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Complete Hyper-elliptic Integrals of the First Kind and the Chebyshev Property*

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Abstract This paper is devoted to study the following complete hyper-elliptic integral of the first kind

$$J(h) = \oint_{\Gamma_h} \frac{\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3}{y} dx,$$

where $\alpha_i \in \mathbb{R}$, Γ_h is an oval contained in the level set $\{H(x,y) = h, h \in (-\frac{5}{36}, 0)\}$ and $H(x,y) = \frac{1}{2}y^2 - \frac{1}{4}x^4 + \frac{1}{9}x^9$. We show that the 3-dimensional real vector spaces of these integrals are Chebyshev for $\alpha_0 = 0$ and Chebyshev with accuracy one for $\alpha_i = 0$ $(i = 1, 2, 3)$.

Keywords Complete hyper-elliptic integral of the first kind, Chebyshev, ECT-system.

MSC(2010) 34C07, 34C05.

1. Introduction and main results

In 1990, Arnold [1] proposed ten problems among which the 7th problem is on the number of zeros of Abelian integrals, which can be stated in the following way: consider the Abelian integral

$$I(h) = \oint_{\Gamma_h} P(x,y)dy + Q(x,y)dx, \quad h \in \mathbb{J},$$

where Γ_h is a family of closed curves of a real polynomial $H(x,y) = h$, $P(x,y)$, $Q(x,y)$ and $H(x,y)$ are polynomials satisfying $\max\{\deg P, \deg Q\} = n$ and $\deg\{H\} = m + 1$, $\mathbb{J}$ is an open interval. How large can the number of isolated zeros of the function $I(h)$ in the open interval $\mathbb{J}$? And for the complete hyper-elliptic integral of the first kind

$$J(h) = \oint_{\Gamma_h} \frac{\alpha_0 + \alpha_1 x + \cdots + \alpha_{g-1} x^{g-1}}{y} dx, \; H(x,y) = y^2 + U(x),$$

where $\deg U = 2g + 1 > 4$, α_i $(i = 1, 2, \cdots g - 1)$ are real parameters. Is the g-dimensional family of $J(h)$ a Chebyshev family in the open interval? Where

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Chebyshev family means that the number of the isolated zeros of $J(h)$ is smaller than $g-1$.

It is known to all that the first part of the 7th problem is so-called the weakened 16th Hilbert problem compare to Hilbert in [13]. On the theme there have been many excellent works, see [2–6,10,11,14–18,20,21,23–28] and the references therein.

However, there are few works on the second part of the 7th problem, especially for $g>2$. Gavrilov and Iliev [8] obtained that the g-dimensional real vector space of $J(h)$ is not Chebyshev for any $g>1$, and when g = 2 and deg $U=5$ there exist exceptional families of ovals $\{\Gamma_h\}$ of $y^2+U(x)=h$ such that every Abelian integral of the form

$$J(h)=\oint_{\Gamma_h}\frac{\alpha_0+\alpha_1 x}{y}dx,\quad \alpha_0^2+\alpha_1^2\neq 0$$

has at most one isolated zero for h in an open interval $\mathbb{I}$. Wang, Wang and Xiao [22] studied the Chebyshev property of the above $J(h)$ for three classes of degenerate families of ovals Γ_h in [8]. It is shown that the three classes of complete hyper-elliptic integrals are Chebyshev, and the exact bounds on the number of zeros of these Abelian integrals are one.

In this paper, motivated by the above results, especially by [1,8,22], we investigate the following hyper-elliptic Hamilton system

$$\dot{x}=y,\quad \dot{y}=-x^3(x^5-1), \tag{1.1}$$

whose Hamiltonian is

$$H(x,y)=\frac{1}{2}y^2-\frac{1}{4}x^4+\frac{1}{9}x^9:=\frac{1}{2}y^2+U(x). \tag{1.2}$$

The oval $H(x,y)=-\frac{5}{36}$ corresponds to the center $C(1,0)$, the oval $H(x,y)=0$ corresponds to the homoclinic through the nilpotent saddle point $O(0,0)$, see Figure 1. It intersects the positive x-axis at point $(\frac{1}{2}\sqrt[5]{72},0)$. The corresponding complete hyper-elliptic integral of the first kind is

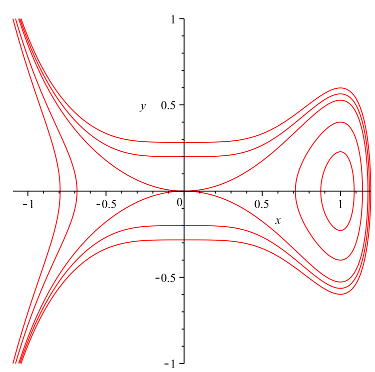

Figure 1. The level curves of $H(x,y)=h$.

$$\begin{aligned}\mathcal{J}(h)=&\oint_{\Gamma_h}\frac{\alpha_0+\alpha_1 x+\alpha_2 x^2+\alpha_3 x^3}{y}dx\\ :=&\alpha_0\mathcal{J}_0(h)+\alpha_1\mathcal{J}_1(h)+\alpha_2\mathcal{J}_2(h)+\alpha_3\mathcal{J}_3(h),\end{aligned}$$

where

$$\Gamma_h = \{(x,y)|H(x,y) = h, h \in (-\frac{5}{36}, 0)\}, \quad \mathcal{J}_i(h) = \oint_{\Gamma_h} \frac{x^i}{y} dx, i = 0, 1, 2, 3.$$

For the sake of convenience, we denote by V_k the 3-dimensional real vector spaces generated by vectors $\{\mathcal{J}_0(h), \mathcal{J}_1(h), \mathcal{J}_2(h), \mathcal{J}_3(h)\} \backslash \{\mathcal{J}_k(h)\}$, $k = 0, 1, 2, 3$. The main result is as follows.

