\
x_0 = \varphi \in \mathscr{B}_\alpha,
\end{cases}
$$
(1.1)

where $-A$ is the infinitesimal generator of an analytic semigroup on a Banach space X, $\Upsilon(t)$ is a closed linear operator defined later, F, G and ρ are given continuous functions to be specified below, and $\mathscr{B}_\alpha$ is an abstract phase space endowed with a seminorm $\|\cdot\|_{\mathscr{B}_\alpha}$.

Partial integro-differential equations can be used to describe a lot of natural phenomena arising from many fields such as fluid dynamics, biological models and

chemical kinetics etc. These are often more accurate than the classical differential equations. A very effective approach to study this kind of equations is to transfer them into integro-differential evolution equations in abstract spaces. Grimmer etc [1–3] proved the existence of solutions of the following integrodifferential evolution equation

$$\begin{cases} v'(t) = Av(t) + \int_0^t \Upsilon(t-s)v(s)ds + g(t), & \text{for } t \geq 0, \\ v(0) = v_0 \in X, \end{cases} \tag{1.2}$$

where $g : \mathbb{R}^+ \to X$ is a continuous function. They obtained the representation of solutions, the existence and uniqueness of solutions via resolvent operator associated to the following linear homogeneous equation

$$\begin{cases} v'(t) = Av(t) + \int_0^t \Upsilon(t-s)v(s)ds, & \text{for } t \geq 0, \\ v(0) = v_0 \in X. \end{cases}$$

That is, the resolvent operator $R(t)$, replacing the role of C_0-semigroup for evolution equations, plays an important role in solving Eq. (1.2) in weak and strict senses. From then on, some topics for nonlinear integro-differential evolution equations, such as existence and regularity, stability, (asymptotic) periodicity of solutions and control problems, have been investigated by many mathematicians through applying the theory of resolvent operators, see [4–10]. Particularly, Lin and Liu [4] investigated existence, uniqueness and regularity of mild solutions for semilinear integrodifferential equations involving nonlocal initial conditions

$$\begin{cases} u'(t) = A\left[u(t) + \int_0^t F(t-s)u(s)ds\right] + f(t, u(t)), & t \in [0, T], \\ u(0) + g(t_1, \cdots, t_p, u(t_1), \cdots, u(t_p)) = u_0, \end{cases}$$

in a Banach space X with A the generator of a strongly continuous semigroup and $F(t)$ a bounded operator for $t \in [0, T]$. f is a C^1 function. In paper [5] the authors studied the following partial functional integro-differential equations with infinite delay in Banach space

$$\begin{cases} u'(t) = Au(t) + \int_0^t \Upsilon(t-s)u(s)ds + f(u_t), & \text{for } t \geq 0, \\ u_0 = \varphi \in \mathscr{B}. \end{cases} \tag{1.3}$$

Under the assumptions that $f : \mathscr{B} \to X$ is continuously differentiable, and f' is locally Lipschitz continuous, the local existence and regularity of mild solutions for Eq. (1.3) were obtained there.

    Meanwhile, the nonlinear neutral functional integro-differential equations with time delay have also been investigated extensively in these years, see [11–18] and the references therein. Ezzinbi etc [11,12] studied the existence of mild solutions for this type of neutral equations by using Banach fixed point theorem, and the regularity of solutions was discussed there under the conditions that the nonlinear functions are continuously differentiable. On the other hand, functional (integro) differential equations with state-dependent delay appear frequently in various models and hence the study of this kind of equations has received great attention in the last years too. Some recent works can be found in [19–28]. Hernández etc [19] investigated

the existence of mild solutions for a class of abstract partial functional differential equations with state-dependent delay by using schauder fixed point theorem. With the help of the theory of resolvent operators and Leray-Schauder Alternative type fixed point theorem, Dos Santos [21] studied the existence result for partial neutral integro-differential equations with state-dependent delay. While in [22], the authors have considered a partial fractional neutral functional integro-differential equation with state-dependent delay and formulated a new set of sufficient conditions proving the existence of mild solutions for considered system with the help of fixed point theorem.

In this paper, inspired by the work in [29, 30], we shall discuss the existence and regularity of solutions and a blowup alternative result for Eq. (1.1) by using the theory of fractional power operators and α-norm, that is, we shall restrict Eq. (1.1) in a Banach space $X_\alpha(\subset X)$. To obtain the existence of mild solutions for System (1.1), we assume that the analytic semigroup $S(t)_{t \geq 0}$ generated by $(-A, D(A))$ is compact for $t > 0$ so that the fixed point principle for condensing maps is applied, which is quite different from the works in [19, 21, 22, 27]. We then study the regularity of mild solutions. We will show that each mild solution may be a strong solution under proper conditions. In particular, we just require that $F(\cdot, \cdot)$ and $G(\cdot, \cdot)$ satisfy the lipschitz conditions, other than continuous differentiability as in [4, 5, 11, 12]. Finally, we obtain a blowup alternative result for Eq. (1.1) by utilizing a piecewise extended method and resolvent operators. We point out here that, so far, there is very few relevant papers that discuss blowup problem for neutral partial integro-differential equation with state-dependent delay. Clearly, our obtained results in this paper extend and develop the existing results in the above mentioned work.

The whole article is arranged as follows: we firstly introduce some preliminaries about analytic resolvent operators and phase space for state-dependent delay in Section 2. Particularly, to make them to be still valid in our situation, we have restated the axioms of phase space on the space X_α. The existence of mild solutions is discussed in Section 3 by applying fixed point theorem. In section 4, we establish some sufficient conditions to guarantee the existence of strong solutions. The blowup alternative result is discussed in section 5. In Section 6, an example is presented to show the applications of the obtained results.

2. Preliminaries

Let X be a Banach space with norm $\| \cdot \|$. Throughout this paper, we always assume that $-A : D(A) \subseteq X \to X$ is the infinitesimal generator of a compact analytic semigroup $(S(t))_{t \geq 0}$. Let Y be the Banach space $(D(A), \| \cdot \|_1)$ with the graph norm $\|x\|_1 = \|Ax\| + \|x\|$, for $x \in D(A)$. We assume $0 \in \rho(A)$, then it is possible to define the fractional power A^α, for $0 < \alpha \leq 1$, as a closed linear operator on its domain $D(A^\alpha)$. Furthermore, the subspace $D(A^\alpha)$ is dense in X and the expression

$$\|x\|_\alpha = \|A^\alpha x\|, \qquad x \in D(A^\alpha),$$

defines a norm on $D(A^\alpha)$. Denoting the space $(D(A^\alpha), \| \cdot \|_\alpha)$ by X_α, then it is well known that for each $0 < \alpha \leq 1$, X_α is a Banach space, $X_\alpha \hookrightarrow X_\beta$ for $0 < \beta < \alpha \leq 1$ and the imbedding is compact whenever $R(\lambda, A) = (\lambda I + A)^{-1}$, the resolvent operator of $-A$, is compact. $\mathscr{L}(X_\alpha, X_\beta)$ will denote the space of bounded linear operators $X_\alpha \to X_\beta$ with norm $\| \cdot \|_{\alpha, \beta}$ and $X_0 = X$. Hereafter we denote

by $C([0, T], X_\alpha)$ the Banach space of continuous functions from $[0, T]$ to X_α with the norm

$$\|x\|_C = \sup_{0 \le t \le T} \|A^\alpha x(t)\|, \quad x \in C([0, T], X_\alpha).$$

For the theory of operator semigroup we refer to [31] and [32].

To study the equation (1.1), we assume that the histories $x_t : (-\infty, 0] \to X$, $x_t(\theta) = x(t + \theta)$, belong to some abstract phase space $\mathscr{B}$, which is defined axiomatically. In this article, we employ an axiomatic definition of the phase space $\mathscr{B}$ introduced by Hale and Kato [33] and follow the terminology used in [34]. Thus, $\mathscr{B}$ will be a linear space of functions mapping $(-\infty, 0]$ into X endowed with a seminorm $\|\cdot\|_\mathscr{B}$. We assume that $\mathscr{B}$ satisfies the following axioms:

(A) If $x : (-\infty, \sigma + a) \to X$, $a > 0$, is continuous on $[\sigma, \sigma + a]$ and $x_\sigma \in \mathscr{B}$, then for every $t \in [\sigma, \sigma + a)$ the followings hold:
(i) x_t is in $\mathscr{B}$;
(ii) $\|x(t)\| \le H\|x_t\|_\mathscr{B}$;
(iii) $\|x_t\|_\mathscr{B} \le K(t - \sigma) \sup\{\|x(s)\| : \sigma \le s \le t\} + M(t - \sigma)\|x_\sigma\|_\mathscr{B}$.
Here $H \ge 0$ is a constant, $K, M : [0, +\infty) \to [0, +\infty)$, $K(\cdot)$ is continuous and $M(\cdot)$ is locally bounded, and H, $K(\cdot)$, $M(\cdot)$ are independent of $x(t)$.

(A_1) For the function $x(\cdot)$ in (A), x_t is a $\mathscr{B}$-valued continuous function on $[\sigma, \sigma + a]$.

(B) The space $\mathscr{B}$ is complete.

We denote by $\mathscr{B}_\alpha$ the set of all the elements in $\mathscr{B}$ which takes values in space X_α, that is,

$$\mathscr{B}_\alpha := \{\varphi \in \mathscr{B} : \ \varphi(\theta) \in X_\alpha \text{ for all } \theta \le 0\}.$$

Then $\mathscr{B}_\alpha$ becomes a subspace of $\mathscr{B}$ endowed with the seminorm $\|\cdot\|_{\mathscr{B}_\alpha}$ which is induced by $\|\cdot\|_\mathscr{B}$ through $\|\cdot\|_\alpha$. More precisely, for any $\varphi \in \mathscr{B}_\alpha$, the seminorm $\|\cdot\|_{\mathscr{B}_\alpha}$ is defined by $\|A^\alpha \varphi(\theta)\|$, instead of $\|\varphi(\theta)\|$. For example, let the phase space $\mathscr{B} = C_r \times L^p(g : X)$, $r \ge 0$, $1 \le p < \infty$ (cf. [34]), which consists of all classes of functions $\varphi \in (\infty, 0] \to X$ such that φ is continuous on $[-r, 0]$, Lebesgue-measurable, and $g\|\varphi(\cdot)\|^p$ is Lebesgue integrable on $(-\infty, -r)$, where $g : (-\infty, -r) \to \mathbb{R}$ is a positive Lebesgue integrable function. The seminorm in $\mathscr{B}$ is defined by

$$\|\varphi\|_\mathscr{B} = \sup\{\varphi(\theta) : -r \le \theta \le 0\} + \left(\int_{-\infty}^{-r} g(\theta)\|\varphi(\theta)\|^p d\theta\right)^{\frac{1}{p}}.$$

Then the seminorm in $\mathscr{B}_\alpha$ is defined by

$$\|\varphi\|_{\mathscr{B}_\alpha} = \sup\{\|A^\alpha \varphi(\theta)\| : -r \le \theta \le 0\} + \left(\int_{-\infty}^{-r} g(\theta)\|A^\alpha \varphi(\theta)\|^p d\theta\right)^{\frac{1}{p}}.$$

See also the space $\mathscr{C}_{g, \frac{1}{2}}$ presented in Section 6. Hence, since X_α is still a Banach space, we will assume that the subspace $\mathscr{B}_\alpha$ also satisfies the following conditions:

(A') If $x : (-\infty, \sigma + a) \to X_\alpha$, $a > 0$, is continuous on $[\sigma, \sigma + a]$ (in $\alpha-$norm) and $x_\sigma \in \mathscr{B}_\alpha$, then for every $t \in [\sigma, \sigma + a)$ the followings hold:
(i) x_t is in $\mathscr{B}_\alpha$;
(ii) $\|x(t)\|_\alpha \le H\|x_t\|_{\mathscr{B}_\alpha}$;
(iii) $\|x_t\|_{\mathscr{B}_\alpha} \le K(t - \sigma) \sup\{\|x(s)\|_\alpha : \sigma \le s \le t\} + M(t - \sigma)\|x_\sigma\|_{\mathscr{B}_\alpha}$.
Here H, $K(\cdot)$ and $M(\cdot)$ are as in $(A)(iii)$ above.

