Journal of Nonlinear Modeling and Analysis
Volume 2, Number 3, September 2020, 333–343
http://jnma.ca; http://jnma.ijournal.cn
DOI:10.12150/jnma.2020.333

Note on Fractional Green's Function*

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Abstract In this paper, we modify some errors on the definition of fractional Green's function in monograph [5], and give the solution of the inhomogenous equation which satisfies the given inhomogenous initial conditions by fractional Green's function.

Keywords Fractional Green's function, Laplace transform, Mittage-Leffler function.

MSC(2010) 26A33, 65L05.

1. Introduction

In the last few decades, fractional differential equations have gained considerable importance and attention due to their applications in science and engineering, such as control, porous media, electrochemistry, viscoelasticity, and electromagnetism theory [1-3, 5-6], ect. There are a large number of papers dealing with the fractional differential equations [7-11]. The Fractional Green's function is a very powerful tool for investigating linear fractional differential equations [3-6]. In Chapter 5 of the monograph [5], the fractional Green's function is defined as follows:

Consider the following equation

$$ {}_0\mathscr{L}_t y(t) \equiv f(t), \tag{1.1}$$

$$ [{}_0\mathscr{D}_t^{\sigma_k - 1} y(t)]_{t=0} = 0, (k = 1, \cdots, n),$$

where

$$ {}_a\mathscr{L}_t y(t) \equiv {}_a \mathscr{D}_t^{\sigma_n} y(t) + \sum_{k=1}^{n-1} p_k(t) {}_a\mathscr{D}_t^{\sigma_{n-k}} y(t) + p_n(t) y(t),$$

$$ {}_a\mathscr{D}_t^{\sigma_k} y(t) \equiv {}_a D_t^{\alpha_k} {}_a D_t^{\alpha_{k-1}} \cdots {}_a D_t^{\alpha_1},$$

$$ {}_a\mathscr{D}_t^{\sigma_k - 1} y(t) \equiv ({}_a D_t^{\alpha_k - 1})({}_a D_t^{\alpha_{k-1}}) \cdots ({}_a D_t^{\alpha_1}),$$

$$ \sigma_k = \sum_{j=1}^{k} \alpha_j, (k = 1, 2, \cdots, n); 0 \le \alpha_j \le 1, (j = 1, 2, \cdots, n). \tag{1.2}$$

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*The authors were supported by the National Natural Science Foundation Fujian Province of China (2016J01032).

Definition 1.1 (see [5]). The function $G(t,\tau)$ satisfying the following conditions
a) ${}_{\tau}\mathscr{L}_t G(t,\tau) = 0 \quad$ for every $\tau \in (0,t)$;
b) $\lim\limits_{\tau \to t-0} ({}_{\tau}\mathscr{D}_t^{\sigma_k - 1} G(t,\tau)) = \delta_{k,n}, \quad k = 0, 1, \cdots, n,$
($\delta_{k,n}$ is Kronecker's delta);
c) $\lim\limits_{\substack{\tau,t \to +0 \\ \tau < t}} ({}_{\tau}\mathscr{D}_t^{\sigma_k} G(t,\tau)) = 0, k = 0, 1, \cdots, n-1$
is called the fractional Green's function of equation (1.1).

The purpose of our paper is to point out that there exist some errors or contradictions in Definition 1.1, and we will provide some examples to illustrate them and modify them. The paper has been organized as follows. In Section 2, we give preliminary facts and provide some basic properties which are needed later. In section 3, we shall point out some errors in Definition 1.1. In section 4 we give some examples to illustrate them and modify Definition 1.1, we also give the solution of the inhomogeneous equation satisfying given inhomogeneous initial conditions by the Laplace transform method and fractional Green's function.

2. Preliminaries and Lemmas

In order to establish our main results we need some preliminary facts and basic lemmas, which we present in this section.