Theorem 1.1. *(i) V_0 is Chebyshev on $(-\frac{5}{36}, 0)$, and the exact bound on the number of zeros of $\mathcal{J}(h) = \alpha_1 \mathcal{J}_1(h) + \alpha_2 \mathcal{J}_2(h) + \alpha_3 \mathcal{J}_3(h)$ is two on $(-\frac{5}{36}, 0)$ for all $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$ (counting the multiplicity).*

(ii) V_1 is Chebyshev with accuracy one on $(-\frac{5}{36}, 0)$, and there exists $(\alpha_0, \alpha_2, \alpha_3) \in \mathbb{R}^3$ such that $\mathcal{J}(h) = \alpha_0 \mathcal{J}_0(h) + \alpha_2 \mathcal{J}_2(h) + \alpha_3 \mathcal{J}_3(h)$ has two zeros on $(-\frac{5}{36}, 0)$ (counting the multiplicity).

(iii) V_2 is Chebyshev with accuracy one on $(-\frac{5}{36}, 0)$, and there exists $(\alpha_0, \alpha_1, \alpha_3) \in \mathbb{R}^3$ such that $\mathcal{J}(h) = \alpha_0 \mathcal{J}_0(h) + \alpha_1 \mathcal{J}_1(h) + \alpha_3 \mathcal{J}_3(h)$ has two zeros on $(-\frac{5}{36}, 0)$ (counting the multiplicity).

(iv) V_3 is Chebyshev with accuracy one on $(-\frac{5}{36}, 0)$, and there exists $(\alpha_0, \alpha_1, \alpha_2) \in \mathbb{R}^3$ such that $\mathcal{J}(h) = \alpha_0 \mathcal{J}_0(h) + \alpha_1 \mathcal{J}_1(h) + \alpha_2 \mathcal{J}_2(h)$ has two zeros on $(-\frac{5}{36}, 0)$ (counting the multiplicity).

2. Preliminaries

For the reader's convenience, we first introduce some helpful results in the literature. For more details, one can see [7, 9, 19].

Definition 2.1. The real vector space of functions V is said to be Chebyshev on an open interval $\mathbb{I} \subset \mathbb{R}$ provided that every function $f \in V \backslash \{0\}$ has at most $\dim V$-1 zeros on $\mathbb{I}$. V is said to be Chebyshev with accuracy m on $\mathbb{I}$ if any function $f \in V \backslash \{0\}$ has at most $\dim V + m - 1$ zeros on $\mathbb{I}$.

Definition 2.2. Let $f_0, f_1, \cdots, f_{n-1}$ be analytic functions on an open interval $\mathbb{I} \subset \mathbb{R}$.

(i) $\{f_0, f_1, \cdots, f_{n-1}\}$ is a Chebyshev system (in short, T-system) on $\mathbb{I}$ if any nontrivial real linear combination

$$\alpha_0 f_0(x) + \alpha_1 f_1(x) + \cdots + \alpha_{n-1} f_{n-1}$$

has at most $n - 1$ zeros on $\mathbb{I}$.

(ii) $\{f_0, f_1, \cdots, f_{n-1}\}$ is a complete Chebyshev system (in short, CT-system) on $\mathbb{I}$ if $\{f_0, f_1, \cdots, f_{k-1}\}$ is a Chbyshev system on $\mathbb{I}$ for each $k = 1, 2, \cdots, n$.

(iii) $\{f_0, f_1, \cdots, f_{n-1}\}$ is an extended complete Chebyshev system (in short, ECT-system) on $\mathbb{I}$ if for each $k = 1, 2, \cdots, n$, any nontrivial real linear combination

$$\alpha_0 f_0(x) + \alpha_1 f_1(x) + \cdots + \alpha_{k-1} f_{k-1}$$

has at most $k - 1$ isolated zeros on $\mathbb{I}$ (counted with multiplicities).

(iv) $\{f_0, f_1, \cdots, f_{n-1}\}$ is a Chebyshev system (in short, T-system) with accuracy m on $\mathbb{I}$ if for any nontrivial real linear combination

$$\alpha_0 f_0(x) + \alpha_1 f_1(x) + \cdots + \alpha_{n-1} f_{n-1}$$

has at most $n + m - 1$ isolated zeros on $\mathbb{I}$.

Remark 2.1. According to Definition 1 and Definition 2, $\{f_0, f_1, \cdots, f_{n-1}\}$ is an ECT-system on $\mathbb{I}$ if and only if the real vector space V generated by the vectors $\{f_0, f_1, \cdots, f_{n-1}\}$ is Chebyshev on $\mathbb{I}$, and $\{f_0, f_1, \cdots, f_{n-1}\}$ is a T-system with accuracy m on $\mathbb{I}$ if and only if the real vector space V generated by the vectors $\{f_0, f_1, \cdots, f_{n-1}\}$ is Chebyshev with accuracy m on $\mathbb{I}$.

