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Threshold Dynamics of an Epidemic Model with Latency and Vaccination in a Heterogeneous Habitat*

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Abstract In this paper, we derive and analyze a nonlocal and time-delayed reaction-diffusion epidemic model with vaccination strategy in a heterogeneous habitat. First, we study the well-posedness of the solutions and prove the existence of a global attractor for the model by applying some existing abstract results in dynamical systems theory. Then we show the global threshold dynamics which predicts whether the disease will die out or persist in terms of the basic reproduction number $\mathfrak{R}_0$ defined by the spectral radius of the next generation operator. Finally, we present the influences of heterogeneous spatial infections, diffusion coefficients and vaccination rate on the spread of the disease by numerical simulations.

Keywords Diffusive epidemic model, Threshold dynamics, Heterogeneous habitat, Vaccination strategy.

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1. Introduction

Since Gumel and Moghadas [6] proposed an epidemic model with nonlinear incidence and vaccination strategy, researchers have done a lot of work on this model and its derived versions (see, e.g., [24]). In these models, the role of vaccination is only to reduce the chance of vaccinated individuals being infected and not fully immunize, which is different from most existing models that follow the assumption that vaccinated individuals will not be infected at all (see, e.g., [30]).

In real world, the nature of disease varies with temperature, humidity and other factors in different environments. The spatial heterogeneity plays an important role in the theory of epidemiology. So far, many mathematical models with spatial dependence have been studied (see, e.g., [13, 29]). On the other hand, many diseases have a latent period before the hosts becoming infectious. For instance, dengue fever is a viral disease, which is transmitted to humans by the Aedes aegypti mosquito feeding during the day. When an infectious mosquito bites a susceptible

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human, the virus is injected into his or her bloodstream and begins an latent period which takes from three to seven days [4]. Suppose that the latency τ is brought into this population, namely, the susceptible or vaccinated individuals will infect other uninfected individuals after being infected τ time, resulting in dividing the population into five epidemiological classes living in the spatial habitat Ω with smooth boundary $\partial\Omega$: susceptible, vaccinated, exposed, infectious and recovered classes, denoted by $S = S(x,t)$, $V = V(x,t)$, $E = E(x,t)$, $I = I(x,t)$ and $R = R(x,t)$, respectively.

Note that if an individual is infected by the disease in one location, and can move freely during the latent period, this individual may appear at any location in the domain when this individual becomes infectious. This means that the mobility of the individuals in the latent period will lead to non-local infection. Some non-local reaction-diffusion models in a spatially continuous habitat have been widely studied (see, e.g., [12,23,25,28]). To incorporate non-local infection into the model properly, we introduce an infection age variable θ, and let $u(x,t,\theta)$ represents the density of infected population with infection age θ at time t and location $x \in \Omega$. Using the standard method on describing age structured population with spatial diffusion [17], we have

$$\frac{\partial u(x,t,\theta)}{\partial t} + \frac{\partial u(x,t,\theta)}{\partial \theta} = D(x,\theta)\Delta u(x,t,\theta) - \mu(x)u(x,t,\theta) - \gamma(x,\theta)u(x,t,\theta), \tag{1.1}$$

where $D(x,\theta)$ and $\gamma(x,\theta)$ are the diffusion rate and the recovery rate at location x and age θ, respectively, and $\mu(x)$ denotes the natural death rate which is independent of the infection age. It easily follows that

$$E(x,t) = \int_0^\tau u(x,t,\theta)d\theta \quad \text{and} \quad I(x,t) = \int_\tau^\infty u(x,t,\theta)d\theta.$$

From the biological considerations, infected individuals connot recover during the latent period. To make the model mathematically tractable yet without losing the main features, we assume that

$$D(x,\theta) = \begin{cases} D_E(x) & \text{for} \quad x \in \Omega, \theta \in [0,\tau), \\ D_I(x) & \text{for} \quad x \in \Omega, \theta \in [\tau,\infty), \end{cases}$$

$$\gamma(x,\theta) = \begin{cases} 0 & \text{for} \quad x \in \Omega, \theta \in [0,\tau), \\ \gamma(x) & \text{for} \quad x \in \Omega, \theta \in [\tau,\infty). \end{cases}$$

Integrating both sides of (1.1) with respect to θ from 0 to τ, and from τ to ∞, respectively, we obtain that

$$\frac{\partial E(x,t)}{\partial t} = D_E(x)\Delta E(x,t) - \mu(x)E(x,t) + u(x,t,0) - u(x,t,\tau) \tag{1.2}$$

and

$$\frac{\partial I(x,t)}{\partial t} = D_I(x)\Delta I(x,t) - (\mu(x)+\gamma(x))I(x,t) + u(x,t,\tau) - u(x,t,\infty). \tag{1.3}$$

Biologically, it can be assumed that $u(x,t,\infty) = 0$ (see e.g., [7]). Let $\beta_1(x)$ and $\beta_2(x)$ be the transmission coefficients of susceptible and vaccinated individuals at

location x, respectively. Note that the susceptible and vaccinated individuals would be infected when they are in contact with the infectious individuals, and the force of infection saturates as the number of the infective individuals increases, by using the Holling-II incidence function, we have

$$u(x,t,0)=\frac{(\beta_1(x)S(x,t)+\beta_2(x)V(x,t))I(x,t)}{1+\alpha(x)I(x,t)}, \tag{1.4}$$

where $\alpha(x)\geq 0$ measures the saturation level, see [1] for more explanation.

Let $v(x,\xi,\theta)=u(x,\theta+\xi,\theta)$ with $\theta\in[0,\tau]$, we have

$$\begin{cases} \dfrac{\partial v(x,\xi,\theta)}{\partial\theta}=\left[\dfrac{\partial u(x,t,\theta)}{\partial t}+\dfrac{\partial u(x,t,\theta)}{\partial\theta}\right]_{t=\theta+\xi} \\ \qquad\qquad =D(x,\theta)\Delta u(x,\theta+\xi,\theta)-(\mu(x)+\gamma(x,a))u(x,\theta+\xi,\theta) \\ \qquad\qquad =D_E(x)\Delta v(x,\xi,\theta)-\mu(x)v(x,\xi,\theta),\quad \theta\in[0,\tau], \\ v(x,\xi,0)=\dfrac{(\beta_1(x)S(x,\xi)+\beta_2(x)V(x,\xi))\,I(x,\xi)}{1+\alpha(x)I(x,\xi)}. \end{cases} \tag{1.5}$$

Regarding ξ as a parameter and letting $\Gamma(\theta,x,y)$ be the fundamental solution associated with the partial differential operator $D_E\Delta-\mu(\cdot)$ with Neumann boundary condition [5], it follows that

$$v(x,\xi,\theta)=\int_\Omega\Gamma(\theta,x,y)\left(\frac{(\beta_1(y)S(y,\xi)+\beta_2(y)V(y,\xi))\,I(y,\xi)}{1+\alpha(y)I(y,\xi)}\right)dy. \tag{1.6}$$

Since $u(x,t,\tau)=v(x,t-\tau,\tau)$, we get

$$u(x,t,\tau)=\int_\Omega\Gamma(\tau,x,y)\left(\frac{(\beta_1(y)S(y,t-\tau)+\beta_2(y)V(y,t-\tau))\,I(y,t-\tau)}{1+\alpha(y)I(y,t-\tau)}\right)dy, \tag{1.7}$$

which, together with (1.2)-(1.4), implies that

$$\begin{aligned}\frac{\partial E(x,t)}{\partial t}=&D_E(x)\Delta E(x,t)-\mu(x)E(x,t)+\frac{(\beta_1(x)S(x,t)+\beta_2(x)V(x,t))I(x,t)}{1+\alpha(x)I(x,t)}\\&-\int_\Omega\Gamma(\tau,x,y)\left(\frac{(\beta_1(y)S(y,t-\tau)+\beta_2(y)V(y,t-\tau))\,I(y,t-\tau)}{1+\alpha(y)I(y,t-\tau)}\right)dy\end{aligned}$$

and

$$\begin{aligned}\frac{\partial I(x,t)}{\partial t}=&D_I(x)\Delta I(x,t)-(\mu(x)+\gamma(x))I(x,t)\\&+\int_\Omega\Gamma(\tau,x,y)\left(\frac{(\beta_1(y)S(y,t-\tau)+\beta_2(y)V(y,t-\tau))\,I(y,t-\tau)}{1+\alpha(y)I(y,t-\tau)}\right)dy.\end{aligned}$$