Definition 2.1. Let $f(t)$ be piecewise continuous on $(0,\infty)$ and $p > 0$, then the fractional integral of order p of $f(t)$ is defined by

$$ {}_aD_t^{-p} f(t) = \frac{1}{\Gamma(p)} \int_a^t (t-\tau)^{p-1} f(\tau) d\tau. $$

Definition 2.2 (see [1-3, 5-6]). Let $f(t)$ be piecewise continuous on $(0,\infty)$ and $0 \le m-1 \le v < m \in N$, then the Riemann-Liouville fractional derivative of f is defined by

$$ {}_aD_t^{v} f(t) = \frac{d^m}{dt^m} [{}_aD_t^{v-m} f(t)]. $$

In Definitions 2.1 and 2.2, if $a = 0$, then we denote ${}_aD_t^{-p} f(t)$ and ${}_aD_t^{v} f(t)$ by $D^{-p} f(t)$ and $D^{v} f(t)$, respectively.

Lemma 2.1 (see [1-3, 5-6]). *Let $\alpha > 0$ and $\beta > 0$, then*

$$ {}_aD_t^{-\alpha}[(t-a)^{\beta-1}] = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (t-a)^{\beta+\alpha-1}, $$

$$ {}_aD_t^{\alpha}[(t-a)^{\beta-1}] = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (t-a)^{\beta-\alpha-1}. $$

Lemma 2.2. (see [5-6]). *If $0 \le m-1 \le v < m \in N$, and σ_m is defined as in (1.2), then we have two classical formulas for the Laplaces transform of the fractional derivative as follows:*

$$ L[{}_0D_t{}^{v} f(t)] = s^{v} F(s) - \sum_{k=0}^{m-1} s^{m-k-1} [{}_0D_t^{k-m+v} f(t)]_{t=0} $$

$$= s^v F(s) - \sum_{k=0}^{m-1} s^k [{}_0D_t^{v-k-1} f(t)]_{t=0}$$

and

$$L[{}_0\mathscr{D}_t{}^{\sigma_m} f(t)] = s^{\sigma_m} F(s) - \sum_{k=0}^{m-1} s^{\sigma_m - \sigma_{m-k}} [{}_0\mathscr{D}_t^{\sigma_{m-k}-1} f(t)]_{t=0},$$

where $L[f(t)] = \int_0^\infty e^{-st} f(t) dt \triangleq F(s)$.

Lemma 2.3 (Initial Value Theorem). *If* $v > 0$ *and* $\lim_{s\to+\infty} s^v L[f(t)] = l$, *then*

$$D^{v-1} f(0) = l.$$

Proof. From Lemma 2.2 we clearly see that

$$\begin{aligned} L[D^v f(t)] &= \int_0^\infty D^v f(t) e^{-st} dt \\ &= s^v F(s) - s^{m-1} D^{-m+v} f(0) - \cdots - D^{v-1} f(0). \end{aligned}$$

Let $s \to \infty$, then $e^{-st} \to 0$, $\int_0^\infty D^v f(t) e^{-st} dt \to 0$
and

$$\lim_{s\to\infty} [s^v F(s) - s^{m-1} D^{-m+v} f(0) - \cdots - D^{v-1} f(0)] = 0.$$

If $\lim_{s\to\infty} s^v L[f(t)] = \lim_{s\to\infty} s^v F(s) = l$, then

$$D^{-m+v} f(0) = \cdots = D^{v-2} f(0) = 0,$$

but

$$D^{v-1} f(0) = l.$$

Lemma 2.4 (see [5]). *The relationships*

$${}_aD_t^p({}_aD_t^q f(t)) = {}_aD_t^q({}_aD_t^p f(t)) = {}_aD_t^{p+q} f(t)$$

hold if and only if

$$f^{(j)}(a) = 0, (j = 0, 1, \cdots, r-1),$$

where

$$m-1 \le p < m, n-1 \le q < n, r = \max(m, n).$$

3. Some Errors on Definition 1.1

In this section, we point out four errors in Definition 1.1 and related results in chapter 5 of the monograph [5].