(A_1') For the function $x(\cdot)$ in (A), x_t is a $\mathscr{B}_\alpha$-valued continuous function on $[\sigma, \sigma+a]$.

(B') The space $\mathscr{B}_\alpha$ is complete.

For any $\varphi \in \mathscr{B}_\alpha$, the notation $\varphi_t, t \leq 0$, represents the function $\varphi_t(\theta) = \varphi(t+\theta)$. Then, for the function $x(\cdot)$ in axiom (A') with $x_0 = \varphi$, we may extend the mapping $t \to x_t$ to the whole interval $(-\infty, T]$ by setting $x_t = \varphi_t$ as $t \leq 0$. On the other hand, for the continuous function $\rho : [0, T] \times \mathscr{B}_\alpha \to (-\infty, T]$, we introduce the set

$$\Re(\rho^-) = \{\rho(s, \psi) : \rho(s, \psi) \leq 0, (s, \psi) \in [0, T] \times \mathscr{B}_\alpha\}$$

and give the following hypothesis on φ_t:

(H_0) The function $t \to \varphi_t$ is continuous from $\Re(\rho^-)$ into $\mathscr{B}_\alpha$ and there exists a continuous and bounded function $J^\varphi : \Re(\rho^-) \to (0, \infty)$ such that, for each $t \in \Re(\rho^-)$,

$$\| \varphi_t \|_{\mathscr{B}_\alpha} \leq J^\varphi(t) \| \varphi \|_{\mathscr{B}_\alpha} .$$

Combining the phase spaces axioms and (H_0), we have the following lemma, see [34].

Lemma 2.1. *If $x : (-\infty, T] \to X_\alpha$ is a function such that $x_0 = \varphi$ and $x \mid_{[0,T]} \in C([0,T], X_\alpha)$, then*

$$\|x_s\|_{\mathscr{B}_\alpha} \leq \left(M_T + \tilde{J}^\varphi\right) \|\varphi\|_{\mathscr{B}_\alpha} + K_T \sup\left\{\|x(\theta)\|; \theta \in [0, \max\{0, s\}]\right\}, s \in \Re(\rho^-) \cup$$

$[0, T],$

where $\tilde{J}^\varphi = \sup\limits_{t \in \Re(\rho^-)} J^\varphi(t), M_T = \sup\limits_{t \in [0,T]} M(t)$ and $K_T = \max\limits_{t \in [0,T]} K(t)$.

The theory of resolvent operator plays an essential role in investigating the existence of solutions of Eq. (1.1). Next we collect the definition and basic results about this theory, see [1–3] for more details.

Definition 2.1. A family of bounded linear operators $R(t) \in \mathscr{L}(X)$ for $t \in [0, T]$ is called resolvent operators for

$$\begin{cases} \dfrac{d}{dt}x(t) = -Ax(t) + \displaystyle\int_0^t \Upsilon(t - s)x(s)ds, \\ x(0) = x_0 \in X, \end{cases} \tag{2.1}$$

if

(i) $R(0) = I$ and $\|R(t)\| \leq N_1 e^{\omega t}$ for some $N_1 \geq 1, \omega \in \mathbb{R}$.

(ii) for all $x \in X$, $R(t)x$ is continuous in $t \in [0, T]$.

(iii) $R(t) \in \mathscr{L}(Y)$, for $t \in [0, T]$. For $x \in Y$, $R(t)x \in C^1([0, T], X) \cap C([0, T], Y)$ and for $t \geq 0$ such that

$$R'(t)x = -AR(t)x + \int_0^t \Upsilon(t - s)R(s)xds$$

$$= -R(t)Ax + \int_0^t R(t - s)\Upsilon(s)xds, \tag{2.2}$$

where Y is the space $D(-A)$ with the graph norm.

We always assume the following hypotheses on operator $-A$ and $\Upsilon(t)$:

$(V1)$ $-A$ generates an analytic semigroup on X. $\Upsilon(t)$ is a closed operator on X with domain at least $D(A)$ a.e. $t \geq 0$, $\Upsilon(t)x$ is strongly measurable for each $x \in D(-A)$ and $\| \Upsilon(t) \|_{1,0} \leq b(t), b(t) \in L^1(0,\infty)$ with $b^*(\lambda)$ absolutely convergent for $Re\lambda > 0$, where $b^*(\lambda)$ denotes the Laplace transform of $b(t)$.

$(V2)$ $\rho(\lambda) = (\lambda I + A - \Upsilon^*(\lambda))^{-1}$ exists as a bounded operator on X which is analytic for λ in the region $\Lambda = \{\lambda \in \mathbb{C} : |arg\lambda| < (\pi/2) + \delta\}$, where $0 < \delta < \pi/2$. In Λ, if $| \lambda | \geq \varepsilon > 0$, then there exists a constant $M = M(\varepsilon) > 0$ so that $\| \rho(\lambda) \| \leq M/|\lambda|$.

$(V3)$ $A\rho(\lambda) \in \mathscr{L}(X)$ for $\lambda \in \Lambda$ and are analytic on Λ into $\mathscr{L}(X)$. $\Upsilon^*(\lambda) \in \mathscr{L}(Y,X)$ and $\Upsilon^*(\lambda)\rho(\lambda) \in \mathscr{L}(Y,X)$ for $\lambda \in \Lambda$. Given $\varepsilon > 0$, there exists $M = M(\varepsilon) > 0$ so that for $\lambda \in \Lambda$ with $| \lambda | \geq \varepsilon$, $\|A\rho(\lambda)\|_{1,0} + \|\Upsilon^*(\lambda)\rho(\lambda)\|_{1,0} \leq M \diagup | \lambda |$, and $\| \Upsilon^*(\lambda) \|_{1,0} \to 0$ as $| \lambda | \to \infty$ in Λ. In addition, $\|A\rho(\lambda)\| \leq M | \lambda |^n$ for some $n > 0, \lambda \in \Lambda$ with $| \lambda | \geq \varepsilon$. Further, there exists $D \subset D(A^2)$ which is dense in Y such that $A(D)$ and $\Upsilon^*(\lambda)D$ are contained in Y and $\| \Upsilon^*(\lambda)x \|_1$ is bounded for each $x \in D, \lambda \in \Lambda, | \lambda | \geq \varepsilon$.

Then, it follows from [2] that, under the conditions $(V1)-(V3)$, there is an analytic resolvent operator $R(t)$ for linear system (2.1) which is given by $R(0) = I$ and

$$R(t)x = (2\pi i)^{-1} \int_\Gamma e^{\lambda t}(\lambda I + A - \Upsilon^*(\lambda))^{-1}xd\lambda, \qquad t > 0,$$

or, equivalently, using the notation of $(V2)$,

$$R(t)x = (2\pi i)^{-1} \int_\Gamma e^{\lambda t}\rho(\lambda)xd\lambda, \qquad t > 0,$$

where Γ is a contour of the type used to obtain an analytic semigroup. We can select contour Γ, included in the region Λ, consisting of Γ_1, Γ_2 and Γ_3, where

$$\Gamma_1 = \left\{ re^{i\varphi} : r \geq 1 \right\}, \quad \Gamma_3 = \left\{ re^{-i\varphi} : r \geq 1 \right\}, \quad \frac{\pi}{2} < \varphi < \frac{\pi}{2} + \delta,$$
$$\Gamma_2 = \left\{ e^{i\theta} : -\varphi \leq \theta \leq \varphi \right\},$$

oriented so that $Im\lambda$ is increasing on Γ_1 and Γ_2. And there also exist $N, C_\alpha > 0$ such that

$$\|R(t)\| \leq N \quad \text{and} \quad \|A^\alpha R(t)\| \leq \frac{C_\alpha}{t^\alpha}, \quad 0 < t \leq T, \alpha > 0. \tag{2.3}$$

We can further get the following lemmas:

Lemma 2.2. *(see [35], Lemma 2.2) $AR(t)$ is continuous for $t > 0$ in the uniform operator topology of $\mathscr{L}(X)$.*

Lemma 2.3. *(see [36], Lemma 2.3) $R(t)$ $(t > 0)$ is continuous in t in uniform operator topology of $\mathscr{L}(X)$.*

Further, for the sake of simplicity, we also require that A^α commute with $R(t)$ for any $0 \leq \alpha \leq 1$, that is, for any $x \in D(A^\alpha)$, there holds

$$A^\alpha R(t)x = R(t)A^\alpha x.$$

Generally speaking, this commutation is not always valid. We point out that, however, this commutation can be realized in many cases. For example, let $\Upsilon(t - s) = b(t - s)A$ with $b(t)$ a scalar function defined on $(0, +\infty)$, then, the linear problem (2.1) becomes

$$\begin{cases} \dfrac{d}{dt}x(t) = -Ax(t) + \displaystyle\int_0^t b(t - s)Ax(s)ds, \\ x(0) = x_0 \in X. \end{cases} \qquad (2.4)$$

If we impose the following conditions on System (2.4),

$(V1')$ The operator $-A$ generates an analytic semigroup on X. In particular

$$\Lambda_1 = \{\lambda \in \mathbb{C} : |arg\lambda| < (\pi/2) + \delta_1\}, 0 < \delta_1 < \pi/2$$

is contained in the resolvent set of $-A$ and $\|(\lambda I + A)^{-1}\| \leq M/|\lambda|$ on Λ_1 for some constant $M > 0$. The scalar function $b(\cdot)$ is in $L^1(0, \infty)$ with $b^*(\lambda)$ absolutely convergent for $Re\lambda > 0$.