Based on the analysis above, we are leading to consider the SVEIR model under

the framework of reaction-diffusion system as follows

$$\begin{cases} \dfrac{\partial S(x,t)}{\partial t} =D_S(x)\Delta S(x,t)+\Pi(x)-(\mu(x)+\omega(x))S(x,t)-\dfrac{\beta_1(x)S(x,t)I(x,t)}{1+\alpha(x)I(x,t)}, \\ \dfrac{\partial V(x,t)}{\partial t} =D_V(x)\Delta V(x,t)-\mu(x)V(x,t)+\omega(x)S(x,t)-\dfrac{\beta_2(x)V(x,t)I(x,t)}{1+\alpha(x)I(x,t)}, \\ \dfrac{\partial E(x,t)}{\partial t} =D_E(x)\Delta E(x,t)-\mu(x)E(x,t)+\dfrac{(\beta_1(x)S(x,t)+\beta_2(x)V(x,t))I(x,t)}{1+\alpha(x)I(x,t)} \\ \qquad\qquad -\displaystyle\int_\Omega \Gamma(\tau,x,y)\frac{(\beta_1(y)S(y,t-\tau)+\beta_2(y)V(y,t-\tau))I(y,t-\tau)}{1+\alpha(y)I(y,t-\tau)}dy, \\ \dfrac{\partial I(x,t)}{\partial t} =D_I(x)\Delta I(x,t)-(\mu(x)+\gamma(x))I(x,t) \\ \qquad\qquad +\displaystyle\int_\Omega \Gamma(\tau,x,y)\frac{(\beta_1(y)S(y,t-\tau)+\beta_2(y)V(y,t-\tau))I(y,t-\tau)}{1+\alpha(y)I(y,t-\tau)}dy, \\ \dfrac{\partial R(x,t)}{\partial t} =D_R(x)R(x,t)+\gamma(x)I(x,t)-\mu(x)R(x,t), \end{cases} \tag{1.8}$$

where $\Pi(x)$ and $\omega(x)$ represent the recruiting rate and the vaccination rate of the susceptible individuals, respectively. Since the components E and R are decoupled with the others, it suffices to study the following subsystem

$$\begin{cases} \dfrac{\partial S(x,t)}{\partial t} =D_S(x)\Delta S(x,t)+\Pi(x)-(\mu(x)+\omega(x))S(x,t)-\dfrac{\beta_1(x)S(x,t)I(x,t)}{1+\alpha(x)I(x,t)}, \\ \dfrac{\partial V(x,t)}{\partial t} =D_V(x)\Delta V(x,t)-\mu(x)V(x,t)+\omega(x)S(x,t)-\dfrac{\beta_2(x)V(x,t)I(x,t)}{1+\alpha(x)I(x,t)}, \\ \dfrac{\partial I(x,t)}{\partial t} =D_I(x)\Delta I(x,t)-(\mu(x)+\gamma(x))I(x,t) \\ \qquad\qquad +\displaystyle\int_\Omega \Gamma(\tau,x,y)\frac{(\beta_1(y)S(y,t-\tau)+\beta_2(y)V(y,t-\tau))I(y,t-\tau)}{1+\alpha(y)I(y,t-\tau)}dy, \end{cases} \tag{1.9}$$

with Neumann boundary conditions

$$\frac{\partial S(x,t)}{\partial n}=\frac{\partial V(x,t)}{\partial n}=\frac{\partial I(x,t)}{\partial n}=0, \quad x\in\partial\Omega, t>0, \tag{1.10}$$

where n is the outward unit normal vector on the boundary $\partial\Omega$, and the initial conditions

$$\begin{aligned} &S(x,\theta)=\phi_1(x,\theta)\geq 0, \quad V(x,\theta)=\phi_2(x,\theta)\geq 0, \\ &I(x,\theta)=\phi_3(x,\theta)\geq,\not\equiv 0, \quad \theta\in[-\tau,0], x\in\Omega. \end{aligned} \tag{1.11}$$

All space dependent parameters in (1.9) are continuous and strictly positive. Note that in our recent work [10], we established a complete threshold result which reveals the existence and non-existence of the strong traveling waves for system (1.9), where all the parameters are spatially homogeneous and the infection mechanism is local. For convenience, we set

$$\begin{aligned} &\overline{\Pi}=\max_{x\in\overline{\Omega}}\Pi(x)>0, \quad \underline{\Pi}=\min_{x\in\overline{\Omega}}\Pi(x)>0, \quad \overline{\beta_1}=\max_{x\in\overline{\Omega}}\beta_1(x)>0, \\ &\overline{\omega}=\max_{x\in\overline{\Omega}}\omega(x)>0, \quad \underline{\omega}=\min_{x\in\overline{\Omega}}\omega(x)>0, \quad \overline{\beta_2}=\max_{x\in\overline{\Omega}}\beta_2(x)>0, \end{aligned}$$

$$\overline{\mu} = \max_{x\in\overline{\Omega}} \mu(x) > 0, \quad \underline{\mu} = \min_{x\in\overline{\Omega}} \mu(x) > 0, \quad \underline{\gamma} = \min_{x\in\overline{\Omega}} \gamma(x) > 0,$$

where $\overline{\Omega}$ is the closure of Ω.

The rest of this paper is organized as follows. In Section 2, we devote to the establishment of the well-posedness of solutions for system (1.9). We show that the solutions for system (1.9) exist globally and have a compact global attractor. In Section 3, We derive a biologically meaningful threshold index, the basic reproduction number $\mathfrak{R}_0$ which is identified as the spectral radius of the next generation operator, and investigate the threshold dynamics in terms of $\mathfrak{R}_0$. We numerically compute $\mathfrak{R}_0$ to study the influences of heterogeneous spatial infections, diffusion coefficients and vaccination rate on the spread of the disease in Section 4. Finally, we present discussions and conclusions in Section 5.

2. The well-posedness

In this section, we focus on the well-posedness of solutions for system (1.9). Let $\mathbb{X} := BUC(\overline{\Omega}, \mathbb{R}^3)$ be the set of all bounded and uniformly continuous functions from $\overline{\Omega}$ to $\mathbb{R}^3$, and $\mathbb{X}_+ := BUC(\overline{\Omega}, \mathbb{R}^3_+)$. Obviously, $\mathbb{X}_+$ is a closed cone of $\mathbb{X}$ and induces a partial ordering on $\mathbb{X}$. In addition, we define a norm $\|\cdot\|_{\mathbb{X}}$ by $\|\varphi\|_{\mathbb{X}} = \sup_{x\in\overline{\Omega}} |\varphi(x)|$, where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^3$. It then follows that $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ is a Banach space. We define $\mathbb{C} := C([-\tau, 0], \mathbb{X})$ with the norm $\|\varphi\| = \sup_{\theta\in[-\tau,0]} \|\varphi(\theta)\|_{\mathbb{X}}$ and $\mathbb{C}_+ := C([-\tau, 0], \mathbb{X}_+)$. Obviously, $(\mathbb{C}, \mathbb{C}_+)$ is an ordered Banach space. For convenience, we regard an element $\phi \in \mathbb{C}_+$ as a function from $\overline{\Omega} \times [-\tau, 0]$ into $\mathbb{R}^3$ defined by $\phi(x, s) = \phi(s)(x)$ for all $x \in \overline{\Omega}$ and $s \in [-\tau, 0]$. For any $\sigma > 0$ and continuous function $u : [-\tau, \sigma) \to \mathbb{X}$, we define $u_t \in \mathbb{C}$ by $u_t(\theta) = u(t + \theta)$ for all $\theta \in [-\tau, 0]$.