Error 1. In Condition (b) of Definition 1.1, one should delete $k = 0$; In Condition (c), when $k = 0$, σ_0 has not been defined. Therefore, one should define that $\sigma_0 \triangleq 0$.

Error 2. In Condition (c) of Definition 1.1, we indicate that it sometimes contradicts with Condition (b) for some $\alpha_j, (j = 1, .., n)$.

For example, if we take $\alpha_n = 1$, then

$$ {}_{\tau}\mathscr{D}_t^{\sigma_n - 1} = ({}_{\tau}D_t^{\alpha_n - 1})({}_{\tau}D_t^{\alpha_{n-1}}) \cdots ({}_{\tau}D_t^{\alpha_1}) = {}_{\tau}D_t^{\alpha_{n-1}} \cdots {}_{\tau}D_t^{\alpha_1} = {}_{\tau}\mathscr{D}_t^{\sigma_{n-1}}. $$

According to Condition (c), one has

$$ \lim_{\substack{\tau, t \to 0+ \\ \tau < t}} [{}_{\tau}\mathscr{D}_t^{\sigma_{n-1}} G(t, \tau)] = 0. $$

But it follows from Condition (b) that

$$ \lim_{\tau \to t-0} [{}_{\tau}\mathscr{D}_t^{\sigma_n - 1} G(t, \tau)] = 1. $$

This is a contradiction.

Error 3. According to Condition (c) of Definition 1.1, if $G(t, \tau) = G(t - \tau)$ is the Green's function of homogeneous linear equation with constant coefficients

$$ {}_0\mathscr{L}_t y(t) = 0, \tag{3.1} $$

then we can prove that $D^{\lambda} G(t)$ is also a solution of the equation for all $\lambda \in (0, 1)$, where

$$ {}_0\mathscr{L}_t y(t) = D^{\alpha_n} D^{\alpha_{n-1}} \cdots D^{\alpha_1} y(t) + \cdots + p_{n-2} D^{\alpha_2} D^{\alpha_1} y(t) + p_{n-1} D^{\alpha_1} y(t) + p_n y(t). \tag{3.2} $$

Proof. It is sufficient to prove

$$ {}_0\mathscr{L}_t [D^{\lambda} G(t)] = D^{\lambda} [{}_0\mathscr{L}_t G(t)] = 0, $$

for any $0 < \lambda < 1$.

In view of (3.2), we have

$$ \mathscr{L}[D^{\lambda} G(t)] = D^{\alpha_n} D^{\alpha_{n-1}} \cdots D^{\alpha_1} (D^{\lambda} G(t)) + \cdots $$
$$ + p_{n-2} D^{\alpha_2} D^{\alpha_1} (D^{\lambda} G(t)) + p_{n-1} D^{\alpha_1} (D^{\lambda} G(t)) + p_n D^{\lambda} G(t). $$

From Condition (c) of Definition 1.1 we have $\lim\limits_{\substack{\tau, t \to 0+ \\ \tau < t}} G(t, \tau) = 0$. It follows from $0 < \lambda < 1, 0 \leq \alpha_1 \leq 1$ and Lemma 2.4 that

$$ D^{\alpha_1} D^{\lambda} G(t) = D^{\lambda} D^{\alpha_1} G(t). \tag{3.3} $$

From Condition (c) of Definition 1.1 we have $\lim\limits_{\substack{\tau, t \to 0+ \\ \tau < t}} D^{\alpha_1} G(t, \tau) = 0$, by (3.3) and then by Lemma 2.4 for function $D^{\alpha_1} G(t)$, we get

$$ D^{\alpha_2} D^{\alpha_1} D^{\lambda} G(t) = D^{\alpha_2} D^{\lambda} D^{\alpha_1} G(t) = D^{\lambda} D^{\alpha_2} D^{\alpha_1} G(t). \tag{3.4} $$