Definition 2.3. Let $f_0, f_1, \cdots, f_{n-1}$ be analytic functions on an open interval $\mathbb{I} \subset \mathbb{R}$. The continuous Wronskian of $(f_0, f_1, \cdots, f_{k-1})$ at $x \in \mathbb{I}$ is

$$W[f_0, f_1, \cdots, f_{k-1}](x) = det(f_j^{(i)}(x))_{0 \leq i,j \leq k-1} = \begin{vmatrix} f_0(x) & f_1(x) & \cdots & f_{k-1}(x) \\ f_0'(x) & f_1'(x) & \cdots & f_{k-1}'(x) \\ \cdots & \cdots & \cdots & \cdots \\ f_0^{(k-1)}(x) & f_1^{(k-1)}(x) & \cdots & f_{k-1}^{(k-1)}(x) \end{vmatrix},$$

where $f'(x)$ is the first order derivative of $f(x)$ and $f^{(i)}(x)$ is the ith order derivative of $f(x)$.

The following relation between an ECT-system and their continuous Wronskian is well known.

Lemma 2.1. *$\{f_0, f_1, \cdots, f_{n-1}\}$ is an ECT-system on $\mathbb{I}$ if and only if for each $k = 1, 2, \cdots, n$, $W[f_k](x) \neq 0$ for all $x \in \mathbb{I}$.*

Let $H(x, y) = A(x) + \frac{y^2}{2}$ be an analytic function in some open subset of $\mathbb{R}^2$ that has a local minimum at the origin, and let $H(0, 0) = 0$. Then there exists a punctured neighborhood $\mathcal{P}$ of the origin foliated by ovals $\Gamma_h \subset \{(x, y)|H(x, y) = h,\ h_0 < h < 0 \text{ or } 0 < h < h_1\}$. The projection of $\mathcal{P}$ on the x-axis is an interval (x_l, x_r) with $x_l < 0 < x_r$. Suppose that $xA'(x) > 0$ for all $x \in (x_l, x_r)\backslash\{0\}$. Then there exists a unique analytic involution function $z(x)$ with $x_l < z(x) < 0$ such that $A(x) = A\big(z(x)\big)$ for $x \in (0, x_r)$.

We consider the Abelian integrals $I_i(h) = \int_{\Gamma_h} g_i(x) y^{2s-1} dx$ for $h \in (h_0, 0)$ or $h \in (0, h_1)$, where g_i $(i = 0, 1, \cdots, n-1)$ are analytic functions on the interval (x_l, x_r) and $s \in \mathbb{N}$.

Define a new analytic function in the interval $(0, x_r)$ as follows

$$l_i(x) = \frac{g_i(x)}{A'(x)} - \frac{g_i(z(x))}{A'(z(x))}. \tag{2.1}$$

Then from Lemma 2.1 we have the following algebraic criterion(see Theorem B in [9] and Theorem A in [19]).

Lemma 2.2. *(i) If $s > n - 2$ and $W[l_0, l_1, \cdots, l_i]$ is different from zero in the interval $(0, x_r)$ for each $i = 0, 1, \cdots, n-1$, then $\{I_0(h), I_1(h), \cdots, I_{n-1}(h)\}$ is an extended complete Chebyshev system on the interval $(h_0, 0)$ or $(0, h_1)$.*

(ii) If $s > n+m-2$ and assume that $W[l_0, l_1, \cdots, l_i]$ is different from zero in $(0, x_r)$ for each $i = 0, 1, \cdots, n-2$ and $W[l_0, l_1, \cdots, l_{n-1}]$ has m zeros on $(0, x_r)$ counted with multiplicities, then $\{I_0(h), I_1(h), \cdots, I_{n-1}(h)\}$ has at most $n+k-1$ isolated zeros on $(0, x_l)$ counted with multiplicities. We call $\{I_0(h), I_1(h), \cdots, I_{n-1}(h)\}$ is a Chebyshev system with accuracy m on the interval $(h_0, 0)$ or $(0, h_1)$.

The following lemma in [9] gave a formula which promotes the power of y in the integrand of Abelian integral $I_i(h)$ into a higher order that we want.

Lemma 2.3. *Let $F(x)$ such that $\frac{F(x)}{A'(x)}$ is analytic at $x = 0$. Then for any $k \in \mathbb{N}$,*

$$\oint_{\Gamma_h} F(x) y^{2s-1} dx = \oint_{\Gamma_h} G(x) y^{2s+1} dx,$$

where $G(x) = \frac{1}{2s+1}\left(\frac{F}{A'}\right)'(x)$.

3. Proof of Theorem 1.1

Since the origin is not the local minimum of $H(x, y)$, we shift the center $C(1, 0)$ of system (1.1) to the origin by the transformation $x = 1-u, y = -v$, and still denote the variable pair by (x, y) after the transformation for the sake of convenience. Then system (1.1) can be written

$$\dot{x} = y, \quad \dot{y} = (x-1)^3 + (x-1)^8, \tag{3.1}$$

which has the Hamiltonian

$$\begin{aligned}\mathcal{H}(x, y) =& \frac{1}{2}y^2 - \frac{1}{9}x^9 + x^8 - 4x^7 + \frac{28}{3}x^6 - 14x^5 + \frac{55}{4}x^4 - \frac{25}{3}x^3 + \frac{5}{2}x^2 \\ :=& \frac{1}{2}y^2 + A(x)\end{aligned}$$

with a local minimum at origin and the continuous family of ovals γ_l surrounding the center $(0, 0)$, where

$$\begin{aligned}A(x) &= -\frac{1}{9}x^9 + x^8 - 4x^7 + \frac{28}{3}x^6 - 14x^5 + \frac{55}{4}x^4 - \frac{25}{3}x^3 + \frac{5}{2}x^2, \\ \gamma_l &= \Big\{(x, y) | \mathcal{H}(x, y) = l, 0 < l < \frac{5}{36}\Big\}.\end{aligned}$$