$(V2')$ There exists $\Lambda = \{\lambda \in \mathbb{C} : |arg\lambda| < (\pi/2) + \delta_2\}, 0 < \delta_2 < \pi/2$, so that $\lambda \in \Lambda$ implies $g_1(\lambda) = 1 + b^*(\lambda)$ exists and is not zero. Further $\lambda g_1^{-1}(\lambda) \in \Lambda_1$ for $\lambda \in \Lambda$.

$(V3')$ In Λ, $b^*(\lambda) \to 0$ as $|\lambda| \to \infty$.

Then, from [2], conditions $(V1) - V3)$ are satisfied and the resolvent operator $R(t)$ is given by

$$R(t)x = (2\pi i)^{-1} \int_\Gamma e^{\lambda t} g_1^{-1}(\lambda) \left(\lambda g_1^{-1}(\lambda)I - A\right)^{-1} xd\lambda,$$

and we see that $A^\alpha R(t) = R(t)A^\alpha$ in this situation.

Now we end this section by stating the following fixed point principle which will be used in Section 3.

Theorem 2.1. *(see [37]) Assume that P is a condensing operator on a Banach space X, i.e., P is continuous and takes bounded sets into bounded sets, and $\alpha(P(B)) \leq \alpha(B)$ for every bounded set B of X with $\alpha(B) > 0$. If $P(H) \subseteq H$ for a convex, closed, and bounded set H of X, then P has a fixed point in H (where $\alpha(\cdot)$ denotes the kuratowski measure of non-compactness).*

Remark 2.1. It is easy to see that, if $P = P_1 + P_2$ with P_1 a contractive operator and P_2 a completely continuous one, then P is a condensing operator on X.

3. Existence of mild solutions

The mild solution of Eq. (1.1) expressed by the resolvent operator is defined as follows.

Definition 3.1. A function $x(\cdot) : (-\infty, T] \to X_\alpha$ is said to be a mild solution of Eq. (1.1), if $x_0 = \varphi$, the restriction of $x(\cdot)$ to interval $[0, T]$ is continuous and the

following integral equation is verified

$$x(t) = R(t)\big[\varphi(0) + F(0,\varphi)\big] - F(t, x_t) - \int_0^t \int_0^s R(t-s)\Upsilon(s-\tau)F(\tau, x_\tau)d\tau ds$$

$$+ \int_0^t R(t-s)G(s, x_{\rho(s,x_s)})ds$$

for $t \in [0, T]$. The last two terms are integrals in sense of Bocher [38].

To guarantee the existence of mild solutions, we impose the following restrictions on Eq. (1.1). Let $\alpha \in (0, 1)$ be given.

(H_1) There exists a constant $\beta \in (0,1)$ with $0 < \alpha + \beta \leq 1$, such that $\Upsilon(t) \in \mathscr{L}(X_{\alpha+\beta}, X)$ for each $t \in [0, T]$. Suppose that there exists a positive number M_1 such that
$$\|\Upsilon(t)\|_{\alpha+\beta, \, 0} \leq M_1 \qquad t \in [0, T].$$

(H_2) The function $F : [0, T] \times \mathscr{B}_\alpha \to X_{\alpha+\beta}$ satisfies the lipschitz condition, i.e., there exists a constant $L > 0$ such that :
$$\big\|F(t_1, \psi_1) - F(t_2, \psi_2)\big\|_{\alpha+\beta} \leq L\big(|t_1 - t_2| + \|\psi_1 - \psi_2\|_{\mathscr{B}_\alpha}\big)$$

for any $0 \leq t_1, t_2 \leq T, \psi_1, \psi_2 \in \mathscr{B}_\alpha$. Moreover, there also holds that :
$$\big\|F(t, \psi)\big\|_{\alpha+\beta} \leq L\big(\|\psi\|_{\mathscr{B}_\alpha} + 1\big)$$

holds for any $(t, \psi) \in [0, T] \times \mathscr{B}_\alpha$, where β is defined by (H_1).

(H_3) The function $G : [0, T] \times \mathscr{B}_\alpha \to X$ satisfies the following conditions:
(i) for each $t \in [0, T]$, the function $G(t, \cdot) : \mathscr{B}_\alpha \to X$ is continuous and for each $\psi \in X_\alpha$ the function $G(\cdot, \psi) : [0, T] \to X$ is strongly measurable;
(ii) for each positive number $r > 0$, there is a positive function $g_r \in C([0, T])$ such that
$$\sup_{\|\psi\|_{\mathscr{B}_\alpha} \leq r} \|G(t, \psi)\| \leq g_r(t)$$

and
$$\liminf_{r \to \infty} \frac{\|g_r\|_C}{r}ds = \delta < \infty.$$

Under these conditions we prove the following existence theorem.

Theorem 3.1. *Let $\varphi \in \mathscr{B}_\alpha$. Assume that assumptions $(H_0) - (H_3)$ hold, then Eq. (1.1) has a mild solution provided that*

$$K_T\left[M_0 L + \frac{C_\alpha}{1-\alpha}T^{2-\alpha}M_1 L + \frac{C_\alpha}{1-\alpha}T^{1-\alpha}\delta\right] < 1, \qquad (3.1)$$

where $M_0 = \|A^{-\beta}\|$.

Proof. Let $\mathbb{D} = \{x \in C([0, T], X_\alpha), x(0) = \varphi(0)\}$ and we define the operator Φ on $\mathbb{D}$ as

$$(\Phi x)(t) = R(t)[\varphi(0) + F(0,\varphi)] - F(t, \bar{x}_t) - \int_0^t \int_0^s R(t-s)\Upsilon(s-\tau)F(\tau, \bar{x}_\tau)d\tau ds$$

$$+ \int_0^t R(t-s)G(s, \bar{x}_{\rho(s,\bar{x}_s)})ds.$$

Here $\bar{x} : (-\infty, T] \to X_\alpha$ satisfies $\bar{x}_0 = \varphi$ and $\bar{x} = x$ on $[0, T]$. Using Axiom (A'), the strong continuity of $R(t)$ and conditions (H_1)-(H_3), we infer that $\Phi x \in C([0, T], X_\alpha)$. We denote $B_r(0, \mathbb{D}) = \{x \in \mathbb{D} : \|x\|_C \leq r\}$.

Clearly, for each $r > 0$, $B_r(0, \mathbb{D})$ is a bounded closed convex subset in $C([0, T], X_\alpha)$. We first prove that there exists $r > 0$ such that $\Phi(B_r(0, \mathbb{D})) \subset B_r(0, \mathbb{D})$. In fact, if this is not true, then for each $r > 0$ there exist $x^r \in B_r(0, \mathbb{D})$ and $t^r \in [0, T]$ such that $r < \|\Phi x^r(t^r)\|_\alpha$. We note that, from Lemma 2.1,

$$\|\bar{x}^r{}_{t^r}\|_{\mathscr{B}_\alpha} \leq rK_T + (M_T + \tilde{J}^\varphi)\|\varphi\|_{\mathscr{B}_\alpha} := r^*,$$

$$\left\|\bar{x}^r_{\rho(s,\bar{x}^r_s)}\right\|_{\mathscr{B}_\alpha} \leq r^*, \quad \liminf_{r \to \infty} \frac{r^*}{r} = K_T,$$

where $0 < s < t^r$. Then, applying (A') and $(H_1) - (H_3)$, we obtain

$$
\begin{aligned}
r < \ & \left\|(\Phi x^r)(t^r)\right\|_\alpha \\
\leq \ & \left\|R(t^r)[\varphi(0) + F(0, \varphi)]\right\|_\alpha + \left\|F(t^r, \bar{x}^r_{t^r})\right\|_\alpha \\
& + \left\|\int_0^{t^r} R(t^r - s)\int_0^s \Upsilon(s - \tau)F(\tau, \bar{x}^r_\tau)d\tau ds\right\|_\alpha + \left\|\int_0^{t^r} R(t^r - s)G(s, \bar{x}^r_{\rho(s,\bar{x}^r_s)})ds\right\|_\alpha \\
\leq \ & N\left[\|\varphi(0)\|_\alpha + \|A^{-\beta}\|\|F(0, \varphi)\|_{\alpha+\beta}\right] + \|A^{-\beta}\|\|F(t^r, \bar{x}^r_{t^r})\|_{\alpha+\beta} \\
& + \int_0^{t^r} \|A^\alpha R(t^r - s)\|\int_0^s \|\Upsilon(s - \tau)\|_{\alpha+\beta,0}\|F(\tau, \bar{x}^r_\tau)\|_{\alpha+\beta}d\tau ds \\
& + \int_0^{t^r} \|A^\alpha R(t^r - s)\|\|G(s, \bar{x}^r_{\rho(s,\bar{x}^r_s)})\|ds \\
\leq \ & N\left[H\|\varphi\|_{\mathscr{B}_\alpha} + M_0 L(\|\varphi\|_{\mathscr{B}_\alpha} + 1)\right] + M_0 L(r^* + 1) \\
& + \int_0^{t^r} \frac{C_\alpha}{(t^r - s)^\alpha}\int_0^s M_1 L(r^* + 1)d\tau ds + \int_0^{t^r} \frac{C_\alpha}{(t^r - s)^\alpha}g_{r^*}(s)ds \\
\leq \ & N\left[(H + M_0 L)\|\varphi\|_{\mathscr{B}_\alpha} + M_0 L\right] + M_0 L(r^* + 1) + \frac{C_\alpha}{1 - \alpha}T^{2-\alpha}M_1 L(r^* + 1) \\
& + \frac{C_\alpha}{1 - \alpha}T^{1-\alpha}\|g_{r^*}\|_C.
\end{aligned}
$$

Dividing both sides by r and taking the lower limit as $r \to \infty$, we have

$$1 \leq K_T\left[M_0 L + \frac{C_\alpha}{1 - \alpha}T^{2-\alpha}M_1 L + \frac{C_\alpha}{1 - \alpha}T^{1-\alpha}\delta\right],$$

which contradicts (3.1). Hence, there exists $r > 0$ such that $\Phi(B_r(0, \mathbb{D})) \subset B_r(0, \mathbb{D})$. In the rest of the proof r is fixed with this property.