Similarly, from Condition (c) of Definition 1.1 we have $\lim\limits_{\substack{\tau, t \to +0 \\ \tau < t}} ({}_{\tau}\mathscr{D}_t^{\sigma_2} G(t, \tau)) = \lim\limits_{\substack{\tau, t \to 0+ \\ \tau < t}} D^{\alpha_2} D^{\alpha_1} G(t, \tau) = 0$, by (3.4) and then by Lemma 2.4 for function $D^{\alpha_2} D^{\alpha_1} G(t)$, we have

$$ D^{\alpha_3} D^{\alpha_2} D^{\alpha_1} D^{\lambda} G(t) = D^{\alpha_3} D^{\lambda} D^{\alpha_2} D^{\alpha_1} G(t) = D^{\lambda} D^{\alpha_3} D^{\alpha_2} D^{\alpha_1} G(t). \tag{3.5} $$

By recurrence method, from Condition (c) and Lemma 2.4, we can obtain

$$D^{\alpha_n} D^{\alpha_{n-1}} \cdots D^{\alpha_1}(D^{\lambda} G(t)) = D^{\lambda}(D^{\alpha_n} D^{\alpha_{n-1}} \cdots D^{\alpha_1} G(t)). \tag{3.6}$$

Equations (3.3)-(3.6) lead to

$${}_0\mathscr{L}_t[D^{\lambda} G(t)] = D^{\lambda}[{}_0L_t G(t)] = 0.$$

Therefore, $D^{\lambda}G(t)$ is also a solution of Eq.(3.1) for any $\lambda \in (0,1)$, it means that the set of the solutions of Eq.(3.1) is infinite dimension.

But, as we know, even in the simplest case when $\alpha_j \equiv 1 (j = 1, ..., n)$, Eq.(3.1) reduces to

$$(D^n + p_1 D^{n-1} + ... + p_{n-1} D + p_n) y(t) = 0,$$

that conclusion is not correct.

Using the Laplace transform method, we can see that the number of the linearly independent solutions for linear fractional differential equation (3.1) with constant coefficients is finite (For detail see section 4). That is to say, the set of the solutions of Eq.(3.1) is finite dimension. Apparently, it is a contradiction.

Error 4. According to Definition 1.1, chapter 5 of the monograph [5] states that for the linear inhomogeneous equation with constant coefficients satisfying given inhomogeneous initial conditions as follows

$$\begin{cases} {}_0\mathscr{L}_t y(t) = f(t), \\ [{}_0\mathscr{D}_t^{\sigma_k - 1} y(t)]_{t=0} = b_k. (k = 1, ..., n) \end{cases} \tag{3.7}$$

where ${}_0\mathscr{L}_t y(t)$ is defined as in (3.2). Then the solution of equation (3.7) has the form

$$y(t) = \sum_{k=1}^{n} b_k \psi_k(t) + \int_0^t G(t-\tau) f(\tau) d\tau, \tag{3.8}$$

where

$$\psi_k(t) = {}_0\mathscr{D}_t^{\sigma_n - \sigma_k} G(t), \quad {}_0\mathscr{D}_t^{\sigma_n - \sigma_k} = ({}_0D_t^{\alpha_n})({}_0D_t^{\alpha_{n-1}}) \cdots ({}_0D_t^{\alpha_{k+1}}). \tag{3.9}$$

From (3.8) we clearly see that the number of the linearly independent solutions of the corresponding homogeneous equation is n. But it seems not correct, we will show that the number is not n by the use of the Laplace transform method (For detail see section 4).

4. Some Examples and Main Results

In this section, we shall provide some examples to illustrate our conclusions, give our main results, in which we will modify Definition 1.1, and prove that the number of the linearly independent solutions of corresponding homogeneous linear equation with constant coefficient is no less than n.