Let $\mathbb{Y} := C(\overline{\Omega}, \mathbb{R})$ and $\mathbb{Y}_+ := C(\overline{\Omega}, \mathbb{R}_+)$. Suppose that $T_i(t)$, $(i = 1, 2, 3)$ are the strongly continuous semigroups associated with $D_S(\cdot)\Delta - (\omega(\cdot) + \mu(\cdot))$, $D_V(\cdot)\Delta - \mu(\cdot)$ and $D_I(\cdot)\Delta - (\mu(\cdot) + \gamma(\cdot))$ subject to Neumann boundary conditions, respectively. It then follows that

$$(T_i(t)\phi)(x) = \int_\Omega \Lambda_i(t, x, y)\phi(y)dy, \quad \forall \phi \in \mathbb{Y}, t \geq 0,$$

where Λ_i are the Green functions associated with $D_S(\cdot)\Delta - (\omega(\cdot) + \mu(\cdot))$, $D_V(\cdot)\Delta - \mu(\cdot)$ and $D_I(\cdot)\Delta - (\mu(\cdot) + \gamma(\cdot))$, respectively, subject to the Neumann boundary condition. By [18, Corollary 7.2.3], $T_i(t) : \mathbb{Y} \to \mathbb{Y}$ is strongly positive and compact for each $t > 0$. Moreover, $T(t) := (T_1(t), T_2(t), T_3(t)) : \mathbb{X} \to \mathbb{X}$, $t \geq 0$ is a strongly continuous semigroup. Let $A_i : D(A_i) \to \mathbb{Y}$ be the generator of $T_i(t)$. Then $T(t) : \mathbb{X} \to \mathbb{X}$ is a semigroup generated by the operator $A = (A_1, A_2, A_3)$ defined on $D(A) = D(A_1) \times D(A_2) \times D(A_3)$.

We now define $F = (F_1, F_2, F_3) : \mathbb{C}_+ \to \mathbb{X}$ by

$$F_1(\phi)(x) = \Pi(x) - \frac{\beta_1(x)\phi_1(0, x)\phi_3(0, x)}{1 + \alpha(x)\phi_3(0, x)},$$

$$F_2(\phi)(x) = \omega(x)\phi_1(0, x) - \frac{\beta_2(x)\phi_2(0, x)\phi_3(0, x)}{1 + \alpha(x)\phi_3(0, x)},$$

$$F_3(\phi)(x) = \int_\Omega \Gamma(\tau, x, y)\frac{(\beta_1(y)\phi_1(-\tau, y) + \beta_2(y)\phi_2(-\tau, y))\phi_3(-\tau, y)}{1 + \alpha(y)\phi_3(-\tau, y)}dy$$

for any $\phi = (\phi_1, \phi_2, \phi_3) \in \mathbb{C}_+$ and $x \in \overline{\Omega}$. Obviously, F is Lipschitz continuous in any bounded subset of $\mathbb{C}_+$. Then system (1.9) can be rewritten as follows

$$\begin{cases} \dfrac{du}{dt} = Au + F(u_t), \quad t \geq 0, \\ u_0 = \phi \in \mathbb{C}_+, \end{cases} \tag{2.1}$$

where $u := (S, V, I)$.

Theorem 2.1. *For any initial value function $\phi = (\phi_1, \phi_2, \phi_3) \in \mathbb{C}_+$, system (1.9) admits a unique mild solution $u(\cdot, t, \phi)$ defined on its maximal interval of existence $[0, \sigma_\phi)$ with $u_0 = \phi$, where $\sigma_\phi \leq \infty$. Further, for any $t \in [0, \sigma_\phi)$, it have $u(\cdot, t, \phi) \in \mathbb{C}_+$. For any $t > \tau$, $u(x, t, \phi)$ is a classical solution of (1.9).*

Proof. For any $\phi = (\phi_1, \phi_2, \phi_3) \in \mathbb{C}_+$ and $\hat{k} > 0$, we have

$$\begin{aligned} &\phi(x,0) + \hat{k}F(\phi)(x) \\ &= \begin{pmatrix} \phi_1(x,0) + \hat{k}\Pi(x) - \hat{k}\frac{\beta_1(x)\phi_1(x,0)\phi_3(x,0)}{1+\alpha(x)\phi_3(x,0)} \\ \phi_2(x,0) + \hat{k}\omega(x)\phi_1(x,0) - \hat{k}\frac{\beta_2(x)\phi_2(x,0)\phi_3(x,0)}{1+\alpha(x)\phi_3(x,0)} \\ \phi_3(x,0) + \hat{k}\int_\Omega \Gamma(\tau, x, y)\frac{(\beta_1(y)\phi_1(y,-\tau)+\beta_2(y)\phi_2(y,-\tau))\phi_3(y,-\tau)}{1+\alpha(y)\phi_3(y,-\tau)}dy \end{pmatrix} \\ &\geq \begin{pmatrix} (1 - \hat{k}\frac{\overline{\beta}_1}{\underline{\alpha}})\phi_1(x,0) \\ (1 - \hat{k}\frac{\overline{\beta}_2}{\underline{\alpha}})\phi_2(x,0) \\ \phi_3(x,0) \end{pmatrix}. \end{aligned}$$

This implies that $\phi(0) + \hat{k}F(\phi) \in \mathbb{C}_+$ when $\hat{k}$ is sufficiently small, and hence,

$$\lim_{\hat{k}\to 0^+} \frac{1}{\hat{k}} \mathrm{dist}\left(\phi(0) + \hat{k}F(\phi), \mathbb{C}_+\right) = 0, \quad \forall \phi \in \mathbb{C}_+.$$

In view of [27, Corollary 8.1.3] (see also [16, Corollary 4]), we can obtain the desired conclusion. □

In the following, we will show that solutions of system (1.9) exist globally on $[0, \infty)$ and converge to a compact attractor in $\mathbb{C}_+$.

Theorem 2.2. *For any $\phi \in \mathbb{C}_+$, system (1.9) has a unique solution $u(\cdot, t, \phi)$ on $[0, \infty)$ with $u_0 = \phi$, and the solution semiflow $\Phi(t) = u_t(\cdot) : \mathbb{C}_+ \to \mathbb{C}_+$ has a global compact attractor for $t \geq 0$.*

Proof. By [14, Lemma 1], the following problem

$$\begin{cases} \dfrac{\partial \nu(x,t)}{\partial t} = D_S(x)\Delta\nu(x,t) + \Pi(x) - (\omega(x) + \mu(x))\nu(x,t), & x \in \Omega, t > 0, \\ \dfrac{\partial \nu(x,t)}{\partial n} = 0, & x \in \partial\Omega, t > 0 \end{cases}$$

admits a unique positive steady state $\nu^*(x)$ which is globally attractive in $\mathbb{Y}$. Since

$$\frac{\partial S(x,t)}{\partial t} \leq D_S(x)\Delta S + \Pi(x) - (\omega(x) + \mu(x)S(x,t), \quad \forall t \in [0, \sigma_\phi), x \in \Omega, \tag{2.2}$$

it then follows from the standard parabolic comparison theorem that $S(\cdot,t,\phi)$ is bounded on $[0,\sigma_\phi)$. Thus, there exists $M_1>0$, such that $S(x,t)\le M_1$, $\forall t\in[0,\sigma_\phi)$. Further, we have

$$\frac{\partial V(x,t)}{\partial t}\le D_V(x)\Delta V+\omega(x)M_1-\mu(x)V(x,t),\quad \forall t\in[0,\sigma_\phi),x\in\Omega.$$

