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Some New Exact Solutions for Time Fractional Thin-film Equation

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Abstract In this paper, the invariant subspace method is utilized to obtain some new exact solutions for the time fractional thin-film equation. The fractional derivative in the considered equation is given in Remain-Liouville and Caputo senses. Some new invariant subspaces have been obtained that are not reported in the literature before.

Keywords Time-fractional thin-film equation, Riemann-Lioville fractional derivative, Caputo fractional derivative, Invariant subspace method, New exact solutions.

MSC(2010) 34K37, 35G20.

1. Introduction

Exact solutions of nonlinear evolution equations play a very important role in the study of nonlinear physical phenomena. Many methods can be utilized for obtaining exact solutions of nonlinear evolution equations such as Backlund transformation method [6, 9], Lie group method [1, 3, 19, 24], the tanh method [2, 5, 22, 29], the $\exp(-\varphi(z))-$expansion method [12, 13, 15, 17], the exp function approach [16] and the invariant subspace method (ISM) [7, 21].

The importance of the invariant subspace method comes from it is not only used for solving nonlinear evolution equations but also it can be used for solving fractional nonlinear evolution equations. Very recently, it is widely utilized for investigating exact solutions of fractional nonlinear evolution equations (see for example [4, 8, 10, 11, 14, 18, 20, 26–28, 30]).

In this paper, we use the ISM to investigate some new solutions of the time-fractional thin-film equation [27]

$$\frac{\partial^\alpha u}{\partial t^\alpha}= -u\left(\frac{\partial^4 u}{\partial x^4}\right)+\beta\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial^3 u}{\partial x^3}\right)+\gamma\left(\frac{\partial^2 u}{\partial x^2}\right)^2 , \qquad t> 0\ ,\ 0 <\alpha\leq 1\ . \tag{1.1}$$

Equation (1.1) can be used as a model for thin film flow on a substrate [7] . Here, $u(x,t)$ denotes the height of the fluid. The invariant subspaces and some exact solutions of the Eq. (1.1) (when $\alpha = 1$) have been obtained in [7, 30]. Some exact solution of Eq. (1.1) have been obtained in [27] using the ISM. The main aim of

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this paper is to obtain some new invariant subspaces and some new exact solutions of Eq. (1.1).
The rest of the paper is organized as follows: The basics and definitions of the ISM are introduced in Section 2. The new inavariant subspaces and exact solutions of Eq. (1.1) are discussed in Section 3. Section 4 discuss the results and conclusion of the paper.

2. ISM: Time Fractional Partial Differential Equations (PDEs)

Consider a time fractional PDE

$$\frac{\partial^\alpha u}{\partial t^\alpha}=F\left[u\right]=F\left(x,t,u,u_x,u_{xx},\ldots\ldots\right),\qquad 0<\alpha\leq 1, \tag{2.1}$$

where $F\left[u\right]$ is a nonlinear differential operator of order k. The n-dimensional invariant subspace $W_n = \mathcal{L}\left\{f_1\left(x\right), f_2\left(x\right),\ldots,f_k\left(x\right)\right\}$ (where $n \leq 2k+1$) can be obtained from the solution of the following nth-order linear ordinary differential equation:

$$L\left[y\right] = y^{(n)}(x) + a_{n-1}y^{(n-1)}(x) + \cdots + a_0 y\left(x\right) = 0, \tag{2.2}$$

where, the constant coefficients $a_{n-1},\ a_{n-2},\ldots,a_0$ can be obtained from the condition [13]

$$L\left[F\left[u\right]\right]\big|_{L[u]=0} = 0.$$

The time fractional PDE (2.1) can be converted into a system of time fractional ordinary differential equations (ODEs) through the following theorem:

Theorem 1 [13]**.** Let W_n be the linear space spanned by n linearly independent functions $\{\ f_i(x), i\ =\ 1,\ 2,...,\ n\}$ and suppose that W_n is invariant under the operator $F\left[u\right]$, which means that

$$F\left[\sum_{i=1}^{n} c_i f_i\left(x\right)\right] = \sum_{i=1}^{n} \overline{F}_i\left(c_1, c_2, \ldots . c_n\right) f_i\left(x\right), \tag{2.3}$$

for whatever constants $c_1, c_2, \ldots,c_n$. Then, the fractional PDE (2.1) has the solution of the form

$$u(x,t) = \sum_{i=1}^{n} c_i\left(t\right) f_i\left(x\right), \tag{2.4}$$

where the coefficients $c_1(t), c_2(t), \ldots . c_n(\text{t})$ satisfy the following system of fractional ODEs

$$\frac{d^\alpha c_i(t)}{dt^\alpha} = \psi_i\ \left(c_1\left(t\right), \ldots .,c_n\left(\text{t}\right)\right),\qquad i\ =\ 1,\ 2,...,\ n. \tag{2.5}$$

Here, the fractional order derivative $\frac{\partial^\alpha u}{\partial t^\alpha}$ will be considered in the Riemann-Liouville sense in some cases. In some other cases the fractional derivative $\frac{\partial^\alpha u}{\partial t^\alpha}$ will be in the Caputo sense according to the availability of exact solutions of the obtained system of fractional ODEs after using the invariant subspace method . Basic definitions and properties of the Riemann- Liouville and Caputo fractional derivatives are given in Appendix A.

3. Invariant subspaces and exact solutions of the time fractional thin-film Eq. (1.1)

In this section, the invariant subspaces and exact solutions of the time fractional thin-film Eq. (1.1) will be investigated. Here, we investigate the invariant subspaces with dimensions $n = \{4, 5, 6, 7, 8, 9\}$.

3.1. Invariant subspaces when $n = 4$

Applying the invariant subspace method to Eq. (1.1) taking into consideration the following auxiliary equation

$$y^{(4)} + a_3 y^{(3)} + a_2 y'' + a_1 y' + a_0 y = 0,$$

we get the following cases:

Case 1: when $\beta = 2, \gamma = -1$. In this case, the invariant subspace is $W_4 = \mathcal{L}\left\{sin\left(a_3 x\right), cos\left(a_3 x\right), e^{-(a_3 x)}, 1\right\}$. The exact solution of Eq. (1.1) can be written in the form

$$u\left(x,t\right) = c_1\left(t\right)\ \sin\left(a_3 x\right) + C_2\left(t\right)\cos(a_3 x)\ + \mathrm{e}^{-a_3 x}\ c_3(t) + C_4(t), \tag{3.1}$$

where a_3 is an arbitrary constant. Substitute Eq. (3.1) into Eq. (1.1) and compare the two sides of Eq. (1.1), to get

$$\frac{d^\alpha C_1(t)}{dt^\alpha} = -a_3^4\ C_1(t)C_4(t), \tag{3.2}$$

$$\frac{d^\alpha C_2\left(t\right)}{dt^\alpha} = -\ a_3^4\ C_2(t)C_4(t), \tag{3.3}$$

$$\frac{d^\alpha C_3\left(t\right)}{dt^\alpha} = -a_3^4\ C_3(t)C_4(t), \tag{3.4}$$

$$\frac{d^\alpha C_4\left(t\right)}{dt^\alpha} = -\ 2\ a_3^4\ \left(C_1\left(t\right)^2 + C_2\left(t\right)^2\right). \tag{3.5}$$