Then we show that Φ is a condensing map from $B_r(0, \mathbb{D})$ into $B_r(0, \mathbb{D})$. To prove this, we decompose $\Phi = \Phi_1 + \Phi_2$ as

$$(\Phi_1 x)(t) = R(t)F(0, \varphi) - F(t, \bar{x}_t) - \int_0^t \int_0^s R(t - s)\Upsilon(s - \tau)F(\tau, \bar{x}_\tau)d\tau ds,$$

$$(\Phi_2 x)(t) = R(t)\varphi(0) + \int_0^t R(t - s)G(s, \bar{x}_{\rho(s,\bar{x}_s)})ds.$$

We will verify that Φ_1 is a contraction while Φ_2 is a completely continuous operator. To prove that Φ_1 is a contraction, we take $x_1, x_2 \in B_r(0, \mathbb{D})$ arbitrarily. For each $t \in [0, T]$, we get that

$$
\begin{aligned}
\big\|(\Phi_1 x_1)(t) &- (\Phi_1 x_2)(t)\big\|_\alpha \\
&\leq \big\|F(t, \bar{x}_{1t}) - F(t, \bar{x}_{2t})\big\|_\alpha \\
&\quad + \left\| \int_0^t \int_0^s R(t-s)\Upsilon(s-\tau)[F(\tau, \bar{x}_{1\tau}) - F(\tau, \bar{x}_{2\tau})]d\tau ds \right\|_\alpha \\
&\leq \|A^{-\beta}\|L\|\bar{x}_{1t} - \bar{x}_{2t}\|_{\mathscr{B}_\alpha} \\
&\quad + \int_0^t \int_0^s \|A^\alpha R(t-s)\|M_1\|F(\tau, \bar{x}_{1\tau}) - F(\tau, \bar{x}_{2\tau})\|_{\alpha+\beta}d\tau ds \\
&\leq \left[M_0 L + \frac{C_\alpha}{1-\alpha}M_1 T^{2-\alpha}L\right] K_T \sup_{0\leq t\leq T}\|x_1(t) - x_2(t)\|_\alpha \\
&= L^* \sup_{0\leq t\leq T}\|x_1(t) - x_2(t)\|_\alpha,
\end{aligned}
$$

where $L^* = \left[M_0 L + \frac{C_\alpha}{1-\alpha}M_1 T^{2-\alpha}L\right] K_T < 1$ by (3.1). Thus,

$$
\big\|\Phi_1 x_1 - \Phi_1 x_2\big\|_C \leq L^*\big\|x_1 - x_2\big\|_C.
$$

Next we prove that Φ_2 is completely continuous in several steps.

(i) Φ_2 is continuous on $B_r(0, \mathbb{D})$.

Let $x^n \subseteq B_r(0, \mathbb{D})$ with $x^n \to x$ ($n \to \infty$) in $C([0,T], X_\alpha)$ for some $x \in B_r(0, \mathbb{D})$. From Axiom (A'), it is easy to see that $\bar{x}_s^n \to \bar{x}_s$ uniformly for $s \in [0, T]$ as $n \to \infty$. By virtue of (H_3) we have

$$
\begin{aligned}
\big\|G(s, \bar{x}_{\rho(s,\bar{x}_s^n)}^n) &- G(s, \bar{x}_{\rho(s,\bar{x}_s)})\big\| \\
&\leq \big\|G(s, \bar{x}_{\rho(s,\bar{x}_s^n)}^n) - G(s, \bar{x}_{\rho(s,\bar{x}_s^n)})\big\| + \big\|G(s, \bar{x}_{\rho(s,\bar{x}_s^n)}) - G(s, \bar{x}_{\rho(s,\bar{x}_s)})\big\|,
\end{aligned}
$$

which implies that $G\big(s, \bar{x}_{\rho(s,\bar{x}_s^n)}^n\big) \to G\big(s, \bar{x}_{\rho(s,\bar{x}_s)}\big)$ as $n \to \infty$ for each $s \in [0, T]$. Note that, by Lemma 2.1 and (H_3),

$$
\big\|G(s, \bar{x}_{\rho(s,\bar{x}_s^n)}^n) - G(s, \bar{x}_{\rho(s,\bar{x}_s)})\big\| \leq 2g_{r^*}(s),
$$

then, by the dominated convergence theorem, we obtain

$$
\begin{aligned}
\big\|\Phi_2 x_n - \Phi_2 x\big\|_C &= \sup_{t\in[0,T]} \left\| \int_0^t R(t-s)\Big[G(s, \bar{x}_{\rho(s,x_s^n)}^n) - G(s, \bar{x}_{\rho(s,\bar{x}_s)})\Big]ds \right\|_\alpha \\
&\leq \sup_{t\in[0,T]} \int_0^t \big\|A^\alpha R(t-s)\big\|\big\|G(s, \bar{x}_{\rho(s,x_s^n)}^n) - G(s, \bar{x}_{\rho(s,\bar{x}_s)})\big\|ds \\
&\longrightarrow 0, \qquad \text{as } n \to \infty,
\end{aligned}
$$

i.e. Φ_2 is continuous.

(ii) $\Phi_2(B_r(0, \mathbb{D})) = \{\Phi_2 x : x \in (B_r(0, \mathbb{D}))\}$ is clearly bounded in $C([0, T], X_\alpha)$.

(iii) $\Phi_2(B_r(0, \mathbb{D})) = \{\Phi_2 x : x \in (B_r(0, \mathbb{D}))\}$ is equicontinuous in $C([0, T], X_\alpha)$.

Let $x \in B_r(0, \mathbb{D}), t_1, t_2 \in (0, T]$ and $\varepsilon > 0$, such that $0 < \varepsilon < t_1 < t_2 \le T$, then

$$
\begin{aligned}
\left\|(\Phi_2 x)(t_1) - (\Phi_2 x)(t_2)\right\|_\alpha &\le \left\|\left[R(t_2) - R(t_1)\right]\varphi(0)\right\|_\alpha \\
&+ \left\|\int_0^{t_1 - \varepsilon}[R(t_2 - s) - R(t_1 - s)]G(s, \bar{x}_{\rho(s, \bar{x}_s)})ds\right\|_\alpha \\
&+ \left\|\int_{t_1 - \varepsilon}^{t_1}[R(t_2 - s) - R(t_1 - s)]G(s, \bar{x}_{\rho(s, \bar{x}_s)})ds\right\|_\alpha \\
&+ \left\|\int_{t_1}^{t_2} R(t_2 - s)G(s, \bar{x}_{\rho(s, \bar{x}_s)})ds\right\|_\alpha \\
&\le \|R(t_2) - R(t_1)\|H\|\varphi\|_{\mathscr{B}_\alpha} \\
&+ \int_0^{t_1 - \epsilon}\|A^{\alpha - 1}\|\|AR(t_2 - s) - AR(t_1 - s)\|g_{r^*}(s)ds \\
&+ \int_{t_1 - \epsilon}^{t_1}\|A^{\alpha - 1}\|\|AR(t_2 - s) - AR(t_1 - s)\|g_{r^*}(s)ds \\
&+ \int_{t_1}^{t_2}\|A^\alpha R(t_2 - s)\|g_{r^*}(s)ds,
\end{aligned}
$$

where $r^* = rK_T + (M_T + \tilde{J}^\varphi)\|\varphi\|_{\mathscr{B}_\alpha}$. As $t_2 \to t_1$ and ε sufficiently small, the right-hand side of the above inequality tends to zero independently of $x \in B_r(0, \mathbb{D})$, since $R(t)$ and $AR(t)$ is continuous in the uniform operator topology on $(0, T]$ by Lemma 2.2 and 2.3. Thus, Φ_2 maps $B_r(0, \mathbb{D})$ into an equicontinuous family of functions on $[0, T]$ (note $(\Phi_2 x)(0) = \varphi(0)$).

(iv) $(\Phi_2 B_r(0, \mathbb{D}))(t) = \{\Phi_2 x(t) : x \in B_r(0, \mathbb{D})\}$ is relatively compact in X_α for each $t \in [0, T]$.

Obviously, $(\Phi_2 x)(0)$ is relatively compact in X_α since $(\Phi_2 x)(0) = \varphi(0)$. Now let $t \in (0, T]$ be fixed, we just need to prove the set

$$
V(t) := \left\{\int_0^t R(t - s)G(s, \bar{x}_{\rho(s, x_s)})ds : x \in B_r(0, \mathbb{D})\right\}
$$

is relatively compact in X_α. Observe that, for $0 < \alpha < \alpha' < 1$,

$$
\begin{aligned}
\left\|A^{\alpha'}\int_0^t R(t - s)G(s, \bar{x}_{\rho(s, \bar{x}_s)})ds\right\| &\le \int_0^t \left\|A^{\alpha'}R(t - s)\right\|\left\|G(s, \bar{x}_{\rho(s, \bar{x}_s)})\right\|ds \\
&\le \frac{C_{\alpha'}T^{1 - \alpha'}}{1 - \alpha'}\|g_{r^*}\|_C,
\end{aligned}
$$

which implies $A^{\alpha'}V(t)$ is bounded in X. Hence we infer that $V(t)$ is relatively compact in X_α by the compactness of operator $A^{-\alpha'} : X \to X_{\alpha'}$ (the imbedding $X_{\alpha'} \hookrightarrow X_\alpha$ is compact). Hence for each $t \in [0, T], (\Phi_2 B_r(0, \mathbb{D}))(t)$ is relatively

compact in X_α. From the Arzela-Ascoli theorem, we deduce that Φ_2 is a completely continuous map.

Therefore, $\Phi = \Phi_1 + \Phi_2$ is a condensing map from $B_r(0, \mathbb{D})$ into $B_r(0, \mathbb{D})$, and by Theorem 2.1, we conclude that there exists a fixed point $x(\cdot)$ for Φ on $C([0, T], X_\alpha)$. Now we define

$$x^*(t) = \begin{cases} x(\cdot) & t \in [0, T]; \\ \varphi(t) & t \in (-\infty, 0), \end{cases}$$

then $x^*(\cdot)$ is the mild solution of equation (1). The proof is completed. $\qquad\square$

4. Regularity of solutions

In this section, we discuss the regularity of mild solutions for Eq. (1.1), that is, we will provide conditions to allow the existence of strong solutions of Eq. (1.1). The strong solutions of Eq. (1.1) are defined as

Definition 4.1. A function $x(\cdot) : (-\infty, T] \to X_\alpha$ is said to be a strong solution of Problem (1.1), if
  (1) $x(t) + F(t, x_t)$ is differentiable a.e. on $(0, T]$ and $x'(\cdot) \in L^1([0, T], X)$;
  (2) $x(\cdot) \in D(A)$ satisfies

$$\frac{d}{dt}\Big[x(t) + F(t, x_t)\Big] = -A\Big[x(t) + F(t, x_t)\Big] + \int_0^t \Upsilon(t - s)x(s)ds + G(t, x_{\rho(t, x_t)}), \ a.e.$$

on $[0, T]$

and

$$x_0 = \varphi \in \mathscr{B}_\alpha.$$

In order to prove the existence of strong solutions for Eq. (1.1), we need to establish the following lemma.