The homoclinic γ_l defined by $l = \frac{5}{36}$ intersects the x-axis at the points $(1 - \frac{1}{2}\sqrt[5]{72}, 1)$. It is easy to check that $xA'(x) > 0$ for $x \in (1 - \frac{1}{2}\sqrt[5]{72}, 1)\setminus\{0\}$. Thus for $x \in (0, 1)$, we can define an involution $z(x)$ with $1 - \frac{1}{2}\sqrt[5]{72} < z(x) < 0$ such that $A(x) = A(z(x))$, where $z(x)$ is implicity defined by $q(x, z) = 0$, here

$$\begin{aligned}q(x, z) =& -90x - 336x^5 - 36x^7 + 4x^8 + 300x^2 - 495x^3 + 504x^4 + 144x^6 + 300z^2 \\ & - 495z^3 + 504z^4 - 336z^5 + 144z^6 - 36z^7 + 4z^8 - 90z + 4z^2x^6 - 36z^2x^5 \\ & + 144z^2x^4 - 336z^2x^3 - 495zx^2 + 300zx + 4x^5z^3 + 504z^2x^2 - 495z^2x \\ & + 4zx^7 - 36zx^6 + 144zx^5 - 336zx^4 + 504zx^3 - 36x^4z^3 + 4x^4z^4 \\ & + 144x^3z^3 - 36x^3z^4 + 4x^3z^5 + 144x^2z^4 - 336x^2z^3 - 36x^2z^5 + 4x^2z^6 \\ & + 504xz^3 - 336xz^4 + 144xz^5 - 36xz^6 + 4xz^7.\end{aligned}$$

And we have

$$\frac{dz}{dx} = -\frac{\Delta_1(x,z)}{\Delta_2(x,z)} =: \Theta, \tag{3.2}$$

where

$$\begin{aligned}
\Delta_1(x,z) =& -90 + 600x + 864x^5 + 32x^7 - 1485x^2 + 2016x^3 - 1680x^4 - 252x^6 \\
& - 495z^2 + 504z^3 - 336z^4 + 144z^5 - 36z^6 + 4z^7 + 300z + 24z^2x^5 - 180z^2x^4 \\
& + 576z^2x^3 + 1512zx^2 - 990zx - 1008z^2x^2 + 1008z^2x + 28zx^6 - 216zx^5 \\
& + 720zx^4 - 1344zx^3 + 12x^2z^5 + 20x^4z^3 - 144x^3z^3 + 16x^3z^4 - 108x^2z^4 \\
& + 432x^2z^3 - 672xz^3 + 288xz^4 - 72xz^5 + 8xz^6, \\
\Delta_2(x,z) =& -90 + 300x + 144x^5 + 4x^7 - 495x^2 + 504x^3 - 336x^4 - 36x^6 - 1485z^2 \\
& + 2016z^3 - 1680z^4 + 864z^5 - 252z^6 + 32z^7 + 600z + 12z^2x^5 - 108z^2x^4 \\
& + 432z^2x^3 + 1008zx^2 - 990zx - 1008z^2x^2 + 1512z^2x + 8zx^6 - 72zx^5 \\
& + 288zx^4 - 672zx^3 + 24x^2z^5 + 16x^4z^3 - 144x^3z^3 + 20x^3z^4 - 180x^2z^4 \\
& + 576x^2z^3 - 1344xz^3 + 720xz^4 - 216xz^5 + 28xz^6.
\end{aligned}$$

Lemma 3.1. *$\{\mathcal{J}_1(h), \mathcal{J}_2(h), \mathcal{J}_3(h)\}$ is an ECT-system on $(-\frac{5}{36}, 0)$.*

Proof. We only need to verify that $\{J_1(l), J_2(l), J_3(l)\}$ is an ECT-system on $(0, \frac{5}{36})$, where $J_i(l) = \oint_{\gamma_l} \frac{x^i}{y} dx, i = 0,1,2,3$. We can not apply Lemma 2.2 directly for $\{J_1(l), J_2(l), J_3(l)\}$, since $n = 3$ and $s = 0$. Note that $\frac{1}{2}y^2 + A(x) = l$ along the oval γ_l with $0 < l < \frac{5}{36}$, then for $i = 0,1,2,3$, by Lemma 2.3, we have

$$\begin{aligned}
J_i(l) &= \frac{1}{l} \oint_{\gamma_l} \frac{x^i\big(\frac{1}{2}y^2 + A(x)\big)}{y} dx \\
&= \frac{1}{l}\Big[\oint_{\gamma_l} \frac{1}{2}x^i y dx + \oint_{\gamma_l} \frac{x^i A(x)}{y} dx\Big] \\
&= \frac{1}{l}\oint_{\gamma_l} \Big[\frac{1}{2}x^i + \Big(\frac{x^i A(x)}{A'(x)}\Big)'\Big] y dx \\
&= \frac{1}{l}\oint_{\gamma_l} f_i(x) y dx.
\end{aligned}$$

where

$$f_i(x) = \frac{1}{2}x^i + \Big(\frac{x^i A(x)}{A'(x)}\Big)'.$$

But by Lemma 2.2 $s > n-2$ is still not satisfied since $n = 3$ and $s = 1$, we need to promote the power of y.