Solving the system of equations (3.2)- (3.5) assuming that the fractional derivatives are in the Riemann–Liouville sense, we get

$$C_4\left(t\right) = \frac{-1}{a_3^4}\frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}\ t^{-\alpha}, \quad \alpha \neq \frac{1}{2},$$

$$C_3\left(t\right) = \ l_3\ t^{-\alpha}\ ,$$

$$C_2\left(t\right) = \pm\sqrt{\frac{1}{2{a_3}^8}\left(\frac{\Gamma\left(1-\alpha\right)}{\Gamma\left(1-2\alpha\right)}\right)^2 - {l_1}^2}\ t^{-\alpha},$$

$$C_1\left(t\right) = l_1\ t^{-\alpha},$$

where l_1 and l_3 are arbitrary constants.
Hence, we obtain the following exact solution of the time fractional thin-film equation (1.1)

$$u\left(x,\,t\right)=\left(l_1\,\sin\left(a_3x\right)\ \pm\sqrt{\frac{1}{2{a_3}^8}\left(\frac{\Gamma\left(1-\alpha\right)}{\Gamma\left(1-2\alpha\right)}\right)^2-{l_1}^2}\cos(a_3x)\ +l_3\mathrm{e}^{-a_3x}\right.$$
$$\left.+\ \frac{-1}{a_3^4}\frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}\right)\,t^{-\alpha}. \quad (3.6)$$

The plot of solution (3.6) when $l_1=1, l_3=1, a_3=1$ and $\alpha=0.9$ is given in Fig. 1. Also, the plot of solution (3.6) when $l_1=1, l_3=0, a_3=1$ and $\alpha=0.9$ is given in Fig. 2.

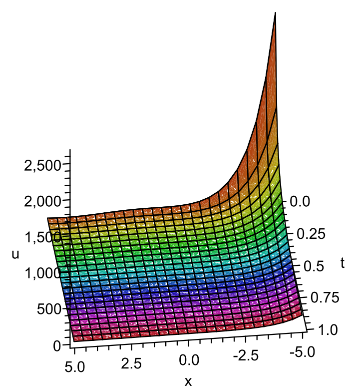

Figure 1. Plot of the solution (3.6) when $l_1=1, l_3=1, a_3=1$ and $\alpha=0.9$.

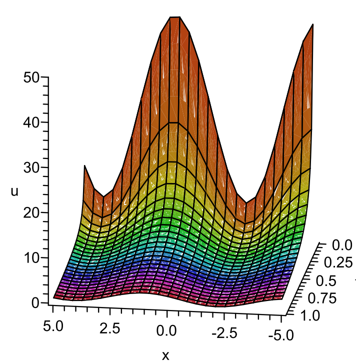

Figure 2. Plot of the solution (3.6) when $l_1=1, l_3=0, a_3=1$ and $\alpha=0.9$.

Case 2: when $\beta=3$ and $\gamma=-2$. In this case, the invariant subspace is $W_4=\ \mathcal{L}\left\{1,\ e^{-\frac{a_3}{2}x}, e^{-\frac{a_3}{4}x}\sin\left(\frac{\sqrt{3}}{4}x\right), e^{-\frac{a_3}{4}x}\cos\left(\frac{\sqrt{3}}{4}x\right)\right\}$ and the exact solution of Eq. (1) can be given by

$$u\,(x,\ t) = C_1\,(t)\,e^{-\frac{a_3}{4}x}\cos\left(\frac{\sqrt{3}}{4}x\right)\ +\ C_2\,(t)\,e^{-\frac{a_3}{4}x}\sin\left(\frac{\sqrt{3}}{4}x\right) + C_3\,(t)\,e^{-\frac{a_3}{2}x} + C_4\,(t)\,, \quad (3.7)$$

where a_3 is an arbitrary constant. Substitute Eq. (3.7) into Eq. (1.1) and compare the two sides of Eq. (1.1), to get

$$\frac{d^{\alpha}C_1(t)}{dt^{\alpha}} = \frac{1}{32}a_3^4\left(C_1\,(t) - \sqrt{3}C_2\,(t)\right)C_4(t), \quad (3.8)$$

$$\frac{d^{\alpha}C_2(t)}{dt^{\alpha}} = \frac{1}{32}a_3^4\left(\sqrt{3}C_1\,(t) + C_2\,(t)\right)C_4(t), \quad (3.9)$$

$$\frac{d^{\alpha}C_3\,(t)}{dt^{\alpha}} = -\frac{1}{32}a_3^4\left(3\left(C_1(t)^2 + C_2(t)^2\right) + 2\ C_3\,(t)\,C_4\,(t)\right), \quad (3.10)$$

$$\frac{d^{\alpha}C_4\,(t)}{dt^{\alpha}} = 0. \quad (3.11)$$

Solving the system of equations (3.8)- (3.11) assuming that the fractional derivatives are in the Caputo sense, we get

$$C_4\,(t) = 0,$$

$$C_3\,(t) = -\frac{3\mathrm{a}_3^4\left(l_1^2 + l_2^2\right)}{32\Gamma\,(1+\alpha)}\ t^{\alpha},$$

$$C_2\,(t) = l_2,$$

$$C_1\,(t) = l_1,$$

where l_1 and l_2 are arbitrary constants.Thus, the exact solution of the time fractional thin-film equation (1.1) is given by

$$u\,(x,\ t) = l_1 e^{-\frac{a_3}{4}x}\cos\left(\frac{\sqrt{3}}{4}x\right)\ +\ l_2\ e^{-\frac{a_3}{4}x}\sin\left(\frac{\sqrt{3}}{4}x\right)\ -\ \frac{3\mathrm{a}_3^4\left(l_1^2 + l_2^2\right)}{32\Gamma\,(1+\alpha)}\ t^{\alpha}\ e^{-\frac{a_3}{2}x}. \quad (3.12)$$

The plot of solution (3.12) when $l_1 = l_2 = 1, a_3 = 4$ and $\alpha = 0.9$ is given in Fig.3.