Lemma 4.1. *Let $-A$ be the infinitesimal generator of an analytic semigroup $S(t)_{t \geq 0}$, $R(t)$ is the resolvent operator introduced in Definition 2.1 and $f \in L^1([0, T], X_\alpha)$ satisfies the lipschitz condition, that is, there is $L^* > 0$, s.t.*

$$\|f(t_1) - f(t_2)\|_\alpha \leq L^*|t_1 - t_2|, \quad t_1, t_2 \in [0, T]. \tag{4.1}$$

Then

$$\int_0^t R(t - s)f(s)ds \in D(A)$$

and

$$A \int_0^t R(t - s)f(s)ds = \int_0^t AR(t - s)f(s)ds, \quad t \in [0, T].$$

Proof.   We write

$$\begin{aligned} k(t) : &= \int_0^t R(t - s)f(s)ds \\ &= \int_0^t R(t - s)[f(s) - f(t)]ds + \int_0^t R(t - s)f(t)ds \\ &:= k_1(t) + k_2(t), \end{aligned}$$

and we consider the two parts separately. For $k_1(t)$, we define

$$k_{1,\epsilon}(t) = \int_0^{t-\epsilon} R(t-s)[f(s) - f(t)]ds, \qquad \text{for } t \geq \epsilon$$

and

$$k_{1,\epsilon}(t) = 0, \qquad \text{for } 0 \leq t < \epsilon.$$

From this definition, it is clear that $k_{1,\epsilon} \to k_1(t)$ as $\epsilon \to 0$, $k_{1,\epsilon} \in D(A)$ and for $t \geq \epsilon$,

$$Ak_{1,\epsilon}(t) = \int_0^{t-\epsilon} AR(t-s)[f(s) - f(t)]ds.$$

On the other hand, utilizing condition (4.1), we have that

$$\left\| Ak_{1,\epsilon} - \int_0^t AR(t-s)[f(s) - f(t)]ds \right\| = \left\| \int_{t-\epsilon}^t AR(t-s)[f(s) - f(t)]ds \right\|$$

$$\leq \int_{t-\epsilon}^t \|A^{1-\alpha}R(t-s)\|\|f(s) - f(t)\|_\alpha ds$$

$$\leq \int_{t-\epsilon}^t \frac{C_{1-\alpha}}{(t-s)^{1-\alpha}} L^* |s-t| ds$$

$$= \frac{C_{1-\alpha}L^*}{1+\alpha} \epsilon^{1+\alpha} \to 0 \quad (\epsilon \to 0),$$

which shows that

$$\lim_{\epsilon \to 0} Ak_{1,\epsilon}(t) = \int_0^t AR(t-s)[f(s) - f(t)]ds.$$

The closedness of A then implies that $k_1(t) \in D(A)$ for $t \in [0,T]$ and

$$Ak_1(t) = \int_0^t AR(t-s)[f(s) - f(t)]ds.$$

To prove the same conclusion for $k_2(t)$, we define similarly that

$$k_{2,\epsilon}(t) = \int_0^{t-\epsilon} R(t-s)f(t)ds, \qquad \text{for } t \geq \epsilon$$

and

$$k_{2,\epsilon}(t) = 0, \qquad \text{for } 0 \leq t < \epsilon.$$

Then, it is also clear that $k_{2,\epsilon} \to k_2(t)$ as $\epsilon \to 0^+$ and $k_{2,\epsilon} \in D(A)$, and for $t \geq \epsilon$

$$Ak_{2,\epsilon}(t) = \int_0^{t-\epsilon} AR(t-s)f(t)ds$$

and

$$\lim_{\epsilon \to 0} Ak_{2,\epsilon}(t) = \int_0^t AR(t-s)f(t)ds.$$

The closeness of A then implies that $k_2(t) \in D(A)$ for $t > 0$ and

$$Ak_2(t) = \int_0^t AR(t-s)f(t)ds.$$

To sum up, we complete the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Remark 4.1. From the proofs of this lemma it is readily seen that, for any $x(\cdot) :$ $(-\infty, T] \to X_\alpha$ with $x\,|_{[0,T]}$ continuous, if $s \to F(s, x_s)$ and $s \to G(s, x_{\rho(s,x_s)})$ both satisfy the lipschitz condition, then we similarly have

$$A \int_0^t R(t-s) G(s, x_{\rho(s,x_s)}) ds ds = \int_0^t A R(t-s) G(s, x_{\rho(s,x_s)}) ds,$$

and

$$A \int_0^t \int_0^s R(t-s) \Upsilon(s-\tau) F(\tau, x_\tau) d\tau ds = \int_0^t \int_0^s A R(t-s) \Upsilon(s-\tau) F(\tau, x_\tau) d\tau ds.$$

To prove the existence of strong solutions for Eq. (1.1), we will gain additional properties of the the phase subspace $\mathscr{B}_\alpha$. Let $\mathcal{BC}_\alpha$ be the set of bounded and continuous functions mapping $(-\infty, 0]$ into X_α, and C_{00} be subset consisting of functions in $\mathcal{BC}_\alpha$ with compact support. If $\mathscr{B}_\alpha$ also satisfies the additional axiom (C):

(C) If a uniformly bounded sequence $\{\varphi^n(\theta)\}$ in C_{00} converges to a function $\varphi(\theta)$ uniformly on every compact set on $(-\infty, 0]$, then $\varphi \in \mathscr{B}_\alpha$ and $\lim\limits_{n \to +\infty} \|\varphi^n - \varphi\|_{\mathscr{B}_\alpha} = 0$.

Then $\mathcal{BC}_\alpha$ is continuously imbedded into $\mathscr{B}_\alpha$. Put

$$\|\varphi\|_\infty = \sup\{\|\varphi(\theta)\|_\alpha : \theta \le 0\},$$

for $\varphi \in \mathcal{BC}_\alpha$, then one has that (see [33, 34]).

Lemma 4.2. *If the phase space $\mathscr{B}_\alpha$ satisfies the axiom (C), then $\mathcal{BC}_\alpha \subset \mathscr{B}_\alpha$, and there exists a constant $J > 0$ such that $\|\varphi\|_{\mathscr{B}_\alpha} \le J\|\varphi\|_\infty$ for all $\varphi \in \mathcal{BC}_\alpha$.*

Theorem 4.1. *Let X be a reflexive Banach space and the phase space $\mathscr{B}_\alpha$ satisfies the axiom (C) additionally. Suppose that condition (H_0) and (H_2) are satisfied. Also the following conditions hold:*

(H_1') *The function $F(\cdot, \cdot)$ maps $[0, T] \times \mathscr{B}_\alpha$ into $D(A^{\alpha+\beta})$ and for any function $x(\cdot) : (-\infty, T] \to X_\alpha$ with $x\,|_{[0,T]}$ continuous, the map $t \to A^{\alpha+\beta} F(t, x_t)$ satisfies the lipschitz condition, i.e. there exist $L_1 > 0$ such that*

$$\left\| F(t_2, x_{t_2}) - F(t_1, x_{t_1}) \right\|_{\alpha+\beta} \le L_1 |t_2 - t_1|.$$

(H_2') *Function $G(\cdot, \cdot)$ satisfies the local lipschitz condition, i.e. for each $(t^0, \varphi^0) \in [0, T] \times \mathscr{B}_\alpha$, there exist a neighborhood W of (t^0, φ^0) and constants $L_2 > 0$, such that*

$$\|G(t_2, \varphi_2) - G(t_1, \varphi_1)\| \le L_2[|t_2 - t_1| + \|\varphi_2 - \varphi_1\|_{\mathscr{B}_\alpha}],$$

for any $(t_i, \varphi_i) \in W$, $i = 1,\ 2$.

(H_3') *The initial function $\varphi \in \mathscr{B}_\alpha$ satisfies the lipschitz condition on $(-\infty, 0]$, together with $\varphi(0) + F(0, \varphi) \in D(A^{\alpha+\beta})$.*

(H_4') *For any function $x(\cdot) : (-\infty, T] \to X_\alpha$ with $x\,|_{[0,T]}$ continuous, the map $t \to \rho(t, x_t)$ satisfies the lipschitz condition on $(-\infty, T]$, i.e.*

$$|\rho(t_2, x_{t_2}) - \rho(t_1, x_{t_1})| \le L_4 |t_2 - t_1|.$$

(H$_5'$) $\Upsilon(t) \in \mathscr{L}(X_{\alpha+\beta}, X)$ *also satisfies the lipschitz condition on* $[0, T]$, *i.e.* $\|\Upsilon(t) - \Upsilon(s)\|_{\alpha+\beta,0} \leq L_5|t - s|$.

Then the equation (1.1) has a strong solution on $(-\infty, T]$.

Proof. Let Φ be the operator defind in the proof of Theorem 3.1. And by Theorem 3.1 , we see that Eq. (1.1) has a mild solution $x(\cdot)$ on $(-\infty, T]$. For this $x(\cdot)$, let

$$f(t) = R(t)[\varphi(0) + F(0, \varphi)],$$

$$p(t) = \int_0^t \int_0^s R(t - s)\Upsilon(s - \tau)F(\tau, x_\tau)d\tau ds,$$

$$q(t) = \int_0^t R(t - s)G(s, x_{\rho(s,x_s)})ds.$$

In the sequel we prove that they all satisfy the lipschitz condition in X_α, and then together with that the $F(t, x_t)$ satisfies the lipschitz condition, we derive $x(\cdot)$ satisfies the lipschitz condition on $[\varepsilon, T]$.

Let $t \in [\varepsilon, T]$ and $h > 0$ small enough, then from (2.2), we find

$$
\begin{aligned}
\left\|f(t + h) - f(t)\right\|_\alpha &= \left\|\int_t^{t+h} R'(s)[\varphi(0) + F(0, \varphi)]ds\right\|_\alpha \\
&\leq \left\|\int_t^{t+h} AR(s)[\varphi(0) + F(0, \varphi)]ds\right\|_\alpha \\
&\quad + \left\|\int_t^{t+h} \int_0^s \Upsilon(s - \tau)R(\tau)[\varphi(0) + F(0, \varphi)]d\tau ds\right\|_\alpha \\
&\leq \int_t^{t+h} \|AR(s)\|\|A^{-\beta}\|\|\varphi(0) + F(0, \varphi)\|_{\alpha+\beta}ds \\
&\quad + \int_t^{t+h} \int_0^s \|\Upsilon(s - \tau)A^{-\beta}R(\tau)\| \|\varphi(0) + F(0, \varphi)\|_{\alpha+\beta}d\tau ds \\
&\leq \int_t^{t+h} \frac{C_1}{s}M_0\|\varphi(0) + F(0, \varphi)\|_{\alpha+\beta}ds \\
&\quad + M_0 M_1 NT\|\varphi(0) + F(0, \varphi)\|_{\alpha+\beta}h \\
&= \left[C_1 M_0[ln(t + h) - lnt] + M_0 M_1 NTh\right]\|\varphi(0) + F(0, \varphi)\|_{\alpha+\beta} \\
&\leq \left[C_1 M_0 M^* + M_0 M_1 NT\right]\|\varphi(0) + F(0, \varphi)\|_{\alpha+\beta}h \\
&= L_{01}h,
\end{aligned}
$$

where

$$M^* = Max\{|ln(T + h)|, |ln\varepsilon|\},$$
$$L_{01} := \left[C_1 M_0 M^* + M_0 M_1 NT\right]\|\varphi(0) + F(0, \varphi)\|_{\alpha+\beta}.$$