By [14, Lemma 1] and the comparison principle again, we see that $V(\cdot,t,\phi)$ is bounded on $[0,\sigma_\phi)$, and hence, there exists $M_2>0$, such that $V(x,t)\le M_2$, $\forall t\in[0,\sigma_\phi)$. Thus, we have

$$\begin{aligned}\frac{\partial I(x,t)}{\partial t}\le& D_I(x)\Delta I+\frac{\beta_1(x)M_1+\beta_2(x)M_2}{\alpha(x)}\\&-(\mu(x)+\gamma(x))I(x,t),\quad \forall t\in[0,\sigma_\phi),\quad x\in\Omega.\end{aligned}$$

Again, by [14, Lemma 1] and the comparison principle, it follows that $I(\cdot,t,\phi)$ is bounded on $[0,\sigma_\phi)$. It then follows that $u(\cdot,t,\phi)=(S(\cdot,t,\phi),V(\cdot,t,\phi),I(\cdot,t,\phi))$ is bounded on $[0,\sigma_\phi)$. Consequently, we know from [27, Theorem 2.1.1] that the solution $u(\cdot,t,\phi)$ exists globally on $[0,\infty)$. Therefore, system (1.9) generates a semiflow $\Phi(t):\mathbb{C}_+\to\mathbb{C}_+$ by

$$(\Phi(t)\phi)(\theta,x)=u(x,t+\theta,\phi),\quad \forall\theta\in[-\tau,0],x\in\overline{\Omega}.$$

For any fixed $\phi\in\mathbb{C}_+$, by (2.2) and the comparison principle, we have some $t_1=t_1(\phi)>0$ such that

$$S(\cdot,t,\phi)\le\frac{2\overline{\Pi}}{\underline{\mu}+\underline{\omega}}:=B_1,\quad \forall t\ge t_1.$$

Similarly, we can prove that there exists B_i and $t_i=t_i(\phi)>0$ with $i=2,3$ such that

$$V(\cdot,t,\phi)\le B_2,\quad \forall t\ge t_2\quad\text{and}\quad I(\cdot,t,\phi)\le B_3,\quad \forall t\ge t_3.$$

Thus, the nonnegative solutions of system (1.9) are ultimately bounded with respect to the maximum norm. It means that the solution semiflow $\Phi(t):\mathbb{C}_+\to\mathbb{C}_+$ is point dissipative. By [27, Theorem 2.2.6], we get that $\Phi(t)$ is compact for any $t>\tau$. Therefore, from [9, Theorem 3.4.8], we conclude that $\Phi(t)$ has a compact global attractor in $\mathbb{C}_+$. This completes the proof. □

3. Threshold dynamics

This section is devoted to the threshold dynamics for the spatially heterogeneous system (1.9) in terms of the basic reproduction number $\mathfrak{R}_0$. To this end, we first define the basic reproduction number for system (1.9). According to [14, Lemma 1], it is easy to see that the following problem

$$\begin{cases}\dfrac{\partial S(x,t)}{\partial t}=D_S(x)\Delta S(x,t)+\Pi(x)-(\omega(x)+\mu(x))S(x,t), & x\in\Omega,\quad t>0,\\ \dfrac{\partial V(x,t)}{\partial t}=D_V(x)\Delta V(x,t)+\omega(x)S(x,t)-\mu(x)V(x,t), & x\in\Omega,\quad t>0,\\ \dfrac{\partial S(x,t)}{\partial n}=\dfrac{\partial V(x,t)}{\partial n}=0, & x\in\partial\Omega,\quad t>0\end{cases}\tag{3.1}$$

admits a unique positive steady state $(S_0(x), V_0(x))$. Thus, system (1.9) has a unique disease-free equilibrium $E_0 := (S_0(x), V_0(x), 0)$.

Linearizing the third equation of system (1.9) at the disease-free equilibrium E_0, we have

$$\begin{cases} \dfrac{\partial I(x,t)}{\partial t} =D_I(x)\Delta I(x,t) - (\mu(x)+\gamma(x))I(x,t) \\ \qquad\qquad + \displaystyle\int_\Omega \Gamma(\tau,x,y)(\beta_1(y)S_0(y)+\beta_2(y)V_0(y))I(y,t-\tau)dy, \quad x\in\Omega, t>0, \\ \dfrac{\partial I(x,t)}{\partial n} =0, \quad x\in\partial\Omega, t>0. \end{cases} \tag{3.2}$$

Substituting $I(t,x) = e^{-\lambda t}\psi(x)$ into (3.2), we obtain the following nonlocal eigenvalue problem

$$\begin{cases} \lambda\psi(x) =D_I(x)\Delta\psi(x) - (\mu(x)+\gamma(x))\psi(x) \\ \qquad\qquad + e^{-\lambda\tau}\displaystyle\int_\Omega \Gamma(\tau,x,y)(\beta_1(y)S_0(y)+\beta_2(y)V_0(y))\psi(y)dy, \quad x\in\Omega, t>0, \\ \dfrac{\partial\psi(x)}{\partial n} =0, \quad x\in\partial\Omega, t>0. \end{cases} \tag{3.3}$$

Note that (3.3) is nonlinear in terms of λ, which is caused by the presence of the delay τ in the item $I(y,t-\tau)$ of (3.2). Since the linear delayed equation (3.2) is monotone, the general results on monotone delay equations (see [22, Theorem 2.2]) suggest that the delay τ plays no role in determining the stability of the trivial solution of (3.2), which motivates us to consider the following associated linear nonlocal system resulting from dropping τ in (3.2):

$$\begin{cases} \dfrac{\partial I(x,t)}{\partial t} =D_I(x)\Delta I(x,t) - (\mu(x)+\gamma(x))I(x,t) \\ \qquad\qquad + \displaystyle\int_\Omega \Gamma(\tau,x,y)(\beta_1(y)S_0(y)+\beta_2(y)V_0(y))I(y,t)dy, \quad x\in\Omega, t>0, \\ \dfrac{\partial I(x,t)}{\partial n} =0, \quad x\in\partial\Omega, t>0. \end{cases} \tag{3.4}$$

Substituting $I(t,x) = e^{-\lambda t}\psi(x)$ into (3.4), we obtain the nonlocal eigenvalue problem as follows

$$\begin{cases} \lambda\psi(x) =D_I(x)\Delta\psi(x) - (\mu(x)+\gamma(x))\psi(x) \\ \qquad\qquad + \displaystyle\int_\Omega \Gamma(\tau,x,y)(\beta_1(y)S_0(y)+\beta_2(y)V_0(y))\psi(y)dy, \quad x\in\Omega, t>0, \\ \dfrac{\partial\psi(x)}{\partial n} =0, \quad x\in\partial\Omega, t>0. \end{cases} \tag{3.5}$$

As direct applications of [18, Theorem 7.6.1] and [22, Theorem 2.2], we have the following conclusion regarding the eigenvalue problems (3.3) and (3.5).

Lemma 3.1. *The eigenvalue problems* (3.3) *and* (3.5) *have principal eigenvalues* $\overline{\lambda}(S_0, V_0, \tau)$ *and* $\lambda(S_0, V_0)$*, respectively, corresponding to which, there exists a unique strongly positive eigenvector. Further,* $\overline{\lambda}(S_0, V_0, \tau)$ *has the same sign as* $\lambda(S_0, V_0)$ *for any* $\tau > 0$.