$$\begin{aligned}
J_i(l) &= \frac{1}{l^2} \oint_{\gamma_l} f_i(x)\Big(\frac{1}{2}y^2 + A(x)\Big) y dx \\
&= \frac{1}{l^2}\Big[\oint_{\gamma_l} \frac{1}{2} f_i(x) y^3 dx + \oint_{\gamma_l} f_i(x)A(x) y dx\Big] \\
&= \frac{1}{l^2}\oint_{\gamma_l} \Big[\frac{1}{2}f_i(x) + \frac{1}{3}\Big(\frac{f_i(x)A(x)}{A'(x)}\Big)'\Big] y^3 dx
\end{aligned}$$

$$= \frac{1}{l^2}\oint_{\gamma_l} g_i(x)y^3dx,$$

where

$$g_i(x) = \frac{1}{2}f_i(x) + \frac{1}{3}\Big(\frac{f_i(x)A(x)}{A'(x)}\Big)'. \tag{3.3}$$

Define

$$\bar{J}_i(l) = \oint_{\gamma_l} g_i(x)y^3dx,\ \ i = 0, 1, 2, 3, 4.$$

It is only to prove $\{\bar{I}_1(l), \bar{I}_2(l), \bar{I}_3(l)\}$ is an ECT-system on $(0, \frac{5}{36})$. For $i = 1, 2, 3$, set

$$\varphi_i(x) = \frac{g_i(x)}{A'(x)} - \frac{g_i(z(x))}{A'(z(x))}.$$

Then we get $\varphi_i(x, z) = \eta_1(x, z)m_i(x, z), i = 1, 2, 3$, where

$$\eta_1(x,z) = \frac{x-z}{11664(x-1)^{11}(z-1)^{11}(z^4-5z^3+10z^2-10z+5)^5(x^4-5x^3+10x^2-10x+5)^5},$$

and $m_i(x, z)$ are polynomials of (x, z). By Lemma 2.2, we only need to assert that for all $1 - \frac{1}{2}\sqrt[5]{72} < z < 0 < x < 1$

(i) $W[m_1](x, z) \neq 0$; (ii) $W[m_1, m_2](x, z) \neq 0$; (iii) $W[m_1, m_2, m_3](x, z) \neq 0$.

In fact, for $i = 1, 2, 3$, let

$$a_i(x, z) = \frac{\partial m_i}{\partial x} + \frac{\partial m_i}{\partial z}\Theta, \quad b_i(x, z) = \frac{\partial a_i}{\partial x} + \frac{\partial a_i}{\partial z}\Theta,$$

where Θ is defined by (3.2). Then

$$W[m_1](x, z) = m_1(x, z),$$

$$W[m_1, m_2](x, z) = \begin{vmatrix} m_1 & m_2 \\ a_1 & a_2 \end{vmatrix} = \frac{\xi_0(x, z)}{\xi(x, z)}\sigma_1(x, z),$$

$$W[m_1, m_2, m_3](x, z) = \begin{vmatrix} m_1 & m_2 & m_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \frac{\xi_0^3(x, z)}{\xi^3(x, z)}\sigma_2(x, z),$$

where $z = z(x)$ satisfies $q(x, z) = 0$, $\sigma_i(x, z)$ are polynomials in (x, z) for $i = 1, 2$, and

$$\xi_0(x, z) = (x - z)(z^4 - 5z^3 + 10z^2 - 10z + 5)(x^4 - 5x^3 + 10x^2 - 10x + 5),$$

$$\begin{aligned}\xi(x, z) =& 4x^7 + 8x^6z + 12x^5z^2 + 16x^4z^3 + 20x^3z^4 + 24x^2z^5 + 28xz^6 + 32z^7 - 36x^6 \\ & - 72x^5z - 108x^4z^2 - 144x^3z^3 - 180x^2z^4 - 216xz^5 - 252z^6 + 144x^5 \\ & + 288x^4z + 432x^3z^2 + 576x^2z^3 + 720xz^4 + 864z^5 - 336x^4 - 672x^3z \\ & - 1008x^2z^2 - 1344xz^3 - 1680z^4 + 504x^3 + 1008x^2z + 1512xz^2 \\ & + 2016z^3 - 495x^2 - 990xz - 1485z^2 + 300x + 600z - 90.\end{aligned} \tag{3.4}$$

Now we rely on the symbolic computation by Maple to compute the resultant between two polynomials and apply Sturm's theorem to assert nonexistence of zeros of two polynomials.

The resultant with respect to x between $q(x,z)$ and $m_1(x,z)$ is $(z^4-5z^3+10z^2-10z+5)^4(x-1)^{30}\zeta_0(z)$ with $\zeta_0(z)$ a polynomial of degree 386 in z. Applying Sturm's theorem, we can assert that $\zeta_0(z)\neq 0$ for all $z\in(1-\frac{1}{2}\sqrt[5]{72},0)$. Hence $W[m_1](x,z)\neq 0$ for all $1-\frac{1}{2}\sqrt[5]{72}<z<0<x<1$.