Hence $f(t)$ satisfies the lipschitz condition. For $p(t)$ we see that

$$
\begin{aligned}
\|p(t + h) - p(t)\|_\alpha &= \left\|\int_0^{t+h} \int_0^s R(t + h - s)\Upsilon(s - \tau)F(\tau, x_\tau)d\tau ds\right. \\
&\quad \left. - \int_0^t \int_0^s R(t - s)\Upsilon(s - \tau)F(\tau, x_\tau)d\tau ds\right\|_\alpha
\end{aligned}
$$

$$\leq \left\| \int_h^{t+h} \int_0^s R(t+h-s)\Upsilon(s-\tau)F(\tau, x_\tau)d\tau ds \right.$$

$$\left. - \int_0^t \int_0^s R(t-s)\Upsilon(s-\tau)F(\tau, x_\tau)d\tau ds \right\|_\alpha$$

$$+ \left\| \int_0^h \int_0^s R(t+h-s)\Upsilon(s-\tau)F(\tau, x_\tau)d\tau ds \right\|_\alpha$$

$$\leq \left\| \int_0^t \int_0^s R(t-s)[\Upsilon(s+h-\tau)-\Upsilon(s-\tau)]F(\tau, x_\tau)d\tau ds \right\|_\alpha$$

$$+ NTM_0M_1 \sup_{0\leq s\leq T} \|F(s, x_s)\|_{\alpha+\beta} h$$

$$\leq L_5 M_0 NT^2 \sup_{0\leq s\leq T} \|F(s, x_s)\|_{\alpha+\beta} h$$

$$+ M_0 M_1 NT \sup_{0\leq s\leq T} \|F(s, x_s)\|_{\alpha+\beta} h$$

$$= L_{02} h.$$

Then $p(t)$ also satisfies the lipschitz condition. Similarly, we have the following estimate with $q(t)$,

$$\|q(t+h) - q(t)\|_\alpha \leq \left\| \int_0^t [R(t+h-s) - R(t-s)]G(s, x_{\rho(s,x_s)})ds \right\|_\alpha$$

$$+ \left\| \int_t^{t+h} R(t+h-s)G(s, x_{\rho(s,x_s)})ds \right\|_\alpha$$

$$\leq \left\| \int_0^t \int_t^{t+h} R'(\tau-s)G(s, x_{\rho(s,x_s)})d\tau ds \right\|_\alpha$$

$$+ \frac{C_\alpha}{1-\alpha} \sup_{0\leq s\leq T} \left\| G(s, x_{\rho(s,x_s)}) \right\| h^{1-\alpha}$$

$$= \left\| \int_0^t \int_t^{t+h} AR(\tau-s)G(s, x_{\rho(s,x_s)})d\tau ds \right\|_\alpha$$

$$+ \left\| \int_0^t \int_t^{t+h} \int_0^{\tau-s} R(\tau-s-\rho)\Upsilon(\rho)G(s, x_{\rho(s,x_s)})d\rho d\tau ds \right\|_\alpha$$

$$+ \frac{C_\alpha}{1-\alpha} \sup_{0\leq s\leq T} \left\| G(s, x_{\rho(s,x_s)}) \right\| h^{1-\alpha}$$

$$\leq \frac{C_{1+\alpha}[t^{1-\alpha} + h^{1-\alpha} - (t+h)^{1-\alpha}]}{\alpha(1-\alpha)} \left\| G(s, x_{\rho(s,x_s)}) \right\|$$

$$+ \frac{C_\alpha M_1[(t+h)^{3-\alpha} - h^{3-\alpha} - t^{3-\alpha}]}{(1-\alpha)(2-\alpha)(3-\alpha)} \sup_{0\leq s\leq T} \left\| G(s, x_{\rho(s,x_s)}) \right\|$$

$$+ \frac{C_\alpha}{1-\alpha} \sup_{0\leq s\leq T} \left\| G(s, x_{\rho(s,x_s)}) \right\| h^{1-\alpha}$$

$$\leq \left(\frac{C_{1+\alpha}}{\alpha(1-\alpha)} + \frac{C_\alpha}{1-\alpha} \right) \sup_{0\leq s\leq T} \left\| G(s, x_{\rho(s,x_s)}) \right\| h^{1-\alpha}$$

$$+ \frac{C_\alpha M_1[(3-\alpha)(T+h)^{2-\alpha} - h^{2-\alpha}]}{(1-\alpha)(2-\alpha)(3-\alpha)} \sup_{0\leq s\leq T} \left\| G(s, x_{\rho(s,x_s)}) \right\| h$$

$$= L_{03}h.$$

From these estimates we see that $f(t), p(t)$ and $q(t)$ all satisfy the lipschitz condition continuous on $[\varepsilon, T]$. So combined condition (H_1'), we deduce that $x(t)$ satisfies the local lipschitz condition on $[\varepsilon, T]$, and it is clear that X_α is a reflexive Banach space since X is so. Hence, $x(t)$ is continuously differentiable a.e. on $[\varepsilon, T]$. Furthermore,

$$
\begin{aligned}
\frac{dp(t)}{dt} &= \lim_{h \to 0} \frac{1}{h} \left[\int_0^{t+h} \int_0^s R(t+h-s)\Upsilon(s-\tau)F(\tau, x_\tau)d\tau ds \right. \\
&\quad \left. - \int_0^t \int_0^s R(t-s)\Upsilon(s-\tau)F(\tau, x_\tau)d\tau ds \right] \\
&= \int_0^t \int_0^s R'(t-s)\Upsilon(s-\tau)F(\tau, x_\tau)d\tau ds \\
&\quad + \lim_{h \to 0} \frac{1}{h} \int_t^{t+h} \int_0^s R(t+h-s)\Upsilon(s-\tau)F(\tau, x_\tau)d\tau ds \\
&= - \int_0^t \int_0^s AR(t-s)\Upsilon(s-\tau)F(\tau, x_\tau)d\tau ds \\
&\quad + \int_0^t \int_0^s \int_0^{t-s} \Upsilon(t-s-\nu)R(\nu)\Upsilon(s-\tau)F(\tau, x_\tau)d\nu d\tau ds \\
&\quad + \int_0^t \Upsilon(t-s)F(s, x_s)ds \\
&= - \int_0^t \int_0^s AR(t-s)\Upsilon(s-\tau)F(\tau, x_\tau)d\tau ds \\
&\quad + \int_0^t \Upsilon(t-s) \int_0^s \int_0^{t-s} R(s-\nu)\Upsilon(\tau-\nu)F(\nu, x_\nu)d\nu d\tau ds \\
&\quad + \int_0^t \Upsilon(t-s)F(s, x_s)ds,
\end{aligned}
$$

$$
\begin{aligned}
\frac{dq(t)}{dt} &= \int_0^t R'(t-s)G(s, x_{\rho(s, x_s)})ds + G(t, x_{\rho(t, x_t)}) \\
&= - \int_0^t AR(t-s)G(s, x_{\rho(s, x_s)})ds + \int_0^t \Upsilon(t-s) \int_0^s R(s-\tau)G(\tau, x_{\rho(\tau, x_\tau)})d\tau ds \\
&\quad + G(t, x_{\rho(t, x_t)})
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{df(t)}{dt} &= R'(t)[\varphi(0) + F(0, \varphi)] \\
&= - AR[\varphi(0) + F(0, \varphi)] + \int_0^t \Upsilon(t-s)R(s)[\varphi(0) + F(0, \varphi)]ds.
\end{aligned}
$$

From above formulas, we obtain that

$$\frac{d}{dt}[x(t) + F(t, x_t)]$$

$$= \frac{d}{dt}\left[f(t) - p(t) + q(t)\right]$$

$$= -A\left[R(t)\left[\varphi(0) + F(0,\varphi)\right] - F(t,x_t) - \int_0^t\int_0^s R(t-s)\Upsilon(s-\tau)F(\tau,x_\tau)d\tau ds\right.$$

$$\left. + \int_0^t R(t-s)G(s,x_{\rho(s,x_s)})ds\right] + \int_0^t\Upsilon(t-s)\left[R(s)\left[\varphi(0) + F(0,\varphi)\right] - F(s,x_s)\right.$$

$$\left. - \int_0^s\int_0^\tau R(s-\nu)\Upsilon(\tau-\nu)F(\nu,x_\nu)d\nu d\tau + \int_0^s R(s-\tau)G(\tau,x_{\rho(\tau,x_\tau)})d\tau\right]ds$$

$$+ G(t,x_{\rho(t,x_t)})$$

$$= -Ax(t) + \int_0^t\Upsilon(t-s)x(s)ds + G(t,x_{\rho(t,x_t)}),$$

which shows $x(t) + F(t,x_t)$ is continuously differentiable a.e. on $[\varepsilon, T]$ and Eq. (1.1) holds.

Taking ε small enough, it indicates that $x(t) + F(t,x_t)$ is continuously differentiable a.e. on $(0,T]$, together with Eq. (1.1) holds. The proof is completed. □

5. Blowup Alternative Result

In this section, based on the local existence of mild solutions for Eq. (1.1) on interval $[0,T]$ (see Theorem 3.1), we will prove that given any $\varphi \in \mathscr{B}_\alpha$, Eq. (1.1) exists a mild solution $x(\cdot) : (-\infty, T_{max}) \longrightarrow X_\alpha$ on a maximal existence interval $(-\infty, T_{max})$ and satisfies the blowup alternative result: either $T_{max} = +\infty$ (i.e. $x(\cdot)$ is a global solution) or else $T_{max} < +\infty$ and $\|x(\cdot)\| \to +\infty$ as $t \to T_{max}^-$ (i.e. $x(\cdot)$ blow up in finite time).

Theorem 5.1. *Assume that assumptions $(H_0)-(H_3)$ hold, then for every $\varphi \in \mathscr{B}_\alpha$, Eq. (1.1) exists a mild solution $x(\cdot)$ on a maximal existence interval $(-\infty, T_{max})$ with $x|_{[0,T_{max})} \in C([0,T_{max}), X_\alpha)$. If $T_{max} < +\infty$, then $\lim\limits_{t \to T_{max}^-} \|x(t)\| = +\infty$.*

Proof. *By Theorem 3.1, we know that there exists a constant $h_0 > 0$ such that Eq. (1.1) has a mild solution $x(\cdot) : (-\infty, h_0] \longrightarrow X_\alpha$. Moreover, $x(\cdot)$ can be extended to a large interval $(-\infty, h_0 + h_1]$ with $h_1 > 0$ by defining $x(t)=y(t)$ on $[h_0, h_0 + h_1]$, where $y(t)$ is the mild solution of the initial value problem*

$$\begin{cases} \dfrac{d}{dt}\left[y(t) + F(t,y_t)\right] = -A[y(t) + F(t,y_t)] + \int_0^t\Upsilon(t-s)y(s)ds + G(t,y_{\rho(t,y_t)}), \\ \quad t \in [h_0, h_0 + h_1], \\ y_0 = x_{h_0}. \end{cases}$$

Hence, repeating the above procedure and using the methods of steps, we can prove that $x(\cdot)$ can be extended to a maximal existence interval $(-\infty, T_{max})$, namely

$x(\cdot) : (-\infty, T_{max}) \longrightarrow X_\alpha$ *is a mild solution of*

$$\begin{cases} \dfrac{d}{dt}\left[x(t) + F(t, x_t)\right] = -A[x(t) + F(t, x_t)] + \displaystyle\int_0^t \Upsilon(t-s)x(s)ds + G(t, x_{\rho(t,x_t)}), \\ t \in [0, T_{max}), \\ x_0 = \varphi \in \mathscr{B}_\alpha. \end{cases}$$

(5.1)

In the following, we prove that the mild solution $x(\cdot)$ of Eq. (5.1) blows up in finite time, i.e. if $T_{max} < +\infty$, then $\lim\limits_{t \to T_{\max}^-} x(t) = +\infty$.