Case 3: when $\boldsymbol{\beta = \frac{9}{2}}$ and $\boldsymbol{\gamma = \frac{-7}{2}}$. The invariant subspace in this case is $W_4 = \mathcal{L}\{1,\ \mathrm{e}^{-\frac{\mathrm{a}_3}{2}x},\ \mathrm{e}^{-\mathrm{a}_3 x},\ \mathrm{e}^{\frac{\mathrm{a}_3}{2}x}\}$ and the exact solution of Eq. (1) can be expressed as

$$u\,(x,\,t) = \mathrm{e}^{-\frac{\mathrm{a}_3}{2}x}C_1\,(t) + \mathrm{e}^{-\mathrm{a}_3 x}\ C_2\,(t) + \mathrm{e}^{\frac{\mathrm{a}_3}{2}x}\ C_3\,(t) + C_4\,(t)\,. \quad (3.13)$$

where a_3 is an arbitrary constant. Substitute Eq. (3.13) into Eq. (1.1) and compare the two sides of Eq. (1.1), to obtain

$$\frac{d^{\alpha}C_1(t)}{dt^{\alpha}} = -\frac{45}{8}a_3^4\ C_2\,(t)\,C_3\,(t) - \frac{1}{16}a_3^4\ C_1\,(t)\,C_4\,(t)\,, \quad (3.14)$$

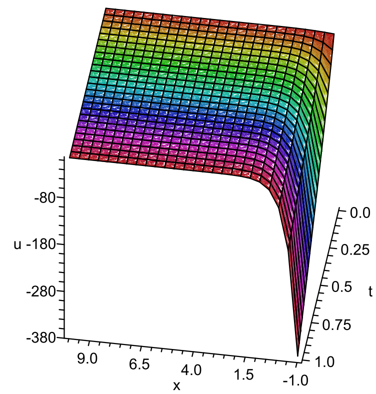

Figure 3. Plot of the solution (3.12) when $l_1 = l_2 = 1, a_3 = 4$ and $\alpha = 0.9$.

$$\frac{d^\alpha C_2(t)}{dt^\alpha} = -a_3^4 \, C_2(t) \, C_4(t), \tag{3.15}$$

$$\frac{d^\alpha C_3(t)}{dt^\alpha} = -\frac{1}{16} a_3^4 \, C_3(t) \, C_4(t), \tag{3.16}$$

$$\frac{d^\alpha C_4(t)}{dt^\alpha} = -\frac{9}{8} a_3^4 \, C_1(t) \, C_3(t). \tag{3.17}$$

Solving the system of equations (3.14)- (3.17) when the fractional derivatives are in the Riemann–Liouville sense, we get

$$C_4(t) = \frac{-1}{{a_3}^4} \frac{\Gamma(1+2b_1+\alpha)}{\Gamma(1+2b_1)} \, t^{-\alpha}, \quad b_1 \neq -\frac{1}{2},$$

$$C_3(t) = \frac{8}{9{a_3}^8 \, l_1} \frac{\Gamma(1-\alpha)\Gamma(1+\alpha+2b_1)}{\Gamma(1-2\alpha)\Gamma(1+2b_1)} \, t^{-2\alpha-b_1}, \quad \alpha \neq \frac{1}{2},$$

$$C_2(t) = l_2 t^{2b_1+\alpha},$$

$$C_1(t) = l_1 \, t^{b_1},$$

where, l_1 is an arbitrary constant and

$$l_2 = \frac{-\,{a_3}^4 \, {l_1}^2 \Gamma(1-2\alpha)\left(16\Gamma(1+b_1)\Gamma(1+2b_1) - l_3\Gamma(1+b_1-\alpha)\right)}{80 l_3 \, \Gamma(1-\alpha)\Gamma(1+b_1-\alpha)},$$

$$l_3 = \Gamma(1+2b_1+\alpha).$$

The value of b_1 can be obtained numerically from the following relation

$$\frac{\Gamma(1-2\alpha-b_1)}{\Gamma(1-3\alpha-b_1)} = \frac{1}{16} \frac{l_3}{\Gamma(1+2b_1)}.$$

Thus, the exact solution of the time fractional thin-film equation (1.1) is given by

$$u(x,t) = l_1 \, t^{b_1} \, \mathrm{e}^{-\frac{a_3}{2} x} + l_2(t^{2b_1+\alpha})\left(\mathrm{e}^{-a_3 x}\right) + \frac{8}{9{a_3}^8 \, l_1} \frac{\Gamma(1-\alpha)\Gamma(1+\alpha+2b_1)}{\Gamma(1-2\alpha)\Gamma(1+2b_1)} \, (t^{-2\alpha-b_1})\left(\mathrm{e}^{\frac{a_3}{2} x}\right) - \frac{1}{{a_3}^4} \frac{\Gamma(1+2b_1+\alpha)}{\Gamma(1+2b_1)} \, t^{-\alpha}. \tag{3.18}$$

Case 4: In this case, the invariant subspace is $W_4 = \mathcal{L}\{1,x,x^2,x^3\}$ and the exact solution of Eq. (1) can be expressed as

$$u(x,\ t) = C_1(t)\,x^3 + C_2(t)\,x^2 + C_3(t)\,x + C_4(t)\,. \tag{3.19}$$

Substitute Eq. (3.19) into Eq. (1.1) and compare the two sides of Eq. (1.1), to get

$$\frac{d^\alpha C_1(t)}{dt^\alpha} = 0, \tag{3.20}$$

$$\frac{d^\alpha C_2(t)}{dt^\alpha} = 18\,(\beta + 2\gamma)\,C_1(t)^2, \tag{3.21}$$

$$\frac{d^\alpha C_3(t)}{dt^\alpha} = 12\,(\beta + 2\gamma)\,C_1(t)\ C_2(t)\,, \tag{3.22}$$

$$\frac{d^\alpha C_4(t)}{dt^\alpha} = 4\gamma C_2(t)^2 + 6\beta C_1(t)\ C_3(t)\,. \tag{3.23}$$

The system of equations (3.20)- (3.23) can be solved when the fractional derivatives are in the Caputo sense in two subcases as follows :

Case 4.1: when β and γ are arbitrary constants.
This case is discussed in [27]. So, we will not discuss it here.

Case 4.2: when $\beta = -2\gamma$

$$C_4(t) = -\frac{12(-48 + l_1 l_3)\gamma}{\Gamma(1+\alpha)}\ t^\alpha,$$

$$C_3(t) = l_3\ ,$$

$$C_2(t) = l_2,$$

$$C_1(t) = l_1,$$

where, l_1, l_2 and l_3 are arbitrary constants. Thus, the exact solution of the time fractional thin-film equation (1.1) is given by

$$u(x,\ t) = l_1 x^3 + l_2 x^2 + l_3 x - \frac{12(-48 + l_1 l_3)\gamma}{\Gamma(1+\alpha)}\ t^\alpha. \tag{3.24}$$

The plot of solution (3.24) when $l_1 = l_2 = l_3 = 1$ and $\alpha = 0.9$ is given in Fig. 4.

3.2. Invariant subspaces when $n = 5$

Applying the ISM to Eq. (1.1) taking into consideration the auxiliary equation $y^{(5)} + a_4 y^{(4)} + a_3 y^{(3)} + a_2 y'' + a_1 y' + a_0 y = 0$, we get the following cases:

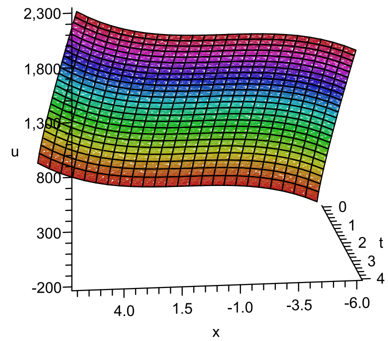

Figure 4. Plot of the solution (3.24) when $l_1 = l_2 = l_3 = 1$ and $\alpha = 0.9$.