The resultant with respect to x between $q(x,z)$ and $\xi(x,z)$ is

$$R(z) = 5135673858195456z^7(z^4-5z^3+10z^2-10z+5)^7(z-1)^{21}.$$

It is easy to check that $R(z)\neq 0$ for all $z\in(1-\frac{1}{2}\sqrt[5]{72},0)$. Thus, $W[m_1,m_2](x,z)$ and $W[m_1,m_2,m_3](x,z)$ are well defined for $1-\frac{1}{2}\sqrt[5]{72}<z<0<x<1$.

The resultant with respect to x between $q(x,z)$ and $\sigma_1(x,z)$ is $557256278016(z^4-5z^3+10z^2-10z+5)^6(z-1)^{64}\zeta_1(z)$ with $\zeta_1(z)$ a polynomial of degree 756 in z. Applying Sturm's theorem, we obtain that $\zeta_1(z)=0$ has a root z^* on $(1-\frac{1}{2}\sqrt[5]{72},0)$, where

$$\begin{aligned} z^* \approx {} & -0.05662614414380414516262935389743654689291513136 29\\ & 7305422922277039097954332795357548772879565736549666. \end{aligned}$$

Substituting $z=z^*$ into $q(x,z)=0$, we get $q(x^*,z^*)=0$, where

$$\begin{aligned} x^* \approx {} & 0.0699178876735929098955533584133572019958457610497 4\\ & 17931437894466464777205353477541488636679644539997 8. \end{aligned}$$

But substituting $x=x^*, z=z^*$ into $W[m_1,m_2](x,z)$, we obtain

$$\begin{aligned} W[m_1,m_2](x^*,z^*) = {} & 4.97793151164503124299575718329130477635845826224012 2\\ & 962780914455864450384677265249171924291484023951\times 10^{21}. \end{aligned}$$

Hence $W[m_1,m_2](x,z)\neq 0$ for all $1-\frac{1}{2}\sqrt[5]{72}<z<0<x<1$.

The resultant with respect to x between $q(x,z)$ and $\sigma_2(x,z)$ is

$$197111727471974165070724487232199 0656(z^4-5z^3+10z^2-10z+5)^6(z-1)^{104}\zeta_2(z)$$

with $\zeta_2(z)$ a polynomial of degree 1092 in z. Applying Sturm's theorem, we can assert that $\zeta_2(z)\neq 0$ for all $z\in(1-\frac{1}{2}\sqrt[5]{72},0)$. Hence $W[m_1,m_2,m_3](x,z)\neq 0$ for all $1-\frac{1}{2}\sqrt[5]{72}<z<0<x<1$. □

Lemma 3.2. *For $h\in(-\frac{5}{36},0)$, each of the following function sequences is a T-system with accuracy one:*

(i) $\{\mathcal{J}_1(h),\mathcal{J}_2(h),\mathcal{J}_0(h)\}$; (ii) $\{\mathcal{J}_0(h),\mathcal{J}_3(h),\mathcal{J}_1(h)\}$; (iii) $\{\mathcal{J}_0(h),\mathcal{J}_3(h),\mathcal{J}_2(h)\}$.

Proof. Without loss of generality, we only prove (i). The others can be shown in a similar way. Note that $\frac{1}{2}y^2+A(x)=l$ along the oval γ_l with $0<l<\frac{5}{36}$, then for $i=0,1,2,3,4$, by Lemma 2.3, we have

$$J_i(l) = \frac{1}{l^3}\oint_{\gamma_l} g_i(x)(\frac{1}{2}y^2+A(x))y^3dx$$

$$
\begin{aligned}
&= \frac{1}{l^3} \oint_{\gamma_l} \Big[\frac{1}{2} g_i(x) + \frac{1}{5}\Big(\frac{g_i(x)A(x)}{A'(x)}\Big)'\Big] y^5 dx \\
&= \frac{1}{l^2} \oint_{\gamma_l} h_i(x) y^5 dx,
\end{aligned}
$$

where $g_i(x)$ is defined by (3.3) and $h_i(x) = \frac{1}{2} g_i(x) + \frac{1}{5}\Big(\frac{g_i(x)A(x)}{A'(x)}\Big)'$. Define

$$
\tilde{J}_i(l) = \oint_{\gamma_l} h_i(x) y^5 dx, i = 0, 1, 2, 3, 4.
$$

We need to prove that $\{\tilde{J}_1(l), \tilde{J}_2(l), \tilde{J}_0(l)\}$ is a T-system with accuracy one on $(0, \frac{5}{36})$. For $i = 0, 1, 2, 3, 4$, let

$$
\psi_i(x) = \frac{h_i(x)}{A'(x)} - \frac{h_i(z(x))}{A'(z(x))}.
$$

Then we get $\psi_i(x, z) = \eta_2(x, z) n_i(x, z),\ i = 0, 1, 2, 3, 4$, where

$$
\eta_2(x, z) = \frac{x - z}{2099520(x-1)^{15}(z-1)^{15}(z^4 - 5z^3 + 10z^2 - 10z + 5)^7 (x^4 - 5x^3 + 10x^2 - 10x + 5)^7},
$$

and $n_i(x, z)$ are rational functions of (x, z). For $i = 1, 2, 3, 4$, let

$$
c_i(x, z) = \frac{\partial n_i}{\partial x} + \frac{\partial n_i}{\partial z}\Theta, \quad d_i(x, z) = \frac{\partial c_i}{\partial x} + \frac{\partial c_i}{\partial z}\Theta,
$$