For this purpose, we consider some examples at first.

Example 1. Consider

$$\begin{cases} {}_0\mathscr{D}_t^{\sigma_2} y(t) = D^{\alpha_2} D^{\alpha_1} y(t) = f(t), (0 < \alpha_1, \alpha_2 < 1) \\ [{}_0\mathscr{D}^{\sigma_k - 1} y(t)] = 0. \qquad (k = 1, 2) \end{cases} \tag{4.1}$$

It is easy to see that the solution of (4.1) is

$$y(t) = D^{-\alpha_1} D^{-\alpha_2} f(t) = D^{-(\alpha_1+\alpha_2)} f(t)$$
$$= \frac{1}{\Gamma(\alpha_1+\alpha_2)} \int_0^t (t-\tau)^{\alpha_1+\alpha_2-1} f(\tau) d\tau,$$

and the fractional Green's function of the equation

$$_0\mathscr{D}_t^{\sigma_2} y(t) = D^{\alpha_2} D^{\alpha_1} y(t) = 0$$

has the form

$$G(t,\tau) = G(t-\tau) = \frac{1}{\Gamma(\alpha_1+\alpha_2)} (t-\tau)^{\alpha_1+\alpha_2-1}. \tag{4.2}$$

Evidently, the Green's function $G(t,\tau)$ satisfies Condition (a) and (b) of Definition 1.1, that is

(a) $_\tau\mathscr{L}_t^{\sigma_2} G(t,\tau) = (_\tau D_t^{\alpha_2})(_\tau D_t^{\alpha_1})[\frac{1}{\Gamma(\alpha_1+\alpha_2)}(t-\tau)^{\alpha_1+\alpha_2-1}] = 0$ for every $\tau \in (0,t)$;

(b) $\lim_{\tau\to t-0} (_\tau\mathscr{D}_t^{\sigma_k-1} G(t,\tau)) = \delta_{k,n}, \quad k=1,2.$

In fact, one has

$$\lim_{\tau\to t-0} (_\tau\mathscr{D}_t^{\sigma_1-1} G(t,\tau)) = \lim_{\tau\to t-0} D^{\alpha_1-1}[\frac{1}{\Gamma(\alpha_1+\alpha_2)}(t-\tau)^{\alpha_1+\alpha_2-1}]$$
$$= \lim_{\tau\to t-0} [\frac{1}{\Gamma(\alpha_2+1)}(t-\tau)^{\alpha_2}] = 0$$

and

$$\lim_{\tau\to t-0} (_\tau\mathscr{D}_t^{\sigma_2-1} G(t,\tau)) = \lim_{\tau\to t-0} D^{\alpha_2+\alpha_1-1}[\frac{1}{\Gamma(\alpha_1+\alpha_2)}(t-\tau)^{\alpha_1+\alpha_2-1}]$$
$$= \lim_{\tau\to t-0} [\frac{1}{\Gamma(1)}(t-\tau)^{0}] = 1.$$

But $G(t,\tau)$ may not satisfy Condition (c) of Definition 1.1, that is

$$\lim_{\substack{\tau,t\to+0 \\ \tau<t}} (_\tau\mathscr{D}_t^{\sigma_k} G(t,\tau)) = 0 \ (k=0,1)$$

does not hold for every $\alpha_1, \alpha_2 \in [0,1]$.