Case 5.1: when $\boldsymbol{\beta = \frac{9}{2}, \gamma = -\frac{7}{2}}$. The invariant subspace is given by

$$W_5 = \mathcal{L}\left\{1, \sin\left(\frac{\sqrt{a_3}x}{\sqrt{5}}\right), \cos\left(\frac{\sqrt{a_3}x}{\sqrt{5}}\right), \sin\left(2\frac{\sqrt{a_3}x}{\sqrt{5}}\right), \cos\left(2\frac{\sqrt{a_3}x}{\sqrt{5}}\right)\right\}.$$

The exact solution of Eq. (1.1) can be written in the form

$$\begin{aligned} u(x,t) =& C_5(t) + C_4(t)\cos\left(\frac{\sqrt{a_3}x}{\sqrt{5}}\right) + C_3(t)\sin\left(\frac{\sqrt{a_3}x}{\sqrt{5}}\right) \\ &+ C_2(t)\cos\left(2\frac{\sqrt{a_3}x}{\sqrt{5}}\right) + C_1(t)\sin\left(2\frac{\sqrt{a_3}x}{\sqrt{5}}\right), \end{aligned} \tag{3.25}$$

where a_3 is an arbitrary constants. Substite Eq. (3.25) into Eq. (1.1) and compare the two sides of Eq. (1.1), to btain

$$\frac{d^\alpha C_1(t)}{dt^\alpha} = -\frac{16}{25}{a_3}^2 C_1(t)C_5(t), \tag{3.26}$$

$$\frac{d^\alpha C_2(t)}{dt^\alpha} = -\frac{16}{25}{a_3}^2 C_2(t)C_5(t), \tag{3.27}$$

$$\frac{d^\alpha C_3(t)}{dt^\alpha} = \frac{1}{25}{a_3}^2(45C_2(t)C_3(t) - 45C_1(t)C_4(t) - C_3(t)C_5(t)), \tag{3.28}$$

$$\frac{d^\alpha C_4(t)}{dt^\alpha} = -\frac{1}{25}{a_3}^2\left(45C_1(t)C_3(t) + C_4(t)(45C_2(t) + C_5(t))\right), \tag{3.29}$$

$$\frac{d^\alpha C_5(t)}{dt^\alpha} = -\frac{9}{50}{a_3}^2\left(16C_1(t)^2 + 16C_2(t)^2 + C_3(t)^2 + C_4(t)^2\right). \tag{3.30}$$

Solving the system of equations (3.26)- (3.30) assuming that the fractional derivatives are in the Riemann–Liouville sense, we get

$$C_5(t) = -\frac{25\Gamma(1-\alpha)}{16\mathrm{a}_3^2\Gamma(1-2\alpha)}\, t^{-\alpha}, \quad \alpha \neq \frac{1}{2},$$

$$C_4(t) = l_4\ t^{-\alpha},$$
$$C_3(t) = l_3\ t^{-\alpha}\ ,$$
$$C_2(t) = l_2\ t^{-\alpha},$$
$$C_1(t) = l_1\ \ t^{-\alpha},$$

where, l_1 is an arbitrary constant and

$$l_2 = \pm\frac{\sqrt{-2304a_3^4 l_1^2(\Gamma(1-2\alpha))^2 + 625(\Gamma(1-\alpha))^2}}{48\mathrm{a}_3^2\Gamma(1-2\alpha)},$$

$$l_3 = \pm\frac{5\sqrt{\pi}\sqrt{\frac{2^{1+4\alpha}}{a_3^4\left(\Gamma\left(\frac{1}{2}-\alpha\right)\right)^2}}\sqrt{25\Gamma(1-\alpha) + \sqrt{-2304a_3^4 l_1^2(\Gamma(1-\alpha))^2 + 625(\Gamma(1-\alpha))^2}}}{24\sqrt{\Gamma(1-\alpha)}},$$

$$l_4 = \frac{\mathrm{l}_3(8l_1^2 + 16l_2^2 - l_3^2)}{8l_1 l_2}.$$

Therefore, we get the following exact solution of the time fractional thin-film equation (1.1)

$$u(x,\,t) = \Big(-\frac{25\Gamma(1-\alpha)}{16\mathrm{a}_3^2\Gamma(1-2\alpha)} + l_4\cos\left(\frac{\sqrt{a_3}x}{\sqrt{5}}\right) + l_3\sin\left(\frac{\sqrt{a_3}x}{\sqrt{5}}\right) + l_2\ \cos\left(2\frac{\sqrt{a_3}x}{\sqrt{5}}\right)$$
$$+ l_1\ \ \sin\left(2\frac{\sqrt{a_3}x}{\sqrt{5}}\right)\Big)t^{-\alpha}.$$

Case 5.2: when β and γ are arbitrary constants.
This case is discussed in [27]. So, we will not discuss it here.

3.3. Invariant subspaces when $n = 6$

Applying the ISM to Eq. (1.1) taking into consideration the auxiliary equation $y^{(6)} + a_5y^{(5)} + a_4y^{(4)} + a_3y^{(3)} + a_2y' + a_1y' + a_0y = 0$, we get the following case:

Case 6: when $\gamma = -\frac{3}{20}(-2+5\beta)$. The invariant subspace is given by $W_6 = \mathcal{L}\{1,\ x,\ x^2,\ x^3,\ x^4,\ x^5\}$. So, the solution of Eq. (1.1) can be expressed as

$$u(x,\,t) = c_1(t) + c_2(t)\,x + c_3(t)\,x^{\,2} + c_4(t)\,x^{\,3} + c_5(t)\,x^4 \ \ + c_6(t)\,x^5. \qquad (3.31)$$

Substitute Eq. (3.31) into Eq. (1.1) and equate different powers of x to zero, to get

$$\frac{d^\alpha C_1(t)}{dt^\alpha} = \left(\frac{6}{5} - 3\beta\right)C_3(t)^2 + 6\beta C_2(t)C_4(t) - 24C_1(t)C_5(t)\,, \qquad (3.32)$$

$$\frac{d^\alpha C_2(t)}{dt^\alpha} = \frac{6}{5}(6-5\beta)C_3(t)C_4(t) + 24((-1+\beta)C_2(t)C_5(t) - 5C_1(t)C_6(t)), \qquad (3.33)$$

$$\frac{d^\alpha C_3(t)}{dt^\alpha} = \left(\frac{54}{5} - 9\beta\right)C_4(t)^2 + \frac{12}{5}(-4+5\beta)\,C_3(t)\,C_5(t) +$$

$$60\,(-2+\beta)\,C_2\,(t)\,C_6\,(t)\,, \quad (3.34)$$

$$\frac{d^{\alpha}C_4\,(t)}{dt^{\alpha}} = -\frac{12}{5}(-8+5\beta)(C_4(t)C_5\,(t) - 5C_3(t)C_6\,(t)), \quad (3.35)$$

$$\frac{d^{\alpha}C_5\,(t)}{dt^{\alpha}} = -\frac{6}{5}(-8+5\beta)(2C_5\,(t)^2 - 5C_6(t)C_4\,(t)), \quad (3.36)$$

$$\frac{d^{\alpha}C_6\,(t)}{dt^{\alpha}} = 0. \quad (3.37)$$

The system of equations (3.32)- (3.37) can be solved in the following special case when the fractional derivatives are in the Caputo sense

$$C_6\,(t) = l_6, \quad C_3\,(t) = C_4\,(t) = C_5\,(t) = 0, \beta = 2,\ \gamma = -\frac{6}{5},$$

$$C_1\,(t) = l_1,$$

$$C_2\,(t) = -\frac{120\ l_1 l_6}{\Gamma\,(1+\alpha)}\ t^{\alpha},$$

where l_1 and l_6 are arbitrary constants. Therefore, we get the following exact solution of the time fractional thin-film equation (1.1)

$$u\,(x,\,t) = l_1 - \frac{120\ l_1 l_6}{\Gamma\,(1+\alpha)}\ t^{\alpha}\ x + l_6\ x^5. \quad (3.38)$$