where Θ is defined by (3.2). Then

$$
W[n_1, n_2, n_0](x, z) = \begin{vmatrix} n_1 & n_2 & n_0 \\ c_1 & c_2 & c_0 \\ d_1 & d_2 & d_0 \end{vmatrix} = \frac{84(x - z)^3}{\xi(x, z)^3}\tau(x, z),
$$

where $z = z(x)$ satisfies $q(x, z) = 0$, $\xi(x, z)$ is defined as (3.4), and $\tau(x, z)$ is a polynomial in (x, z). By Lemma 3.1, we obtain that $W[n_1](x, z) \neq 0$, $W[n_1, n_2](x, z) \neq 0$, and $W[n_1, n_2](x, z)$ and $W[n_1, n_2, n_0](x, z)$ are well defined.

Now we assert that $W[n_1, n_2, n_0](x, z)$ has only one zero for $1 - \frac{1}{2}\sqrt[5]{72} < z < 0 < x < 1$. In fact, the resultant with respect to x between $q(x, z)$ and $\tau(x, z)$ is

$$
660123187103394274955004739584(z^4 - 5z^3 + 10z^2 - z + 5)^{18}(z-1)^{140} p(z)
$$

with $p(z)$ a polynomial of degree 1528 in z. Applying Sturm's theorem, we can get that $p(z) = 0$ has only one root z_1^* on $(1 - \frac{1}{2}\sqrt[5]{72}, 0)$, where

$$
\begin{aligned}
z_1^* \approx\ & -0.172496110743238959188185788130729469073712042401 \\
& 831776671852810730938333990191880025962871453310000 0.
\end{aligned}
$$

Substituting $z = z_1^*$ into $q(x, z) = 0$, we get $q(x_1^*, z_1^*) = 0$, where

$$
\begin{aligned}
x_1^* \approx\ & 0.588172423311321752449006910150667645476389013926 88 \\
& 22395545058507080769650831951383791271107371609875.
\end{aligned}
$$

Substituting $x = x_1^*, z = z_1^*$ into $W[n_1, n_2, n_0](x, z)$ gives $W[n_1, n_2, n_0](x_1^*, z_1^*) = 0$. $\square$

Using the similar arguments as in the proof of Lemma 3.1, we get the following lemma.

Lemma 3.3. *$\{\mathcal{J}_0(h), \mathcal{J}_1(h)\}$, $\{\mathcal{J}_0(h), \mathcal{J}_2(h)\}$ and $\{\mathcal{J}_0(h), \mathcal{J}_3(h)\}$ are ECT-systems for $h \in (-\frac{5}{36}, 0)$.*

Lemma 3.4. *For $0 < -h \ll 1$, $\mathcal{J}_k(h)$, $k = 0, 1, 2, 3$, have the following asymptotic expansions*

$$\begin{aligned}
\mathcal{J}_0(h) =& \sqrt{2} r_1 (-h)^{-\frac{1}{4}} + O\big((-h)^{\frac{5}{4}}\big),\\
\mathcal{J}_1(h) =& \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \ln(-h) + O\big((-h)^{\frac{5}{4}}\big),\\
\mathcal{J}_2(h) =& \frac{2\pi \sqrt[5]{9\sqrt{2}}\sqrt{\pi}}{5\Gamma(\frac{4}{5})\Gamma(\frac{7}{10})\sin(\frac{1}{5}\pi)} + \sqrt{2} r_2 (-h)^{\frac{1}{4}} + O\big((-h)^{\frac{5}{4}}\big),\\
\mathcal{J}_3(h) =& \frac{\pi \sqrt[5]{648\sqrt{2}}\sqrt{\pi}}{5\Gamma(\frac{3}{5})\Gamma(\frac{9}{10})\sin(\frac{2}{5}\pi)} - \frac{6\sqrt{2}}{3} r_1 (-h)^{\frac{3}{4}} + O\big((-h)^{\frac{5}{4}}\big),
\end{aligned}$$

where $r_1 > 0$ and $r_2 < 0$ are constants and $\Gamma(\cdot)$ is the Gamma function.

Proof. Noting that

$$\mathcal{J}_i(h) = \frac{dI_i(h)}{dh}, \tag{3.5}$$

where

$$I_i(h) = \oint_{\Gamma_h} x^k y dx, \ \ i = 0, 1, 2, 3.$$

We first calculate asymptotic expansions of $I_i(h)$ $(i = 0, 1, 2, 3)$ for $0 < -h \ll 1$. From the reference [12], we have

$$\begin{aligned}
I(h) =& \alpha_0 I_0(h) + \alpha_1 I_1(h) + \alpha_2 I_2(h) + \alpha_3 I_3(h)\\
=& c_1 + c_2(-h)^{\frac{3}{4}} + c_3 h \ln(-h) + c_4 h + c_5(-h)^{\frac{5}{4}} + c_6(-h)^{\frac{7}{4}} + O\big((-h)^{\frac{9}{4}}\big),
\end{aligned}$$