In fact, it is easy to see that

$$\lim_{\substack{\tau,t\to+0 \\ \tau<t}} G(t,\tau) = \lim_{\substack{\tau,t\to+0 \\ \tau<t}} [\frac{1}{\Gamma(\alpha_1+\alpha_2)}(t-\tau)^{\alpha_1+\alpha_2-1}] \neq 0$$

for $\alpha_1+\alpha_2 \leq 1$, and

$$\lim_{\substack{\tau,t\to+0 \\ \tau<t}} (_\tau\mathscr{D}_t^{\sigma_1} G(t,\tau)) = \lim_{\substack{\tau,t\to+0 \\ \tau<t}} D^{\alpha_1}[\frac{1}{\Gamma(\alpha_1+\alpha_2)}(t-\tau)^{\alpha_1+\alpha_2-1}]$$
$$= \lim_{\substack{\tau,t\to+0 \\ \tau<t}} [\frac{1}{\Gamma(\alpha_2)}(t-\tau)^{\alpha_2-1}] \neq 0.$$

Therefore, we confirm that Condition (c) of Definition is a superfluous condition, and we modify Definition 1.1 as follows.

Definition 4.1. The function $G(t,\tau)$ is called the fractional Green's function of equation (1.1) if it satisfies the following conditions

a) ${}_\tau\mathscr{L}_t G(t,\tau) = 0 \quad$ for every $\tau \in (0,t)$;

b) $\lim\limits_{\tau\to t-0}({}_\tau\mathscr{D}_t^{\sigma_k-1}G(t,\tau)) = \delta_{k,n}, \quad k = 1,\cdots,n.$

($\delta_{k,n}$ is Kronecker's delta).

Example 2. Consider

$$\begin{cases} D^{\alpha_2}D^{\alpha_1}y(t) + aD^{\alpha_1}y(t) + by(t) = f(t), \ (0 < \alpha_1, \alpha_2 < 1) \\ \qquad [{}_0\mathscr{D}^{\sigma_k-1}y(t)] = b_k. \qquad (k = 1, 2) \end{cases} \tag{4.3}$$

Applying Laplace transform of a sequential fractional derivative to equation (4.3), by Lemma 2.2, we obtain

$$s^{\sigma_2}Y(s) - b_2 - b_1 s^{\alpha_2} + a[s^{\sigma_1}Y(s) - b_1] + bY(s) = F(s)$$

or

$$Y(s) = \frac{F(s)}{P(s)} + \frac{b_2 + b_1 a}{P(s)} + \frac{b_1 s^{\alpha_2}}{P(s)}, \tag{4.4}$$

where $Y(s)$ and $F(s)$ denote the Laplace transforms of $y(t)$ and $f(t)$ respectively, and

$$P(s) = s^{\sigma_2} + as^{\sigma_1} + b.$$

Therefore, the fractional Green's function $G(t)$ can be obtained from the inverse Laplace transform with the following expression:

$$P^{-1}(s) = \frac{1}{s^{\sigma_2} + as^{\sigma_1} + b},$$

namely,

$$G(t) = L^{-1}[P^{-1}(s)].$$

That is

$$L[G(t)] = P^{-1}(s).$$

By Lemma 2.2, we have

$$L[D^{\alpha_2}G(t)] = s^{\alpha_2}P^{-1}(s) - D^{\alpha_2-1}G(0).$$

Because of

$$\lim_{s\to\infty} s^{\alpha_2}L[G(t)] = \lim_{s\to\infty}\frac{s^{\alpha_2}}{s^{\sigma_2} + as^{\sigma_1} + b} = 0,$$

therefore, by Lemma 2.3, let $v = \alpha_2 - 1$, we have $D^{\alpha_2-1}G(0) = 0$. So that

$$L[D^{\alpha_2}G(t)] = s^{\alpha_2}P^{-1}(s),$$

and then the inverse Laplace transform of (4.4) gives the solutions of Eq.(4.3)

$$y(t) = \int_0^t G(t-\tau)f(\tau)d\tau + (b_2 + ab_1)G(t) + b_1 D^{\alpha_2}G(t). \tag{4.5}$$