Assume that the populations of all classes are near the disease-free equilibrium E_0. Let $\psi(x)$ be the spatial distribution of infectious individuals at $t = 0$, and as time evolves, those distributions at time $t > 0$ is $T_3(t)\psi$. We define a positive linear operator $Q : \mathbb{Y} \to \mathbb{Y}$ by

$$Q(\psi)(x) = \int_\Omega \Gamma(\tau, x, y)(\beta_1(y)S_0(y) + \beta_2(y)V_0(y))\psi(y)dy, \quad \forall \psi \in \mathbb{Y}, x \in \overline{\Omega}.$$

Then the distribution of new infected individuals reads that

$$\begin{aligned} L(\psi)(x) :=& \left[\int_0^\infty Q(T_3(t)\psi)(x)dt\right] = \int_\tau^\infty Q(T_3(t-\tau)\psi)(x)dt \\ =& \int_\tau^\infty \int_\Omega \Gamma(\tau, x, y)(\beta_1(y)S_0(y) + \beta_2(y)V_0(y))(T_3(t-\tau)\psi)(y)dydt \\ =& \int_0^\infty \int_\Omega \Gamma(\tau, x, y)(\beta_1(y)S_0(y) + \beta_2(y)V_0(y))(T_3(t)\psi)(y)dydt \\ =& \int_0^\infty Q(T_3(t)\psi)(x)dt \\ =& Q\left(\int_0^\infty T_3(t)\psi dt\right)(x), \end{aligned} \tag{3.6}$$

which is indeed the next infection operator. Following the work of Diekmann et al. [3], we see that $\mathfrak{R}_0 := r(L)$ with $r(L)$ being the spectral radius of L. Further, we can obtain the following lemma by using the general results in [21] and the same arguments as in [26, Lemma 2.2].

Lemma 3.2. *$\mathfrak{R}_0 - 1$ has the same sign as $\lambda(S_0, V_0)$.*

Remark 3.1. When all the parameters of system (1.9) are assumed to be spatially independent, we then have

$$\mathfrak{R}_0 = e^{-\mu\tau}\frac{\beta_1 S_0 + \beta_2 V_0}{\mu + \gamma} = e^{-\mu\tau}\frac{\Pi(\beta_1\mu + \beta_2\omega)}{\mu(\mu + \omega)(\mu + \gamma)},$$

where $S_0 = \dfrac{\Pi}{\omega + \mu}$ and $V_0 = \dfrac{\omega\Pi}{\mu(\omega + \mu)}$.

Next, we turn our attention to the uniform persistence for system (1.9). To this end, we prove the following lemmas.

Lemma 3.3. *Let $u(x, t, \phi)$ be the solution of system (1.9) with $u_0 = \phi \in \mathbb{C}_+$. Then the following statements are valid:*

(i) *If there exists some $t_0 > 0$ such that $I(\cdot, t_0, \phi) \geq, \not\equiv 0$, then $I(x, t, \phi) \geq 0$ for all $t > t_0$ and $x \in \overline{\Omega}$.*

(ii) *$S(x, t, \phi) > 0$ and $V(x, t, \phi) > 0$ for any $t > 0$ and $x \in \overline{\Omega}$. Further, we have*

$$\liminf_{t\to\infty} S(x, t, \phi) \geq \frac{\underline{\Pi}}{\overline{\mu} + \overline{\omega} + \frac{\overline{\beta}_1}{\underline{\alpha}}} \quad \text{and} \quad \liminf_{t\to\infty} V(x, t, \phi) \geq \frac{\underline{\omega\Pi}}{\left(\overline{\mu} + \overline{\omega} + \frac{\overline{\beta}_1}{\underline{\alpha}}\right)\left(\overline{\mu} + \frac{\overline{\beta}_2}{\underline{\alpha}}\right)}$$

uniformly for $x \in \overline{\Omega}$.

Proof. (i) By [14, Lemma 1] and the third equation of system (1.9), it is easy to get that $I(x,t,\phi)$ satisfies

$$\begin{cases} \dfrac{\partial I(x,t)}{\partial t} \geq D_I(x)\Delta I - (\mu(x)+\gamma(x))I(x,t), & x\in\Omega, t>0,\\ \dfrac{\partial I(x,t)}{\partial n} = 0, & x\in\partial\Omega, t>0. \end{cases}$$

If $I(\cdot,t_0,\phi)\geq,\not\equiv 0$ for some $t_0>0$, the comparison principle implies that $I(x,t,\phi)>0$ for any $t>t_0$ and $x\in\overline{\Omega}$.

(ii) By the first equation of (1.9), it follows that

$$\begin{cases} \dfrac{\partial S(x,t)}{\partial t} \geq D_S(x)\Delta S + \underline{\Pi} - \left(\overline{\mu}+\overline{\omega}+\dfrac{\overline{\beta}_1}{\underline{\alpha}}\right)S(x,t), & x\in\Omega, t>0,\\ \dfrac{\partial S(x,t)}{\partial n} = 0, & x\in\partial\Omega, t>0. \end{cases}$$

Let $\upsilon_1(x,t,\phi)$ be the solution of

$$\begin{cases} \dfrac{\partial \upsilon_1(x,t)}{\partial t} = D_S(x)\Delta \upsilon_1 + \underline{\Pi} - \left(\overline{\mu}+\overline{\omega}+\dfrac{\overline{\beta}_1}{\underline{\alpha}}\right)\upsilon_1(x,t), & x\in\Omega, t>0,\\ \dfrac{\partial \upsilon_1(x,t)}{\partial n} = 0, & x\in\partial\Omega, t>0,\\ \upsilon_1(x,0)=\phi_1(x,0)=S(x,0), & x\in\Omega. \end{cases}$$

Then the comparison principle for scalar parabolic equations implies that $S(x,t,\phi)\geq \upsilon_1(x,t,\phi_1)>0$ for all $t>0$ and $x\in\overline{\Omega}$. Further, by [14, Lemma 1], we obtain that

$$\liminf_{t\to\infty} S(x,t,\phi) \geq \frac{\underline{\Pi}}{\overline{\mu}+\overline{\omega}+\frac{\overline{\beta}_1}{\underline{\alpha}}} := B_4$$

uniformly for $x\in\overline{\Omega}$. Similarly, let $\upsilon_2(x,t,\phi)$ be the solution of

$$\begin{cases} \dfrac{\partial \upsilon_2(x,t)}{\partial t} = D_V(x)\Delta \upsilon_2 + \underline{\omega}B_4 - \left(\overline{\mu}+\dfrac{\overline{\beta}_2}{\underline{\alpha}}\right)\upsilon_2(x,t), & x\in\Omega, t>0,\\ \dfrac{\partial \upsilon_2(x,t)}{\partial n} = 0, & x\in\partial\Omega, t>0,\\ \upsilon_2(x,0)=\phi_2(x,0)=V(x,0), & x\in\Omega. \end{cases}$$

We then have $V(x,t,\phi)\geq \upsilon_2(x,t,\phi_2)>0$, $\forall t>0$, $x\in\overline{\Omega}$, and

$$\liminf_{t\to\infty} V(x,t,\phi) \geq \frac{\underline{\omega}B_4}{\overline{\mu}+\frac{\overline{\beta}_2}{\underline{\alpha}}} = \frac{\underline{\omega}\underline{\Pi}}{\left(\overline{\mu}+\overline{\omega}+\frac{\overline{\beta}_1}{\underline{\alpha}}\right)\left(\overline{\mu}+\frac{\overline{\beta}_2}{\underline{\alpha}}\right)}$$

uniformly for $x\in\overline{\Omega}$. □

Lemma 3.4. *Assume $\mathfrak{R}_0>1$. Then there exists $\delta>0$ such that for any $\phi\in\mathbb{C}_+$ with $\phi_3(\cdot,0)\not\equiv 0$, the solution $u(\cdot,t,\phi)$ of system (1.9) satisfies*

$$\limsup_{t\to\infty} \|u(\cdot,t,\phi)-(S_0(\cdot),V_0(\cdot),0)\|_{\mathbb{X}} \geq \delta.$$

Proof. *For any given $\phi \in \mathbb{C}_+$ with $\phi_3(\cdot,0) \not\equiv 0$, by the parabolic maximum principle and Lemma 3.3, it follows that $I(x,t,\phi) > 0$ for all $t > 0$ and $x \in \overline{\Omega}$. When $\mathfrak{R}_0 > 1$, by Lemmas 3.1 and 3.2 as well as the continuous dependence of principal eigenvalue on parameters, we can conclude that there exists $\delta > 0$ such that $\overline{\lambda}(S_0 - \delta, V_0 - \delta, \tau) > 0$.*