The plot of solution (3.38) when $l_1 = l_6 = l$ and $\alpha = 0.9$ is given in Fig. 5.

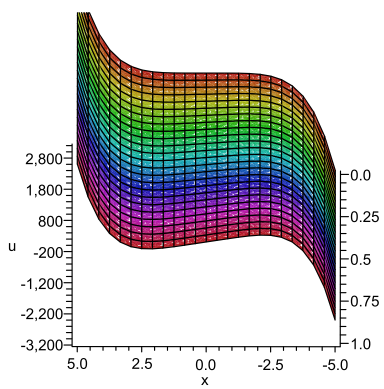

Figure 5. Plot of the solution (3.38) when $l_1 = l_6 = l$ and $\alpha = 0.9$.

3.4. Invariant subspaces when $n = 7$

Applying the ISM to Eq. (1.1) taking into consideration the following auxiliary equation

$$y^{(7)} + a_6\ y^{(6)} + a_5 y^{(5)} + a_4 y^{(4)} + a_3 y^{(3)} + a_2 y'' + a_1 y' + a_0 y = 0,$$

we get the following case:

Case 7: when $\beta = \left(\frac{1}{2} - \frac{5}{4}\gamma\right)$**.** The invariant subspace is given by $W_7 = \mathcal{L}\{1,\ x,\ x^2,\ x^3,\ x^4,\ x^5,\ x^6\}$. Therefore, the solution of Eq. (1.1) can be expressed as

$$u(x,t) = c_1(t) + c_2(t)\,x + c_3(t)\,x^{2} + c_4(t)\,x^{3} + c_5(t)\,x^4 + c_6(t)\,x^5 + c_7(t)\,x^6, \tag{3.39}$$

Substitute Eq. (3.39) into Eq. (1.1) and equate different powers of x to zero, to get

$$\frac{d^\alpha C_1(t)}{dt^\alpha} = 4\gamma C_3(t)^2 - \frac{3}{2}(-2+5\gamma)\,C_2(t)\,C_4(t) - 24C_1(t)C_5(t), \tag{3.40}$$

$$\frac{d^\alpha C_2(t)}{dt^\alpha} = 3(2+3\gamma)\,C_3(t)\,C_4(t) - 6(2+5\gamma)\,C_2(t)\,C_5(t) - 120C_1(t)C_6(t), \tag{3.41}$$

$$\frac{d^\alpha C_3(t)}{dt^\alpha} = \frac{9}{2}(2+3\gamma)\,C_4(t)^2 - 12\gamma C_3(t)\,C_5(t) - 15(6+5\gamma)\,C_2(t)\,C_6(t) - 360C_1(t)C_7(t), \tag{3.42}$$

$$\frac{d^\alpha C_4(t)}{dt^\alpha} = 24(1+\gamma)\,C_4(t)\,C_5(t) - 10(6+7\gamma)C_3(t)C_6(t) - 150(2+\gamma)C_2(t)\,C_7(t), \tag{3.43}$$

$$\frac{d^\alpha C_5(t)}{dt^\alpha} = 24(1+\gamma)\,C_5(t)^2 - \frac{15}{2}(2+3\gamma)\,C_4(t)\,C_6(t) - 60(4+3\gamma)C_3(t)\,C_7(t), \tag{3.44}$$

$$\frac{d^\alpha C_6(t)}{dt^\alpha} = 3(6+5\gamma)(2C_5(t)\,C_6(t) - 9C_4(t)\,C_7(t)), \tag{3.45}$$

$$\frac{d^\alpha C_7(t)}{dt^\alpha} = (6+5\gamma)(5C_6(t)^2 - 12C_5(t)\,C_7(t)). \tag{3.46}$$

Solving the system (3.40)- (3.46) assuming that the fractional derivatives are in the Riemann–Liouville sense, we get

$$\gamma = -\frac{2}{3},\quad C_1(t) = C_2(t) = C_3(t) = 0,$$

$$C_4(t) = \frac{l_6\left(-\frac{\Gamma\left(1-\frac{5\alpha}{4}-\frac{b_1}{4}\right)}{\Gamma\left(1-\frac{9\alpha}{4}-\frac{b_1}{4}\right)} + \frac{2\Gamma\left(\frac{1}{4}(4-3\alpha+b_1)\right)}{\Gamma\left(\frac{1}{4}(4-7\alpha+b_1)\right)}\right)}{72l_7}\, t^{\frac{1}{4}(-3\alpha+b_1)}$$

$$C_5(t) = \frac{\Gamma\left(\frac{1}{4}(4-3\alpha+b_1)\right)}{8\Gamma\left(\frac{1}{4}(4-7\alpha+b_1)\right)}t^{-\alpha},$$

$$C_6(t) = l_6 t^{\frac{1}{4}(-5\alpha-b_1)},$$

$$C_7(t) = l_7\, t^{\frac{1}{2}(-3\alpha-b_1)},$$

where l_7 is an arbitrary constant and

$$l_6 = \pm\frac{\sqrt{\frac{3}{10}}\sqrt{l_7\left(\Gamma(d_1)\,\Gamma(d_2) + 4\Gamma(d_3)\,\Gamma(d_4)\right)}}{2\sqrt{\Gamma(d_3)}\sqrt{\Gamma(d_2)}},$$

$$d_1 = \frac{1}{2}(2-3\alpha-b_1),\quad d_2 = \frac{1}{4}(4-7\alpha+b_1),$$

$$d_3 = \frac{1}{2}\left(2-5\alpha-b_1\right), \quad d_4 = \frac{1}{4}\left(4-3\alpha+b_1\right).$$

The value of b_1 can be obtained from the relation

$$\frac{\Gamma\left(1-\alpha\right)\Gamma\left(\frac{1}{4}\left(4-7\alpha+b_1\right)\right)}{\Gamma\left(1-2\alpha\right)} - \Gamma\left(\frac{1}{4}\left(4-3\alpha+b_1\right)\right) = 0.$$