To do so, we first show that $T_{\max} < +\infty$ implies $\limsup\limits_{t \to T_{max}^-} \|x(t)\|_\alpha = +\infty$. In fact, if $T_{\max} < +\infty$ and $\limsup\limits_{t \to T_{\max}^-} \|x(t)\|_\alpha < +\infty$, we can assume that $\|R(t)\| \le N$, $\|x(t)\|_\alpha \le k$, and the conditions (H_1)-(H_3) are satisfied for $t \in [0, T_{max})$. For $\varepsilon < t_1 < t_2 < T_{max}$ with $\varepsilon > 0$ small enough, one has that

$\|x(t_2) - x(t_1)\|_\alpha$

$\le \left\|\left[R(t_2) - R(t_1)\right]\left[\varphi(0) + F(0, \varphi)\right]\right\|_\alpha + \left\|F(t_2, x_{t,2}) - F(t_1, x_{t_1})\right\|_\alpha$

$\quad + \left\|\displaystyle\int_0^{t_2} R(t_2 - s)G(s, x_{\rho(s,x_s)})ds - \int_0^{t_1} R(t_1 - s)G(s, x_{\rho(s,x_s)})ds\right\|_\alpha$

$\quad + \left\|\displaystyle\int_0^{t_2} R(t_2 - s)\int_0^s \Upsilon(s - \tau)F(\tau, x_\tau)d\tau ds \right.$

$\qquad \left. - \displaystyle\int_0^{t_1} R(t_1 - s)\int_0^s \Upsilon(s - \tau)F(\tau, x_\tau)d\tau ds\right\|_\alpha$

$\le \left\|\left[R(t_2) - R(t_1)\right]\left[\varphi(0) + F(0, \varphi)\right]\right\|_\alpha + M_0 L\left(|t_2 - t_1| + \|x_{t,2} - x_{t,1}\|_{\mathscr{B}_\alpha}\right)$

$\quad + \left\|\displaystyle\int_0^{t_1 - \varepsilon}\left[R(t_2 - s) - R(t_1 - s)\right]G(s, x_{\rho(s,x_s)})ds\right\|_\alpha$

$\quad + \left\|\displaystyle\int_{t_1 - \varepsilon}^{t_1}\left[R(t_2 - s) - R(t_1 - s)\right]G(s, x_{\rho(s,x_s)})ds\right\|_\alpha$

$\quad + \left\|\displaystyle\int_{t_1}^{t_2} R(t_2 - s)G(s, x_{\rho(s,x_s)})ds\right\|_\alpha$

$\quad + \left\|\displaystyle\int_0^{t_1 - \varepsilon}\left[R(t_2 - s) - R(t_1 - s)\right]\int_0^s \Upsilon(s - \tau)F(\tau, x_\tau)d\tau ds\right\|_\alpha$

$\quad + \left\|\displaystyle\int_{t_1 - \varepsilon}^{t_1}\left[R(t_2 - s) - R(t_1 - s)\right]\int_0^s \Upsilon(s - \tau)F(\tau, x_\tau)d\tau ds\right\|_\alpha$

$\quad + \left\|\displaystyle\int_{t_1}^{t_2} R(t_2 - s)\int_0^s \Upsilon(s - \tau)F(\tau, x_\tau)d\tau ds\right\|_\alpha$

$\le \left\|\left[R(t_2) - R(t_1)\right]\left[\varphi(0) + F(0, \varphi)\right]\right\|_\alpha + M_0 L\left(|t_2 - t_1| + \|x_{t,2} - x_{t,1}\|_{\mathscr{B}_\alpha}\right)$

$\quad + \displaystyle\int_0^{t_1 - \varepsilon} \|A^{\alpha - 1}\|\|AR(t_2 - s) - AR(t_1 - s)\|\|G(s, x_{\rho(s,x_s)})\|ds$

$$+ \int_{t_1-\varepsilon}^{t_1} \|A^{\alpha-1}\| \|AR(t_2 - s) - AR(t_1 - s)\| \|G(s, x_{\rho(s,x_s)})\| ds$$

$$+ \int_{t_1}^{t_2} \|A^\alpha R(t_2 - s)\| \|G(s, x_{\rho(s,x_s)})\| ds$$

$$+ \int_0^{t_1-\varepsilon} \|A^{\alpha-1}\| \|AR(t_2 - s) - AR(t_1 - s)\| \int_0^s \|\Upsilon(s - \tau)F(\tau, x_\tau)\| d\tau ds$$

$$+ \int_{t_1-\varepsilon}^{t_1} \|A^{\alpha-1}\| \|AR(t_2 - s) - AR(t_1 - s)\| \int_0^s \|\Upsilon(s - \tau)F(\tau, x_\tau)\| d\tau ds$$

$$+ \int_{t_1}^{t_2} \|A^\alpha R(t_2 - s)\| \int_0^s \|\Upsilon(s - \tau)F(\tau, x_\tau)\| d\tau ds$$

$$\to 0, \quad as \ t_1, t_2 \to T_{max}^-, \quad \varepsilon \to 0.$$

Therefore by Cauchy criteria we know that $\lim\limits_{t \to T_{max}^-} x(t) = x(T_{max})$ *exists and by the first part of the proof the solution $x(t)$ can be extended beyond T_{max}, contradicting the maximality of T_{max}. Hence we get that $\limsup\limits_{t \to T_{max}^-} x(t) = +\infty$.*

To conclude the proof we will show that actually $\lim\limits_{t \to T_{max}^-} x(t) = +\infty$. *If this is not true then there is sequence $t_n \to T_{max}^-$ and a constant $k > 0$ such that $\|x(t_n)\|_\alpha \le k$ for all n. We still assume that $\|R(t)\| \le N$ and the conditions (H_1)-(H_3) are fulfilled for $t \in [0, T_{max}]$. Since $t \longrightarrow \|x(t)\|_\alpha$ is continuous and $\limsup\limits_{t \to T_{max}^-} \|x(t)\|_\alpha = +\infty$ we can find a sequence $\{h_n\}$ having the following properties:*

(i) $h_n \to 0$ *as* $n \to +\infty$;

(ii) $\|x(t)\|_\alpha \le N(k + 1)$ *for* $t_n \le t \le t_n + h_n$;

(iii) $\|x(t_n + h_n)\|_\alpha = N(k + 1)$.

So we have

$$
\begin{aligned}
N(k + 1) = & \ \big\|x(t_n + h_n)\big\|_\alpha \\
\le & \ \big\|R(h_n)\big[x(t_n) + F(t_n, x_{t_n})\big] - F(t_n + h_n, x_{t_n+h_n})\big\|_\alpha \\
& + \left\|\int_{t_n}^{t_n+h_n} R(t_n + h_n - s)G(s, x_{\rho(s,x_s)})ds\right\|_\alpha \\
& + \left\|\int_{t_n}^{t_n+h_n} R(t_n + h_n - s)\int_0^s \gamma(s - \tau)F(\tau, x_\tau)d\tau ds\right\|_\alpha \\
\le & \ Nk + M_0\big\|R(h_n)F(t_n, x_{t_n}) - F(t_n + h_n, x_{t_n+h_n})\big\|_{\alpha+\beta} \\
& + \frac{C_\alpha}{1 - \alpha}\|g_k\|_C h_n^{1-\alpha} + \frac{C_\alpha}{1 - \alpha}M_1 T_{max} \sup_{0 \le t \le T_{max}} \big\|F(t, x_t)\big\|_{\alpha+\beta} h_n^{1-\alpha}.
\end{aligned}
$$

By the continuity of $R(t)$, $F(t, x_t)$, the last three terms can be smaller than 1 when $n \to +\infty$. Therefore the proof is complete. $\square$

Remark 5.1. It is easy to see that when X is a reflexive Banach space and the phase space $\mathscr{B}_\alpha$ satisfies the axiom (C), the strong solution of Eq. (1.1) also has a

blowup alternative result, i.e. If assumptions $(H_0), (H_2)$ and $(H_1') - (H_5')$ hold, then for every $\varphi \in \mathscr{B}_\alpha$, Eq. (1.1) exists a strong solution $x(\cdot)$ on a maximal existence interval $(-\infty, T_{max}^-)$. If $T_{max} < +\infty$, then $\lim\limits_{t \to T_{max}^-} \|x(t)\| = +\infty$.

6. An example

In order to show the applications of above theorems, we consider the following system

$$
\begin{cases}
\dfrac{\partial}{\partial t}\left[z(t,x) + \displaystyle\int_{-\infty}^t \int_0^\pi a\left(s-t, x,\ z(s,y) + \dfrac{\partial}{\partial y} z(s,y) \right) dy\, ds \right] = \dfrac{\partial^2}{\partial x^2}\left[z(t,x) \right. \\[3mm]
\left. + \displaystyle\int_{-\infty}^t \int_0^\pi a\left(s-t, x,\ z(s,y) + \dfrac{\partial}{\partial y} z(s,y) \right) dy\, ds \right] + \displaystyle\int_0^t b(t-s) \dfrac{\partial^2}{\partial x^2} z(s,x) ds \\[3mm]
+ \displaystyle\int_{-\infty}^t c(s-t)\left[z\left(s - \sigma(\|z(t,x)\|), x \right) + \dfrac{\partial}{\partial x} z(s,x) \right] ds, \\[3mm]
z(t,0) = z(t,\pi) = 0, \qquad 0 \le t \le T, \\[1mm]
z(\theta, x) = \varphi(\theta, x), \theta \le 0,\ \ 0 \le x \le \pi,
\end{cases}
$$

$$(6.1)$$

where the functions $a(\cdot,\cdot,\cdot)$, $b(\cdot)$, $c(\cdot)$ and $\varphi(\cdot,\cdot)$ will be described below. Let $X = L^2([0,\pi])$ and operator A be defined by

$$Af = -f''$$

with the domain

$$D(A) = H_0^2([0,\pi]) = \left\{ f(\cdot) \in X :\ f',\ f'' \in X,\ f(0) = f(\pi) = 0 \right\}.$$

Then $-A$ generates a strongly continuous semigroup $(S(\cdot))_{t \ge 0}$ which is analytic, compact and self-adjoint. Furthermore, $-A$ has a discrete spectrum, the eigenvalues are $-n^2$, $n \in \mathbb{N}$, with the corresponding normalized eigenvectors $e_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx)$. Then the following properties hold:

(a) If $f \in D(A)$, then

$$Af = \sum_{n=1}^\infty n^2 \langle f, e_n \rangle e_n.$$

(b) For every $f \in X$,

$$S(t)f = \sum_{n=1}^\infty e^{-n^2 t} \langle f, e_n \rangle e_n,$$

$$A^{-\frac{1}{2}} f = \sum_{n=1}^\infty \frac{1}{n} \langle f, e_n \rangle e_n.$$

In particular, $\|S(t)\| \le e^{-t}$, $\|A^{-1}\| = \|A^{-\frac{1}{2}}\| = 1$.