Suppose, by contradiction, there exists some $\phi_0 = (\phi_{10}, \phi_{20}, \phi_{30}) \in \mathbb{C}_+$ with $\phi_{30}(\cdot,0) \not\equiv 0$ such that

$$\limsup_{t\to\infty} \|u(\cdot,t,\phi_0) - (S_0(x), V_0(x), 0)\|_{\mathbb{X}} < \delta.$$

This indicates that there exists $t_4 > 0$ large enough such that for any $t > t_4$ and $x \in \overline{\Omega}$, there holds

$$\begin{aligned} S_0(x) - \delta < S(x,t,\phi_0) < S_0(x) + \delta,\\ V_0(x) - \delta < V(x,t,\phi_0) < V_0(x) + \delta,\\ \text{and} \qquad I(x,t,\phi_0) < \delta, \end{aligned}$$

which, together with the third equation of (1.9), implies that $I(x,t,\phi_0)$ satisfies

$$\begin{aligned} \frac{\partial I(x,t)}{\partial t} \geq & \int_\Omega \Gamma(\tau,x,y) \frac{(\beta_1(y)(S_0(y)-\delta) + \beta_2(y)(V_0(y)-\delta))\, I(y,t-\delta)}{1+\alpha(y)\delta} dy \\ & + D_I(x)\Delta I(x,t) - (\mu(x)+\gamma(x))I(x,t), \quad \forall t > t_4, x \in \Omega. \end{aligned}$$

Let $\chi_1(x)$ be the positive eigenfunction corresponding to $\overline{\lambda}_0(S_0 - \delta, V_0 - \delta, \tau)$ associated with the problem (3.5). Then $\upsilon_3(x,t) = e^{\overline{\lambda}(S_0-\delta,V_0-\delta,\tau)t}\chi_1(x)$ solves

$$\begin{aligned} \frac{\partial \upsilon_3(x,t)}{\partial t} = & \int_\Omega \Gamma(\tau,x,y \frac{(\beta_1(y)(S_0(y)-\delta) + \beta_2(y)(V_0(y)-\delta))\, \upsilon_3(y,t-\delta)}{1+\alpha(y)\delta} dy \\ & + D_I(x)\Delta \upsilon_3(x,t) - (\mu(x)+\gamma(x))\upsilon_3(x,t), \quad \forall t > t_4, x \in \Omega. \end{aligned}$$

Since $I(x,t,\phi_0) > 0$ for all $t > 0$ and $x \in \overline{\Omega}$, there exists $\eta > 0$ such that

$$I(x,t,\phi_0) \geq \eta \upsilon_3(x,t), \quad \forall t \in [t_4 - \tau, t_4], x \in \overline{\Omega}.$$

By virtue of the comparison principle, we have

$$I(x,t,\phi_0) \geq \eta e^{\overline{\lambda}(S_0-\delta,V_0-\delta,\tau)t}\chi_1(x), \quad \forall t > t_4, x \in \overline{\Omega},$$

which, combining with $\overline{\lambda}(S_0 - \delta, V_0 - \delta, \tau) > 0$, yields that $I(x,t,\phi_0)$ is unbounded. This contradiction thus ends the proof. □

Now we are in a position to prove the main result which indicates that $\mathfrak{R}_0$ is a threshold index for the disease persistence.

Theorem 3.1. *Let $u(x,t,\phi)$ be the solution of (1.9) with $u_0 = \phi \in \mathbb{C}_+$. Then the following two statements hold.*

(i) *If $\mathfrak{R}_0 < 1$, the disease-free equilibrium E_0 is globally attractive in $\mathbb{C}_+$ for system (1.9).*

(ii) *If $\mathfrak{R}_0 > 1$, system (1.9) admits at least one positive steady state $u^*(x) = (S^*(x), V^*(x), I^*(x))$, and there exists $\delta > 0$ such that for any $\phi \in \mathbb{C}_+$ with $\phi_3(\cdot,0) \not\equiv 0$ such that*

$$\liminf_{t\to\infty} S(x,t) \geq \delta, \quad \liminf_{t\to\infty} V(x,t) \geq \delta \quad \text{and} \quad \liminf_{t\to\infty} I(x,t) \geq \delta$$

uniformly for all $x \in \overline{\Omega}$.

Proof. (i) If $\mathfrak{R}_0 < 1$, we have $\lambda(S_0, V_0) < 0$ by Lemma 3.2. Since

$$\lim_{\varepsilon \to 0} \lambda(S_0 + \varepsilon, V_0) = \lambda(S_0, V_0) < 0,$$

there exists a sufficiently small number $\varepsilon_0 > 0$ such that $\lambda(S_0 + \varepsilon_0, V_0) < 0$. Note that $S(x,t)$ satisfies

$$\begin{cases} \dfrac{\partial S(x,t)}{\partial t} \le D_S(x)\Delta S + \Pi(x) - (\mu(x) + \omega(x))S(x,t), & x \in \Omega, t > 0, \\ \dfrac{\partial S(x,t)}{\partial n} = 0, & x \in \partial\Omega, t > 0. \end{cases}$$

For fixed $\varepsilon_0 > 0$, by [14, Lemma 1] and the comparison principle, it follows that there exists $t_5 = t_5(\phi) > 0$ such that

$$S(x,t,\phi) \le S_0(x) + \varepsilon_0, \quad \forall t \ge t_5, x \in \overline{\Omega},$$

and hence,

$$\begin{cases} \dfrac{\partial I(x,t)}{\partial t} \le \displaystyle\int_\Omega \Gamma(\tau,x,y)(\beta_1(y)(S_0(y) + \varepsilon_0) + \beta_2(y)V_0(y))I(y,t-\tau)dy \\ \qquad\qquad + D_I(x)\Delta I(x,t) - (\mu(x) + \gamma(x))I(x,t), & x \in \Omega, t \ge t_5, \\ \dfrac{\partial I(x,t)}{\partial n} = 0, & x \in \partial\Omega, t \ge t_5. \end{cases}$$

In view of Lemma 3.1, let χ_2 be the strongly positive eigenfunction corresponding to $\overline{\lambda}(S_0 + \varepsilon_0, V_0, \tau) < 0$. Then $\upsilon_4(x,t) = e^{\overline{\lambda}(S_0+\varepsilon_0,V_0,\tau)t}\chi_2(x)$ solves the following linear system

$$\begin{cases} \dfrac{\partial \upsilon_4(x,t)}{\partial t} = \displaystyle\int_\Omega \Gamma(\tau,x,y)(\beta_1(y)(S_0(y) + \varepsilon_0) + \beta_2(y)V_0(y))\upsilon_4(y,t-\tau)dy \\ \qquad\qquad + D_I(x)\Delta \upsilon_4(x,t) - (\mu(x) + \gamma(x))\upsilon_4(x,t), & x \in \Omega, t \ge t_5, \\ \dfrac{\partial \upsilon_4(x,t)}{\partial n} = 0, & x \in \partial\Omega, t \ge t_5. \end{cases}$$

Since for any given $\phi \in \mathbb{C}_+$, there exists some $\kappa > 0$ such that

$$I(\cdot,t,\phi) \le \kappa \upsilon_4(\cdot,t), \quad t \in [t_5 - \tau, t_5].$$

Then the comparison principle implies that

$$I(x,t,\phi) \le \kappa e^{\overline{\lambda}(S_0+\varepsilon_0,V_0,\tau)t}\chi_2(x), \quad \forall t \ge t_5, x \in \overline{\Omega},$$

and hence, $\lim_{t\to\infty} I(x,t,\phi) = 0$ uniformly for $x \in \overline{\Omega}$. It then follows that the first equation of (1.9) is asymptotic to

$$\begin{cases} \dfrac{\partial w_1(x,t)}{\partial t} = D_S(x)\Delta w_1 + \Pi(x) - (\mu(x) + \omega(x))w_1(x,t), & x \in \Omega, t > 0, \\ \dfrac{\partial w_1(x,t)}{\partial n} = 0 & x \in \partial\Omega, t > 0. \end{cases}$$

By the theory for asymptotically autonomous semiflows (see e.g., [20, Corollary 4.3]), we have $\lim_{t\to\infty} S(x,t,\phi) = S_0(x)$ uniformly for $x \in \overline{\Omega}$. Further, the second equation of (1.9) is asymptotic to

$$\begin{cases} \dfrac{\partial w_2(x,t)}{\partial t} = D_V(x)\Delta w_2 + \omega(x)S_0(x) - \mu(x)w_2(x,t), \quad x \in \Omega, t > 0, \\ \dfrac{\partial w_2(x,t)}{\partial n} = 0 \quad x \in \partial\Omega, t \geq 0 \end{cases}$$

and then $\lim_{t\to\infty} V(x,t,\phi) = V_0(x)$ uniformly for $x \in \overline{\Omega}$. Therefore, the disease-free equilibrium E_0 is globally attractive.