Thus, the exact solution of Eq. (1.1) is

$$u\left(x,\ t\right) = \frac{l_6\left(-\frac{\Gamma\left(1-\frac{5\alpha}{4}-\frac{b_1}{4}\right)}{\Gamma\left(1-\frac{9\alpha}{4}-\frac{b_1}{4}\right)} + \frac{2\Gamma\left(\frac{1}{4}(4-3\alpha+b_1)\right)}{\Gamma\left(\frac{1}{4}(4-7\alpha+b_1)\right)}\right)}{72l_7} t^{\frac{1}{4}(-3\alpha+b_1)}\ x^{\ 3}$$
$$+\frac{\Gamma\left(\frac{1}{4}\left(4-3\alpha+b_1\right)\right)}{8\Gamma\left(\frac{1}{4}\left(4-7\alpha+b_1\right)\right)} t^{-\alpha}\ x^4 \quad + l_6 t^{\frac{1}{4}(-5\alpha-b_1)}\ x^5 + l_7\ t^{\frac{1}{2}(-3\alpha-b_1)}\ x^6.$$

3.5. Invariant subspaces when $n = 8$

Applying the ISM to Eq. (1.1) taking into consideration the following auxiliary equation

$$y^{(8)} + a_7\ y^{(7)} + a_6\ y^{(6)} + a_5 y^{(5)} + a_4 y^{(4)} + a_3 y^{(3)} + a_2 y'' + a_1 y' + a_0 y = 0,$$

we get the following case:

Case 8: when $\beta = \frac{16}{7}$, $\gamma = -\frac{10}{7}$. The invariant subspace is given by $W_8 = \mathcal{L}\{1,\ x,\ x^2,\ x^3,\ x^4,\ x^5,\ x^6, \mathrm{x}^7\}$. Therefore, the solution of Eq. (1.1) can be expressed as

$$u\left(x,\, t\right) = c_1\left(t\right) + c_2\left(t\right)x + c_3\left(t\right)x^{\ 2} + c_4\left(t\right)x^{\ 3} + c_5\left(t\right)x^4 \ + c_6\left(t\right)x^5 + c_7\left(t\right)x^6 + c_8\left(t\right)x^7. \quad (3.47)$$

Substitute Eq. (3.47) into Eq. (3.11) and equate different powers of x to zero, to get

$$\frac{d^\alpha C_1(t)}{dt^\alpha} = -\frac{40}{7}C_3\left(t\right)^2 + \frac{96}{7}\ C_2(t)C_4\left(t\right) - 24C_1(t)C_5\left(t\right), \quad (3.48)$$

$$\frac{d^\alpha C_2\left(t\right)}{dt^\alpha} = -12\ C_3(t)C_4\left(t\right) + 48C_2(t)C_5\left(t\right) - 120C_1\left(t\right)C_6\left(t\right), \quad (3.49)$$

$$\frac{d^\alpha C_3\left(t\right)}{dt^\alpha} = -\frac{72}{7}\ C_4\left(t\right)^2 + \frac{120}{7}\ C_3\left(t\right)C_5\left(t\right) + \frac{120}{7}\ C_2\left(t\right)C_6\left(t\right) - 360\ C_1(t)C_7\left(t\right), \quad (3.50)$$

$$\frac{d^\alpha C_4\left(t\right)}{dt^\alpha} = -\frac{72}{7}C_4\left(t\right)C_5\left(t\right) + 40C_3\left(t\right)C_6\left(t\right) - \frac{600}{7}C_2\left(t\right)C_7\left(t\right) - 840C_1(t)C_8\left(t\right), \quad (3.51)$$

$$\frac{d^\alpha C_5\left(t\right)}{dt^\alpha} = -\frac{72}{7}\ C_5\left(t\right)^2 + \frac{120}{7}\ C_6\left(t\right)C_4\left(t\right) + \frac{120}{7}\ C_3\left(t\right)C_7\left(t\right) - 360\ C_2(t)C_8\left(t\right), \quad (3.52)$$

$$\frac{d^\alpha C_6\left(t\right)}{dt^\alpha} = -\frac{48}{7}\ C_5\left(t\right)C_6\left(t\right) + \frac{216}{7}\ C_4\left(t\right)C_7\left(t\right) - 120C_3(t)C_8\left(t\right)\ , \quad (3.53)$$

$$\frac{d^\alpha C_7\left(t\right)}{dt^\alpha} = -\frac{40}{7}C_6\left(t\right)^2 \ + \frac{96}{7}\ C_5\left(t\right)C_7\left(t\right) - 24C_4\left(t\right)C_8\left(t\right), \quad (3.54)$$

$$\frac{d^{\alpha}C_8(t)}{dt^{\alpha}}=0. \tag{3.55}$$

Solving the system (3.48)- (3.55) assuming that the fractional derivatives are in the Caputo sense, we get

$$C_1(t)=C_2(t)=C_3(t)=C_4(t)=C_5(t)=0,$$

$$C_6(t)=l_6,$$

$$C_7(t)=-\frac{40l_6^2}{7\Gamma(\alpha+1)}t^{\alpha},$$

$$C_8(t)=l_8,$$

where l_6 and l_8 are constants. Therefore, the exact solution of Eq. (1.1) is

$$u(x,t)=l_6x^5-\frac{40l_6^2}{7\Gamma(\alpha+1)}t^{\alpha}x^6+l_8x^7.$$

3.6. Invariant subspaces when $n=9$

Applying the ISM to Eq. (1.1) taking into consideration the following auxiliary equation

$$y^{(9)}+a_8y^{(8)}+a_7\,y^{(7)}+a_6\,y^{(6)}+a_5y^{(5)}+a_4y^{(4)}+a_3y^{(3)}+a_2y''+a_1y'+a_0y=0,$$

we get the following case:

Case 9: when $\gamma=-\frac{45}{28},\beta=\frac{5}{2}$. The invariant subspace is given by $W_9=\mathcal{L}\{1,\ x,\ x^2,\ x^3,\ x^4,\ x^5,\ x^6\ \ x^7\ \ x^8\}$. Therefore, the solution of Eq. (1.1) can be expressed as

$$\begin{aligned}u(x,t)=&\ c_1(t)+c_2(t)x+c_3(t)x^2+c_4(t)x^3+c_5(t)x^4+c_6(t)x^5\\&+c_7(t)x^6+c_8(t)x^7+c_9(t)x^8,\end{aligned} \tag{3.56}$$