Example 3. Consider

$$\begin{cases} D^{\alpha_3}D^{\alpha_2}D^{\alpha_1}y(t)+aD^{\alpha_2}D^{\alpha_1}y(t)+bD^{\alpha_1}y(t)+cy(t)=f(t), \; (0<\alpha_1,\alpha_2,\alpha_3<1) \\ \left[{}_0\mathscr{D}^{\sigma_k-1}y(t)\right]_{t=0}=b_k. \qquad (k=1,2,3) \end{cases} \tag{4.6}$$

Applying the Laplace transform of a sequential fractional derivative to equation (4.6), we get

$$s^{\sigma_3}Y(s)-b_3-b_2s^{\alpha_3}-b_1s^{\alpha_3+\alpha_2}+a[s^{\sigma_2}Y(s)-b_2-b_1s^{\alpha_2}]+$$

$$b[s^{\sigma_1}Y(s)-b_1]+cY(s)=F(s).$$

So

$$Y(s)=\frac{F(s)}{P(s)}+\frac{b_3+ab_2+bb_1}{P(s)}+\frac{b_2s^{\alpha_3}}{P(s)}+\frac{b_1s^{\alpha_3+\alpha_2}}{P(s)}+\frac{ab_1s^{\alpha_2}}{P(s)}, \tag{4.7}$$

where

$$P(s)=s^{\sigma_3}+as^{\sigma_2}+bs^{\sigma_1}+c.$$

Hence the fractional Green's function $G(t)$ can be obtained from the inverse Laplace transform of (4.7) with the following expression:

$$P^{-1}(s)=\frac{1}{s^{\sigma_3}+as^{\sigma_2}+bs^{\sigma_1}+c}.$$

Using the same arguments as in equation (4.3), we can obtain

$$L[D^{\alpha_2}G(t)]=s^{\alpha_2}P^{-1}(s),$$

$$L[D^{\alpha_3}G(t)]=s^{\alpha_3}P^{-1}(s).$$

From Lemma 2.2, we have

$$L[D^{\alpha_3+\alpha_2}G(t)]=s^{\alpha_3+\alpha_2}P^{-1}(s)-D^{\alpha_3-1}D^{\alpha_2}G(0)-s^{\alpha_3}D^{\alpha_2-1}G(0).$$

Because of

$$\lim_{s\to\infty}s^{\alpha_3}L[D^{\alpha_2}G(t)]=\lim_{s\to\infty}\frac{s^{\alpha_3}s^{\alpha_2}}{s^{\sigma_3}+as^{\sigma_2}+bs^{\sigma_1}+c}=0,$$

therefore by Lemma 2.3, let $v=\alpha_3-1$, for function $D^{\alpha_2}G(t)$, we obtain

$$D^{\alpha_3-1}D^{\alpha_2}G(0)=0.$$

Similarly, by Lemma 2.3, we have

$$D^{\alpha_2-1}G(0)=0.$$

Therefore

$$L[D^{\alpha_3+\alpha_2}G(t)]=s^{\alpha_3+\alpha_2}P^{-1}(s).$$

The inverse Laplace transform of equation (4.7) leads to

$$y(t)=\int_0^t G(t-\tau)f(\tau)d\tau+(b_3+ab_2+bb_1)G(t)$$

$$+b_2 D^{\alpha_3} G(t) + ab_1 D^{\alpha_2} G(t) + b_1 D^{\alpha_3} D^{\alpha_2} G(t). \tag{4.8}$$

In Eq.(1.1), if let $p_k(t) = p_k$ $(k = 1, \cdots, n)$, namely

$$\begin{cases} {}_0\mathscr{L}_t y(t) = f(t), \\ [{}_0\mathscr{D}_t^{\sigma_k - 1} y(t)]_{t=0} = b_k, \ (k = 1, \cdots, n) \end{cases} \tag{4.9}$$

where

$$ {}_a\mathscr{L} y(t) \equiv {}_a\mathscr{D}_t^{\sigma_n} y(t) + \sum_{k=1}^{n-1} p_k [{}_a\mathscr{D}_t^{\sigma_{n-k}} y(t)]] + p_n y(t).$$