(ii) If $\mathfrak{R}_0 > 1$, we have $\lambda(S_0, V_0) > 0$. It then follows from Lemma 3.2 that $\overline{\lambda}(S_0, V_0, \tau) > 0$. Define

$$\mathbb{X}_0 = \{\phi \in \mathbb{C}_+ : \phi_3(\cdot, 0) \not\equiv 0\} \quad \text{and} \quad \partial\mathbb{X}_0 = \mathbb{C}_+ \backslash \mathbb{X}_0 = \{\phi \in \mathbb{C}_+ : \phi_3(\cdot, 0) \equiv 0\}.$$

In view of Lemma 3.3, we obtain that for any $\phi \in \mathbb{X}_0$, $I(x,t,\phi) > 0$, $\forall x \in \overline{\Omega}$, $t > 0$, that is $\Phi(t)\mathbb{X}_0 \subseteq \mathbb{X}_0$.

Set

$$G_\partial := \{\phi \in \partial\mathbb{X}_0 : \Phi(t)\phi \in \partial\mathbb{X}_0, \forall t \geq 0\}$$

and let $\omega(\phi)$ be the omega limit set of the orbit $O^+(\phi) := \{\Phi(t)\phi : t \geq 0\}$. Then for any given $\phi \in G_\partial$, it has $\Phi(t)\phi \in \partial\mathbb{X}_0$, and hence, $I(\cdot, t, \phi) \equiv 0$ for all $t \geq 0$. By using the first and second equations of system (1.9) and the theory of asymptotically autonomous semiflow (see, e.g., [20, Corollary 4.3]), we obtain that for any $\phi \in G_\partial$, there hold that $\lim_{t\to+\infty} S(x,t,\phi) = S_0(x)$ and $\lim_{t\to+\infty} V(x,t,\phi) = V_0(x)$ uniformly for $x \in \Omega$. This indicates that

$$\omega(\phi) = \{S_0(x), V_0(x), 0\}, \quad \forall \phi \in G_\partial. \tag{3.7}$$

Define a continuous function $p : \mathbb{C}_+ \to \mathbb{R}_+$ by

$$p(\phi) := \min_{x\in\overline{\Omega}} \phi_3(x, 0), \quad \forall \phi \in \mathbb{C}_+.$$

From Lemma 3.3, it is easy to see that $p^{-1}(0,\infty) \subseteq \mathbb{X}_0$. Moreover, if $p(\phi) > 0$ or $p(\phi) = 0$ with $\phi \in \mathbb{X}_0$, we have $p(\Phi(t)(\phi)) > 0$, $\forall t > 0$. In other words, p is a generalized distance function for the semiflow $\Phi(t) : \mathbb{C}_+ \to \mathbb{C}_+$ (see e.g., [19]). Note that any forward orbit of $\Phi(t)$ in G_∂ converges to $(S_0(x), V_0(x), 0)$ by (3.7), and Lemma 3.4 implies that $(S_0(x), V_0(x), 0)$ is isolated in $\mathbb{C}_+$ and $W^s(S_0(x), V_0(x), 0) \cap \mathbb{X}_0 = \emptyset$, where $W^s(S_0(x), V_0(x), 0) = \{x \in \overline{\Omega} : \lim_{t\to\infty} d(\Phi(t), \{S_0(x), V_0(x), 0\})\}$ is the stable set of $(S_0(x), V_0(x), 0)$. Meanwhile, we easily observe that there is no cycle in G_∂ from $(S_0(x), V_0(x), 0)$ to $(S_0(x), V_0(x), 0)$. Then, by [19, Theorem 3], there exists $\delta' > 0$ such that

$$\min\{p(\phi) : \phi \in \omega(\phi)\} > \delta', \quad \forall \phi \in \mathbb{X}_0.$$

This implies that $\liminf_{t\to\infty} I(\cdot, t, \phi) \geq \delta'$, $\forall \phi \in \mathbb{X}_0$. From Lemma 3.3, we can choose $\delta \in (0, \delta']$ such that $\liminf_{t\to\infty} S(\cdot, t, \phi) \geq \delta$ and $\liminf_{t\to\infty} V(\cdot, t, \phi) \geq \delta$, $\forall \phi \in \mathbb{X}_0$. Thus, the uniform persistence stated in the conclusion (ii) holds.

By using [15, Theorem 3.7], we obtain that $\Phi(t) : \mathbb{X}_0 \to \mathbb{X}_0$ admits a global attractor $\mathcal{A}_0$. It then follows from [15, Theorem 4.7] that $\Phi(t)$ has an equilibrium $u^*(x) \in \mathbb{X}_0$. Clearly, Lemma 3.3 implies that $u^*(x)$ is a positive steady state of system (1.9). We then complete all the proof. □

4. Numerical simulations

In this section, we study the influences of heterogeneous spatial infections, diffusion coefficients and vaccination rate on the spread of the disease by the numerical method. For the spatially heterogeneous system (1.9), we cannot give the explicit formula for $\mathfrak{R}_0 = r(L)$, which is the spectral radius of the next infection operator. We can numerically compute $\mathfrak{R}_0$ by using the orthogonal projection method in computation of eigenvalues for compact linear operators [2, Section 3.1]. To use the n-th order Fourier projection (see e.g., [11]), we assume the spatial domain $\Omega = (0, 1)$ and the orthonormal basis is $e_k(x) = e^{2k\pi xi}$ with k being a nonnegative constant.

In order to estimate the spatial heterogeneity effect on $\mathfrak{R}_0$, we take $\beta_1(x)$ and $\beta_2(x)$, which describe the variation of transmission coefficients of susceptible and vaccinated individuals, respectively, for an example. Then by [8], the Green function associated with $D\Delta$, subjecting to a homogeneous Neumann boundary condition, reads that

$$\Gamma(D, t, x, y) = 1 + 2\sum_{n=1}^{\infty} e^{-n^2\pi^2 Dt}\cos(n\pi x)\cos(n\pi y).$$

For the operator L defined in (3.6), the Galerkin matrix $A_n = (a_{jk})_{n\times n}$ with

$$a_{jk} = \int_0^1 \overline{e_j(x)} \int_0^1 K(x, y)e_k(y)dydx,$$

where

$$K(x, y) = (\beta_1(x)S_0(x) + \beta_2(x)V_0(x)) \left(\frac{1}{\mu+\gamma} + 2\sum_{n=1}^{\infty} \frac{\cos(n\pi x)\cos(n\pi y)}{D_I n^2\pi^2 + \mu + \gamma} \right).$$

Set $\mu = 0.04$, $\alpha = 0.04$, $\tau = 4$, $\Pi = 1$, $\gamma = 0.3$, $\omega = 0.1$, $D_S = D_V = D_I = 1$ and

$$\beta_1(x) = 0.009(1 + c\cos(\pi x)), \qquad \beta_2(x) = 0.006(1 + c\cos(\pi x)),$$

where $0 \le c \le 1$ is the magnitude of infection heterogeneity. Note that $\int_0^1 \beta_1(x)\mathrm{d}x = 0.009$ and $\int_0^1 \beta_2(x)\mathrm{d}x = 0.006$. It follows that for any c, the spatial average of $\beta_1(x)$ and $\beta_2(x)$ on the interval $[0, 1]$ are same to the spatial homogeneous case (i.e. $c = 0$). Numerically, we find that $\mathfrak{R}_0$ is increasing with respect to c, which reveals that the spatially heterogeneous infection may improve $\mathfrak{R}_0$ (see Figure 1(a)). This indicates that the spatial heterogeneity may induce the persistence of disease compared to the homogeneous habitat.