Substitute Eq. (3.56) into Eq. (3.11) and equate different powers of x to zero, to get

$$\frac{d^{\alpha}C_1(t)}{dt^{\alpha}}=-\frac{45}{7}C_3(t)^2+15\,C_2(t)C_4(t)-24C_1(t)C_5(t), \tag{3.57}$$

$$\frac{d^{\alpha}C_2(t)}{dt^{\alpha}}=-\frac{60}{7}C_3(t)C_4(t)+36C_2(t)C_5(t)-120C_1(t)C_6(t), \tag{3.58}$$

$$\frac{d^{\alpha}C_3(t)}{dt^{\alpha}}=-\frac{90}{7}C_4(t)^2+\frac{132}{7}C_3(t)C_5(t)+30\,C_2(t)C_6(t)-360\,C_1(t)C_7(t), \tag{3.59}$$

$$\begin{aligned}\frac{d^{\alpha}C_4(t)}{dt^{\alpha}}=&-\frac{108}{7}C_4(t)C_5(t)+\frac{360}{7}C_3(t)C_6(t)-60\,C_2(t)C_7(t)\\&-840C_1(t)C_8(t)\end{aligned} \tag{3.60}$$

$$\frac{d^{\alpha}C_5(t)}{dt^{\alpha}}=-\frac{108}{7}C_5(t)^2+\frac{135}{7}C_6(t)C_4(t)+\frac{330}{7}C_3(t)C_7(t)-315\,C_2(t)C_8(t)$$

$$-1680\ C_1(t)C_9\left(t\right), \quad (3.61)$$

$$\frac{d^{\alpha}C_6\left(t\right)}{dt^{\alpha}} = -\frac{108}{7}\ C_5\left(t\right)C_6\left(t\right) + \frac{360}{7}\ C_4\left(t\right)C_7\left(t\right) - 60C_3\left(t\right)C_8\left(t\right) - 840C_2\left(t\right)C_9\left(t\right), \quad (3.62)$$

$$\frac{d^{\alpha}C_7\left(t\right)}{dt^{\alpha}} = -\ \frac{90}{7}C_6\left(t\right)^2\ + \frac{132}{7}\ C_5\left(t\right)C_7\left(t\right) + 30C_4\left(t\right)C_8\left(t\right) - 360C_3\left(t\right)C_9\left(t\right), \quad (3.63)$$

$$\frac{d^{\alpha}C_8\left(t\right)}{dt^{\alpha}} = -\ \frac{60}{7}C_6\left(t\right)C_7\left(t\right) + 36\ C_5\left(t\right)C_8\left(t\right) - 120\ C_4\left(t\right)C_9\left(t\right), \quad (3.64)$$

$$\frac{d^{\alpha}C_9\left(t\right)}{dt^{\alpha}} = -\ \frac{45}{7}C_7\left(t\right)^2\ + 15\ C_6\left(t\right)C_8\left(t\right) - 24\ C_5\left(t\right)C_9\left(t\right). \quad (3.65)$$

Solve the system (3.57)- (3.65) assuming that the fractional derivatives are in the Riemann–Liouville sense, to get

$$C_1\left(t\right) = C_2\left(t\right) = C_3\left(t\right) = C_4\left(t\right) = 0,$$

$$C_5\left(t\right) = l_5\ t^{-\alpha},$$

$$C_6\left(t\right) = l_6\ t^{\frac{1}{4}\left(-5\alpha - b_1\right)},$$

$$C_7\left(t\right) = l_7\ t^{\frac{1}{2}\left(-3\alpha - b_1\right)},$$

$$C_8\left(t\right) = l_8\ t^{\frac{1}{4}\left(-7\alpha - 3b_1\right)},$$

$$C_9\left(t\right) = l_9\ t^{-2\alpha - b_1},$$

where l_6 is an arbitrary constant and

$$l_5 = -\frac{7\Gamma\left(m_5\right)}{108\Gamma\left(m_6\right)},$$

$$l_7 = \frac{16^2\Gamma\left(m_5\right)\Gamma\left(m_2\right)}{2\left(11\Gamma\left(m_5\right)\Gamma\left(m_2\right) + 9\Gamma\left(m_6\right)\Gamma\left(m_4\right)\right)},$$

$$l_8 = -\frac{90l_6^3\Gamma\left(m_6\right)\Gamma\left(m_5\right)\Gamma\left(m_1\right)\Gamma\left(m_2\right)}{7\left(7\Gamma\left(m_5\right)\Gamma\left(m_1\right) + 3\Gamma\left(m_6\right)\Gamma\left(m_3\right)\right)\left(11\Gamma\left(m_5\right)\Gamma\left(m_2\right) + 9\Gamma\left(m_6\right)\Gamma\left(m_4\right)\right)},$$

$$l_9 = -\frac{15\left(3l_7^2 - 7l_6l_8\right)\Gamma\left(1 - 3\alpha - b_1\right)}{7\left(24l_5\Gamma\left(1 - 3\alpha - b_1\right) + \Gamma\left(1 - 2\alpha - b_1\right)\right)},$$

$$m_1 = 1 - \frac{11\alpha}{4} - \frac{3b_1}{4},$$

$$m_2 = 1 - \frac{5\alpha}{2} - \frac{b_1}{2},$$

$$m_3 = 1 - \frac{7\alpha}{4} - \frac{3b_1}{4},$$

$$m_4 = 1 - \frac{3\alpha}{2} - \frac{b_1}{2},$$

$$m_5 = 1 - \alpha,$$

$$m_6 = 1 - 2\alpha.$$

The value of b_1 can be obtained from the relation

$$\frac{\Gamma\left(1 - \frac{5\alpha}{4} - \frac{b_1}{4}\right)}{\Gamma\left(1 - \frac{9\alpha}{4} - \frac{b_1}{4}\right)} - \frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)} = 0.$$

Thus, we get the following exact solution of Eq. (1.1)

$$u(x,t) = l_5\, t^{-\alpha} x^4 \ + l_6\, t^{\frac{1}{4}(-5\alpha - b_1)} x^5 + l_7\, t^{\frac{1}{2}(-3\alpha - b_1)} x^6 + l_8\, t^{\frac{1}{4}(-7\alpha - 3b_1)} x^7 + l_9\, t^{-2\alpha - b_1} x^8. \quad (3.66)$$

4. Conclusion

In this paper, we have investigated the time-fractional thin-film equation (1.1) using the ISM. We have concentrated on higher dimensional invariant subspaces (when $n = 4, 5, 6, 7, 8, 9$). These higher dimensional invariant subspaces are usually ignored in research works due to the difficulty of solving the resulting system of fractional ODEs. Many new exact solutions of Eq. (1.1) have been obtained which have not been reported in [27]. Also, a new invariant subspace (namely, when $n = 5$, $\beta = \frac{9}{2}$ and $\gamma = -\frac{7}{2}$) have been derived.

Appendix A

In this Appendix, the definition of the Riemann-Liouville and Caputo fractional derivatives are introduced. Also, we introduce some basic properties of these fractional derivatives.

Definition 1 [23, 25]. The Riemann-Liouville fractional derivative is given by

$${}_0D_x^{\alpha} f(x) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \left(\frac{d}{dx}\right)^m \int_0^x (x-t)^{m-\alpha-1} f(t)\, dt, & m-1 < \alpha < m \\ \left(\frac{d}{dx}\right)^{m-1} f(x), & \alpha = m-1, \end{cases}$$

where m is an integer number.

Definition 2 [23, 25]. The Caputo fractional derivative is given by

$${}_0^C D_x^{\alpha} f(x) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)\, dt, & m-1 < \alpha < m \\ \left(\frac{d}{dx}\right)^{m-1} f(x), & \alpha = m-1, \end{cases}$$

where m is an integer number.