(c) The operator $A^{1/2}$ is given by

$$A^{1/2}f = \sum_{n=1}^{\infty} n\langle f, e_n\rangle e_n,$$

on the space $D(A^{1/2}) = \{f(\cdot) \in X, \sum_{n=1}^{\infty} n\langle f, e_n\rangle e_n \in X\}$.

Here we take $\alpha = \beta = \frac{1}{2}$ and the phase space $\mathscr{B} = \mathscr{C}_g$, where the space $\mathscr{C}_g$ is defined as: let g be a continuous function on $(-\infty, 0]$ with $g(0) = 1$, $\lim\limits_{\theta \to -\infty} g(\theta) = \infty$, and g is decreasing on $(-\infty, 0]$, then

$$\mathscr{C}_g = \left\{\varphi \in C((-\infty, 0]; X) : \sup_{s \leq 0} \frac{\|\varphi(s)\|}{g(s)} < \infty\right\},$$

and the norm is given by, for $\varphi \in \mathscr{C}_g$,

$$|\varphi|_g = \sup_{s \leq 0} \frac{\|\varphi(s)\|}{g(s)}.$$

It is known that $\mathscr{C}_g$ satisfies the axioms (A), (A_1), and (B), see [34]. Further, the subspace $\mathscr{C}_{g,\frac{1}{2}}$ is defined by

$$\mathscr{C}_{g,\frac{1}{2}} = \left\{\varphi \in C((-\infty, 0]; X_{\frac{1}{2}}) : \sup_{s \leq 0} \frac{\|A^{\frac{1}{2}}\varphi(s)\|}{g(s)} < \infty\right\},$$

endowed with the norm $|\varphi|_{g,\frac{1}{2}} = \sup\limits_{s \leq 0} \frac{\|A^{\frac{1}{2}}\varphi(s)\|}{g(s)}$. Clearly, $\mathscr{C}_{g,\frac{1}{2}}$ satisfies correspondingly the axioms (A'), (A'_1), and (B'), and we may choose a proper g such that H, $K(\cdot)$, $M(\cdot) \leq 1$ (see [34]). Thus we obtain $K_T \leq 1$.

Considering the following conditions:

(i) The function $(\partial^2/\partial x^2)a(\theta, x, y)$ is measurable with $a(\cdot, 0, \cdot) = a(\cdot, \pi, \cdot) \equiv 0$, and there is a function $a_1(\cdot, \cdot) \in L^1((-\infty, 0] \times \mathbb{R}, \mathbb{R}^+)$ such that, for $\theta \in (-\infty, 0]$, x, $y \in \mathbb{R}$,

$$\left|\frac{\partial^2}{\partial x^2}a(\theta, x, y_2) - \frac{\partial^2}{\partial x^2}a(\theta, x, y_1)\right| < a_1(\theta, x)|y_2 - y_1|,$$

$$\left|\frac{\partial^2}{\partial x^2}a(\theta, x, y)\right| < a_1(\theta, x)(|y| + 1),$$

and

$$L := \left(2\pi \int_0^\pi \left(\int_{-\infty}^0 g(\theta)a_1(\theta, x)d\theta\right)^2 dx\right)^{\frac{1}{2}} < \infty. \tag{6.2}$$

(ii) The function $\sigma : [0, \infty) \to [0, \infty)$ is continuously differentiable. The function $c : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies

$$\delta := \left(2 \int_{-\infty}^0 g(\theta)c^2(\theta)d\theta\right)^{\frac{1}{2}} < \infty. \tag{6.3}$$

(iii) The function φ defined by $\varphi(\theta)(x) = \varphi(\theta, x)$ belongs to $\mathscr{C}_{g,\frac{1}{2}}$.

(iv) The function $b(\cdot) \in C^1[0, T]$.

Now define the abstract functions F, G on $\mathscr{C}_{g,\frac{1}{2}}$, $\rho : [0, T] \times \mathscr{C}_{g,\frac{1}{2}} \to (-\infty, T]$ and operator $\Upsilon(t) : D(A) \to X$ by

$$F(\varphi)(x) = \int_{-\infty}^{0} \int_{0}^{\pi} a\left(\theta, x, \ \varphi(\theta)(y) + \varphi(\theta)'(y)\right) dy d\theta,$$

$$G(\varphi)(x) = \int_{-\infty}^{0} c(\theta) \left[\varphi(\theta)(x) + \varphi(\theta)'(x)\right] d\theta,$$

$$\rho(t, \varphi) = t - \sigma(\|\varphi(0)\|),$$

$$\Upsilon(t)z(s, x) = b(t)\frac{\partial^2}{\partial x^2}z(s, x).$$

Then the system (6.1) is rewritten as the abstract form (1.1). It is known that there exists $\vartheta \in (0, \pi/2)$ such that

$$\Lambda = \left\{\lambda \in \mathbb{C} : |arg\lambda| < \frac{\pi}{2} + \vartheta\right\} \subset \rho(-A)$$

and then $(V1') - (V3')$ hold. Hence the linear system of system (6.1) has an analytic resolvent operator $(R(t))_{t \geq 0}$ which is given by $R(0) = I$ and

$$R(t)x = \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} g_1^{-1}(\lambda) \left[\lambda g_1^{-1}(\lambda)I - A\right]^{-1} x d\lambda, \qquad t > 0,$$

where $g_1(\lambda) = 1 + b^*(\lambda)$ and the contour Γ is that described in Section 2.

By condition (i) we know that $R(F) \subset D(A)$ since

$$\langle F(\varphi), e_n\rangle = \frac{1}{n}\left\langle \int_{-\infty}^{0} \int_{0}^{\pi} \frac{\partial}{\partial x}a(\theta, x, \varphi(\theta)(y) + \varphi(\theta)'(y))dy d\theta, \ \tilde{e}_n(x)\right\rangle$$

$$= -\frac{1}{n^2}\left\langle \int_{-\infty}^{0} \int_{0}^{\pi} \frac{\partial^2}{\partial x^2}a(\theta, x, \varphi(\theta)(y) + \varphi(\theta)'(y))dy d\theta, \ e_n(x)\right\rangle,$$

where $\tilde{e}_n(x) = \sqrt{\frac{2}{\pi}}\cos(nx)$, $n = 1, 2, \cdots$. Observe now that, for any $\theta \in (-\infty, 0]$,

$$\|\varphi_2(\theta)(x) - \varphi_1(\theta)(x)\|^2 = \sum_{n=1}^{\infty} \langle\varphi_2 - \varphi_1, e_n\rangle^2$$

$$\leq \sum_{n=1}^{\infty} n^2\langle\varphi_2 - \varphi_1, e_n\rangle^2$$

$$\leq \|\varphi_2(\theta)(x) - \varphi_1(\theta)(x)\|_{\frac{1}{2}}^2,$$

and

$$\|\varphi_2(\theta)'(x) - \varphi_1(\theta)'(x)\|^2 = \sum_{n=1}^{\infty} \langle \varphi_2' - \varphi_1', e_n \rangle^2$$

$$= \sum_{n=1}^{\infty} \langle \varphi_2 - \varphi_1, e_n' \rangle^2$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} n^2 \langle \varphi_2 - \varphi_1, e_n \rangle \langle \varphi_2 - \varphi_1, e_m \rangle \langle -e_n'', e_m' \rangle$$

$$\leq \|\varphi_2(\theta)(x) - \varphi_1(\theta)(x)\|_{\frac{1}{2}}^2,$$

we get

$$|\varphi_2(\cdot) - \varphi_1(\cdot)|_g \leq |\varphi_2(\cdot) - \varphi_1(\cdot)|_{g,\frac{1}{2}},$$

$$|\varphi_2(\cdot)' - \varphi_1(\cdot)'|_g \leq |\varphi_2(\cdot) - \varphi_1(\cdot)|_{g,\frac{1}{2}}.$$

Thus, under conditions (i) and (ii), F and G satisfy the assumptions (H_2) and (H_3) with L and δ given by (6.2) and (6.3) respectively. Hence, by Theorem 3.1 the system (6.1) admits a mild solution on $(-\infty, T]$ provided that (3.1) is fulfilled (here K_T, $M_0 \leq 1$).

Furthermore, if take $\mathscr{B}_{\frac{1}{2}} = \mathscr{C}_{g,\frac{1}{2}}^0$, where

$$\mathscr{C}_{g,\frac{1}{2}}^0 = \left\{ \varphi \in \mathscr{C}_{g,\frac{1}{2}} : \lim_{s \leq 0} \frac{\|A^{\frac{1}{2}}\varphi(s)\|}{g(s)} = 0 \right\},$$

so that Axiom (C) is satisfied (see [34]). We impose some restrictions on the function $a(\cdot, \cdot, \cdot)$ that $a(\cdot, \cdot, \cdot) \in C^2$, then for any $z(t, y)$ with $z|_{[0,T]}$ is continuous, the maps

$$t \to F(z_t) = \int_{-\infty}^{t} \int_0^{\pi} a\left(s - t, x, \ z(s, y) + \frac{\partial}{\partial y} z(s, y) \right) dy ds$$

and

$$t \to \rho(t, z_t) = t - \sigma(\|z(\cdot)\|)$$

are continuously differentiable on $[0, T]$ and hence Lipschitz continuous on this interval. Meanwhile $G(\cdot)$ is locally lipschitz continuous on $\mathscr{C}_{g,\frac{1}{2}}^0$ since it is linear. Thus the conditions (H_1'), (H_2'), (H_4') and (H_5') are all satisfied at this moment. Therefore, if $\varphi(\cdot, x)$ is uniformly Lipschitz continuous and $\varphi(0, x) \in D(A)$, then the system (6.1) has a strong solution on $(-\infty, T]$. By Theorem 5.1, the solution $z(\cdot, x)$ exists on a maximal existence interval $(-\infty, T_{max})$ and satisfies the blowup alternative: either $T_{max} = +\infty$ or else $T_{max} < +\infty$ and $\|z(\cdot, x)\| \to +\infty$ as $t \to T_{max}^-$.

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