Next, we plan to consider the effect of dispersal rate D_I of the infected population on $\mathfrak{R}_0$. Let

$$\beta_1(x) = 0.01(1 + 0.5\cos(\pi x)), \qquad \beta_2(x) = 0.005(1 + 0.5\cos(\pi x))$$

and other parameters are the same to those in Figure 1(a) besides D_I. Then our numerical result suggests that $\mathfrak{R}_0$ is a decreasing function of D_I (see Figure 1(b)), which means that the random movements of the infected population is conducive to inhibit the spread of disease.

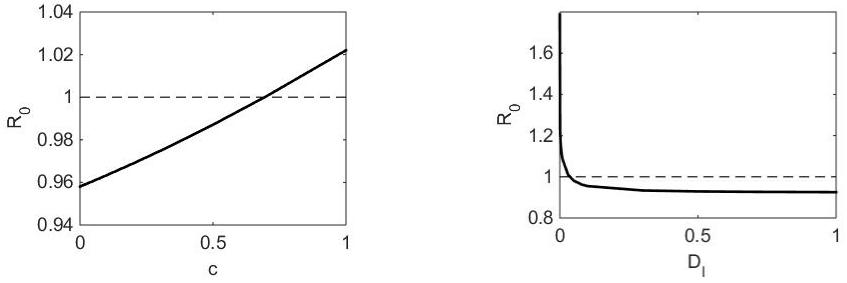

Figure 1. (a) $\mathfrak{R}_0$ is an increasing function of c with $\mu = 0.04$, $\alpha = 0.04$, $\tau = 4$, $\Pi = 1$, $\gamma = 0.3$, $\omega = 0.1$, $D_S = D_V = D_I = 1$, $\beta_1(x) = 0.009(1 + c\cos(\pi x))$, $\beta_2(x) = 0.006(1 + c\cos(\pi x))$. (b) $\mathfrak{R}_0$ is a decreasing function of D_I with $\beta_1(x) = 0.01(1 + 0.5\cos(\pi x))$, $\beta_2(x) = 0.005(1 + 0.5\cos(\pi x))$, and the other parameters are the same to Figure 1(a).

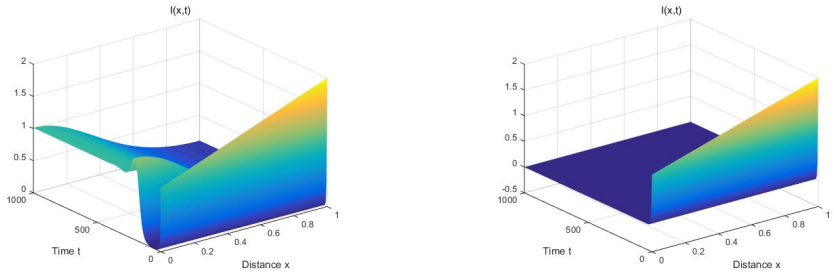

Figure 2. (a) The disease will persist if $\mathfrak{R}_0 = 1.0841 > 1$. (b) The disease will extinct if $\mathfrak{R}_0 = 0.9254 < 1$.

From Figure 1(b), we can see that $\mathfrak{R}_0 = 1.0841 > 1$ when $D_I = 0.01$, and $\mathfrak{R}_0 = 0.9254 < 1$ when $D_I = 1$. It follows from Theorem 3.1 that the disease is persistent when $D_I = 0.01$ (see Figure 2(a)), and extinct when $D_I = 1$ (see Figure 2(b)).

In addition, we observe that $\mathfrak{R}_0$ is a decreasing function of ω (see Figure 3(a)). Hence, in biology, under some suitable parameters, when the vaccination rate increases beyond a critical value, the disease will be extinct for the spatially heterogeneous model.

Finally, we investigate the influences of different distributions of spatially heterogeneous infections by taking

$$\beta_1(x) = 0.01(1 + 0.5\cos(5\pi(x + \eta))), \qquad \beta_2(x) = 0.006(1 + 0.5\cos(5\pi(x + \eta))),$$

where $0 \le \eta \le 1$ defines the phase shift of spatially heterogeneous infections, and keeping the other parameters are the same to those in Figure 3(a). Numerical computations show that $\mathfrak{R}_0$ is periodic in η (see Figure 3(b)). This means that a suitable shift of spatial phase can reduce $\mathfrak{R}_0$ to a value less than unity so that the disease dies out.

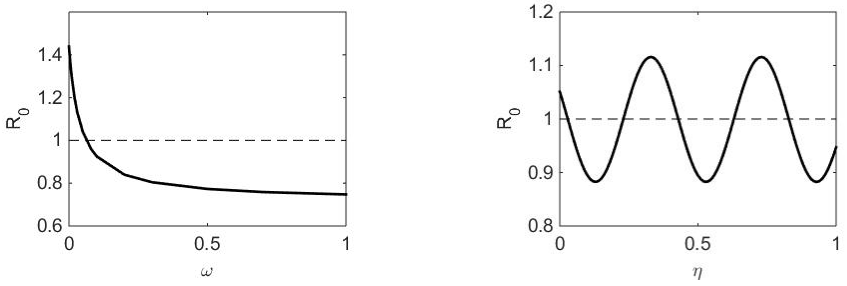

Figure 3. (a) $\mathfrak{R}_0$ is a decreasing function of ω with $\mu = 0.04$, $\alpha = 0.04$, $\tau = 4$, $\Pi = 1$, $\gamma = 0.3$, $\beta_1(x) = 0.01(1 + 0.5\cos(\pi x))$, $\beta_2(x) = 0.005(1 + 0.5\cos(\pi x))$. (b) Periodic fluctuation of $\mathfrak{R}_0$ when η varies with $\beta_1(x) = 0.01(1 + 0.5\cos(5\pi(x + \eta)))$, $\beta_2(x) = 0.006(1 + 0.5\cos(5\pi(x + \eta)))$ and the other parameters are the same to Figure 3(a).

5. Discussions and conclusions

In this paper, we propose and study a diffusive epidemic model with latent period and vaccination strategy in spatially heterogeneous habitat. We define the basic reproduction number $\mathfrak{R}_0$ by the spectral radius of the next infection operator. By the comparison arguments and persistence theory, we show that $\mathfrak{R}_0$ can be served as a threshold which predicts the extinction or persistence of the disease. Specifically, the disease will extinct if $\mathfrak{R}_0 < 1$ and persist in the population if $\mathfrak{R}_0 > 1$. Moreover, our numerical results indicate that: (i) the greater the degree of spatial heterogeneity, the greater the basic reproduction number and hence more infection risk; (ii) the dispersal of the infected population may reduce the values of $\mathfrak{R}_0$ in heterogeneous habitat, which means that a more random mobility could result in less infection risk. (iii) the vaccination rate reduce $\mathfrak{R}_0$, which shows that increasing the vaccination rate can effectively control the disease; (iv) a suitable shift of spatial phase of disease transmission coefficients can lower the basic reproduction number to a value less than unity, which, in return, leads to the extinction of the disease.

It is needful to point that system (1.9) admits at least one positive steady state if $\mathfrak{R}_0 > 1$, but its uniqueness and global asymptotic stability still remain open. Meanwhile, if $\mathfrak{R}_0 = 1$, system 3.1 may have more complex dynamical properties, which require further investigation in the future.

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