The Riemann-Liouville and Caputo fractional derivatives satisfy the following properties [23, 25]:

$${}_0D_x^{\alpha}(f(x) + g(x)) = {}_0D_x^{\alpha} f(x) + {}_0D_x^{\alpha} g(x),$$
$${}_0^C D_x^{\alpha}(f(x) + g(x)) = {}_0^C D_x^{\alpha} f(x) + {}_0^C D_x^{\alpha} g(x),$$
$${}_0D_x^{\alpha}(x^{\beta}) = \begin{cases} 0, \quad if \quad \beta > -1 \quad and \quad \alpha - \beta \in \{0, 1, 2, ..., m-1\}, \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}, \quad if \quad \beta > -1 \quad and \quad \alpha - \beta \notin \mathbb{N}, \end{cases}$$
$${}_0^C D_x^{\alpha}(x^{\beta}) = \begin{cases} 0, \quad if \quad \beta \in \{0, 1, 2, ..., m-1\}, \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}, \quad if \quad \beta \in \mathbb{N} \quad and \quad \beta \geq m, \quad or \quad \beta \notin \mathbb{N} \quad and \quad \beta > m-1. \end{cases}$$

References

[1] A. H. Abdel Kader, M. S. Abdel Latif, F. El Bialy and A. Elsaid, *Symmetry analysis and some new exact solutions of some nonlinear kdv-like equations*, Asian-European Journal of Mathematics, 2018, 11(03), 1850040.

[2] M. S. Abdel Latif, *Some exact solutions of kdv equation with variable coefficients*, Communications in nonlinear science and numerical simulation, 2011, 16(4), 1783–1786.

[3] M. S. Abdel Latif, E. El-Shazly, D. Baleanu et al., *Some new soliton-like and doubly periodic-like solutions of fisher equation with time-dependent coefficients*, Modern Physics Letters B, 2018, 32(33), 1850413.

[4] S. Choudhary and V. Daftardar-Gejji, *Solving systems of multi-term fractional pdes: Invariant subspace approach*, International Journal of Modeling, Simulation, and Scientific Computing, 2019, 10, 1941010.

[5] E. Fan, *Extended tanh-function method and its applications to nonlinear equations*, Physics Letters A, 2000, 277(4-5), 212–218.

[6] E. Fan and Z. Hongqing, *Backlund transformation and exact solutions for whitham-broer-kaup equations in shallow water*, Applied Mathematics and Mechanics, 1998, 19(8), 713–716.

[7] V. A. Galaktionov and S. R. Svirshchevskii, *Exact solutions and invariant subspaces of nonlinear partial differential equations in mechanics and physics*, Chapman and Hall/CRC, 2006.

[8] R. K. Gazizov and A. A. Kasatkin, *Construction of exact solutions for fractional order differential equations by the invariant subspace method*, Computers & Mathematics with Applications, 2013, 66(5), 576–584.

[9] B. K. Harrison, *Bäcklund transformation for the ernst equation of general relativity*, Physical Review Letters, 1978, 41(18), 1197.

[10] M. Hashemi, *Invariant subspaces admitted by fractional differential equations with conformable derivatives*, Chaos, Solitons & Fractals, 2018, 107, 161–169.

[11] K. Hosseini, Z. Ayati and R. Ansari, *Application of the invariant subspace method in conjunction with the fractional sumudus transform to a nonlinear conformable time-fractional dispersive equation of the fifth order*, Computational Methods for Differential Equations, 2019, 7(3), 359–369.

[12] K. Hosseini, A. Bekir and R. Ansari, *Exact solutions of nonlinear conformable time-fractional boussinesq equations using the* $\exp(-\varphi(z))-$*expansion method*, Optical and Quantum Electronics, 2017, 49(4), 131.

[13] K. Hosseini, A. Bekir, M. Kaplan and Ö. Güner, *On a new technique for solving the nonlinear conformable time-fractional differential equations*, Optical and Quantum Electronics, 2017, 49(11), 343.

[14] K. Hosseini, M. Inc, M. Shafiee et al., *Invariant subspaces, exact solutions and stability analysis of nonlinear water wave equations*, Journal of Ocean Engineering and Science, 2019.

[15] K. Hosseini, A. Korkmaz, A. Bekir et al., *New wave form solutions of nonlinear conformable time-fractional zoomeron equation in (2+ 1)-dimensions*, Waves in Random and Complex Media, 2019, 1–11.

[16] K. Hosseini, J. Manafian, F. Samadani et al., *Resonant optical solitons with perturbation terms and fractional temporal evolution using improved tan (ϕ (η)/2)-expansion method and exp function approach*, Optik, 2018, 158, 933–939.

[17] K. Hosseini, P. Mayeli, A. Bekir and O. Guner, *Density-dependent conformable space-time fractional diffusion-reaction equation and its exact solutions*, Communications in Theoretical Physics, 2018, 69(1), 1.

[18] J. Jiang, Y. Feng and S. Li, *Exact solutions to the fractional differential equations with mixed partial derivatives*, Axioms, 2018, 7(1), 10.

[19] A. A. Kader, M. A. Latif and H. Nour, *Some new exact solutions of the modified kdv equation using lie point symmetry method*, International Journal of Applied and Computational Mathematics, 2017, 3(1), 1163–1171.

[20] E. Lashkarian and S. R. Hejazi, *Exact solutions of the time fractional nonlinear schrödinger equation with two different methods*, Mathematical Methods in the Applied Sciences, 2018, 41(7), 2664–2672.

[21] W.-X. Ma and Y. Liu, *Invariant subspaces and exact solutions of a class of dispersive evolution equations*, Communications in Nonlinear Science and Numerical Simulation, 2012, 17(10), 3795–3801.

[22] W. Malfliet, *The tanh method: a tool for solving certain classes of non-linear pdes*, Mathematical methods in the applied sciences, 2005, 28(17), 2031–2035.

[23] K. S. Miller and B. Ross, *An introduction to the fractional calculus and fractional differential equations*, Wiley, 1993.

[24] P. J. Olver, *Applications of Lie groups to differential equations*, 107, Springer Science & Business Media, 2012.

[25] I. Podlubny, *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, 198, Elsevier, 1998.

[26] W. Rui, *Idea of invariant subspace combined with elementary integral method for investigating exact solutions of time-fractional npdes*, Applied Mathematics and Computation, 2018, 339, 158–171.

[27] R. Sahadevan and P. Prakash, *Exact solution of certain time fractional nonlinear partial differential equations*, Nonlinear Dynamics, 2016, 85(1), 659–673.

[28] R. Sahadevan and P. Prakash, *On lie symmetry analysis and invariant subspace methods of coupled time fractional partial differential equations*, Chaos, Solitons & Fractals, 2017, 104, 107–120.

[29] A.-M. Wazwaz, *The tanh method for traveling wave solutions of nonlinear equations*, Applied Mathematics and Computation, 2004, 154(3), 713–723.

[30] L. Zhang and Y. Lin, *Symbolic computation of exact solutions for nonlinear evolution equations*, Nonlinear Dynamics, 2015, 79(2), 823–833.