We conclude that

Theorem 4.1. *The solution of the equation (4.9) has the form*

$$y(t) = \int_0^t G(t-\tau) f(\tau) d\tau + \sum_{k=1}^{n} b_k \mathscr{D}^{\sigma_n - \sigma_k} G(t) + \sum_{k=1}^{n-1} p_k \sum_{r=1}^{n-k} b_r \mathscr{D}^{\sigma_{n-k} - \sigma_r} G(t). \tag{4.10}$$

Proof. Using the same method as above, we can obtain the Laplace transform of (4.9) is

$$Y(s) = \frac{F(s)}{P(s)} + \frac{\sum\limits_{k=1}^{n} b_k s^{\sigma_n - \sigma_k} + \sum\limits_{k=1}^{n-1} p_k \sum\limits_{r=1}^{n-k} s^{\sigma_{n-k} - \sigma_r} b_r}{P(s)}, \tag{4.11}$$

where

$$P(s) = s^{\sigma_n} + \sum_{k=1}^{n-1} p_k s^{\sigma_{n-k}} + p_n.$$

The inverse Laplace transform of equation (4.11) leads to

$$y(t) = \int_0^t G(t-\tau) f(\tau) d\tau + \sum_{k=1}^{n} b_k \mathscr{D}^{\sigma_n - \sigma_k} G(t) + \sum_{k=1}^{n-1} p_k \sum_{r=1}^{n-k} b_r \mathscr{D}^{\sigma_{n-k} - \sigma_r} G(t).$$

Remark. We shall mention that from (4.10), besides the term $\int_0^t G(t-\tau) f(\tau) d\tau$, $\mathscr{D}^{\sigma_n - \sigma_k} G(t)$ may be linearly independent of $\mathscr{D}^{\sigma_{n-k} - \sigma_r} G(t)$, so the number l of terms of linearly independent solution of Eq.(3.7) or Eq.(4.9) may not be n, it satisfies

$$n \le l \le \frac{1}{2} n(n+1).$$

But (3.8) is quite different from (4.10). Hence (3.8) may not be correct for equation (3.7) or (4.9).

Further, in the equation (1.1), if we take $p_k(t) = 0 (k = 1, ..., n-1)$ and $p_n(t) = p_n$, then

$$\begin{cases} {}_0\mathscr{D}_t^{\sigma_n} y(t) + p_n y(t) = f(t), \\ [{}_0\mathscr{D}_t^{\sigma_k - 1} y(t)]_{t=0} = b_k \ (k = 1, \cdots, n). \end{cases} \tag{4.12}$$

Directly from Theorem 4.1 we have

Corollary 4.1. *The solution of the equation (4.12) has the form*

$$y(t) = \int_0^t G(t-\tau)f(\tau)d\tau + \sum_{k=1}^{n} b_k\psi_k(t), \tag{4.13}$$

where

$$\psi_k(t) = {}_0\mathscr{D}_t^{\sigma_n-\sigma_k}G(t), \quad {}_0\mathscr{D}_t^{\sigma_n-\sigma_k} = ({}_0D_t^{\alpha_n})({}_0D_t^{\alpha_{n-1}})\cdots({}_0D_t^{\alpha_{k+1}}). \tag{4.14}$$

Let's consider another special case: In Eq.(1.1), Let $\alpha_1 = \alpha_2 = ... = \alpha_n = \alpha$ and $p_k(n) = p_k$ $(k = 1, ..., n)$. If $k = j + r$, then

$${}_0\mathscr{D}_t^{\sigma_n-\sigma_k}G(t) = {}_0\mathscr{D}_t^{\sigma_{n-j}-\sigma_r}G(t)$$

for $k = 1, ..., n$, $j = 1, ..., n-1$ and $r = 1, ..., n-j$. Therefore, Corollary 4.1 is also valid from (4.10) of Theorem 4.1.

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