where

$$\begin{aligned}
c_1 =& \frac{15\sqrt[5]{6\sqrt{2}}\Gamma(\frac{3}{5})\Gamma(\frac{9}{10})\sin(\frac{1}{10}\pi)}{11\sqrt{\pi}}\alpha_0 + \frac{5\sqrt[5]{432\sqrt{2}}\Gamma(\frac{4}{5})\Gamma(\frac{7}{10})\sin(\frac{3}{10}\pi)}{26\sqrt{\pi}}\alpha_1 +\\
& \frac{3\sqrt{2}}{10}\alpha_2 + \frac{9\pi\sqrt[5]{9\sqrt{2}}\sqrt{\pi}}{238\Gamma(\frac{4}{5})\Gamma(\frac{7}{10})\sin(\frac{1}{5}\pi)}\alpha_3,\\
c_2 =& -\frac{4\sqrt{2} r_1}{3}\alpha_0, \ \ c_3 = \frac{\sqrt{2}}{2}\alpha_1, \ \ c_5 = -\frac{4\sqrt{2} r_2}{5}\alpha_2, \ \ c_6 = \frac{24\sqrt{2}}{7} r_1 \alpha_3,
\end{aligned}$$

and c_4 can be computed to be

$$\frac{2\pi\sqrt[5]{9\sqrt{2}}\sqrt{\pi}}{5\Gamma(\frac{4}{5})\Gamma(\frac{7}{10})\sin(\frac{1}{5}\pi)}\alpha_2 + \frac{\pi\sqrt[5]{648\sqrt{2}}\sqrt{\pi}}{5\Gamma(\frac{3}{5})\Gamma(\frac{9}{10})\sin(\frac{2}{5}\pi)}\alpha_3,$$

if $c_2 = 0$ and $c_3 = 0$, where $r_1 > 0$ and $r_2 < 0$ are constants. The conclusion follows from (3.5). □

In order to get the asymptotic expansions of $\mathcal{J}_i(h)$ $(i = 0, 1, 2, 3)$ for $0 < h + \frac{5}{36} \ll 1$, we consider the following system

$$\dot{x} = y, \quad \dot{y} = x^3 - x^8 + \epsilon(\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3) y. \tag{3.6}$$

Lemma 3.5. *For* $0 < h+\frac{5}{36} \ll 1$, $\mathcal{J}_i(h)$ $(i = 0, 1, 2, 3)$ *have the following asymptotic expansions*

$$\begin{aligned}
\mathcal{J}_0(h) =&\frac{2\sqrt{5}}{25}\pi + \frac{151\sqrt{5}}{750}\pi(h+\frac{5}{36}) + \frac{170401\sqrt{5}}{180000}\pi(h+\frac{5}{36})^2 + +O\Big((h+\frac{5}{36})^3\Big),\\
\mathcal{J}_1(h) =&\frac{2\sqrt{5}}{25}\pi + \frac{91\sqrt{5}}{750}\pi(h+\frac{5}{36}) + \frac{17381\sqrt{5}}{36000}\pi(h+\frac{5}{36})^2 + +O\Big((h+\frac{5}{36})^3\Big),\\
\mathcal{J}_2(h) =&\frac{2\sqrt{5}}{25}\pi + \frac{43\sqrt{5}}{750}\pi(h+\frac{5}{36}) + \frac{33529\sqrt{5}}{180000}\pi(h+\frac{5}{36})^2 + +O\Big((h+\frac{5}{36})^3\Big),\\
\mathcal{J}_3(h) =&\frac{2\sqrt{5}}{25}\pi + \frac{7\sqrt{5}}{750}\pi(h+\frac{5}{36}) + \frac{3073\sqrt{5}}{180000}\pi(h+\frac{5}{36})^2 + +O\Big((h+\frac{5}{36})^3\Big).
\end{aligned}$$

Proof. We move the center $C(1,0)$ into the origin by letting $x = u+1$, $y = \sqrt{5}v$ and $t = \frac{1}{\sqrt{5}}\tau$, then system (3.6) becomes

$$\begin{aligned}
\frac{du}{d\tau} =&v,\\
\frac{dv}{d\tau} =&-u - \frac{1}{5}u^8 - \frac{8}{5}u^7 - \frac{28}{5}u^6 - \frac{56}{5}u^5 - 14u^4 - 11u^3 - 5u^2\\
&+\epsilon\frac{1}{\sqrt{5}}\Big(\alpha_0 + \alpha_1(u+1) + \alpha_2(u+1)^2 + \alpha_3(u+1)^3\Big)v.
\end{aligned} \tag{3.7}$$

For $\epsilon = 0$, the corresponding Hamiltonian of (3.7) is

$$\Gamma_{\bar{h}}:\ \bar{H}(u,v) = \frac{1}{2}(u^2+v^2) + \frac{1}{45}u^9 + \frac{1}{5}u^8 + \frac{4}{5}u^7 + \frac{28}{5}u^6 + \frac{14}{5}u^5 + \frac{11}{4}u^4 + \frac{5}{3}u^3 = \bar{h},$$

where $\bar{h} = \frac{1}{5}(h+\frac{5}{36})$, $0 < \bar{h} < \frac{1}{36}$. Let $u = r\cos\theta$, $v = r\sin\theta$. Then $\Gamma_{\bar{h}}$ can be transformed into for $0 < \bar{h} \ll 1$

$$\begin{aligned}
&r\sqrt{1+\frac{10}{3}r\cos^3\theta + \frac{11}{2}r^2\cos^4 + \frac{28}{5}r^3\cos^5\theta + \frac{56}{15}r^4\cos^6\theta + \frac{8}{5}r^5\cos^7\theta + \frac{2}{5}r^6\cos^8\theta + \frac{2}{45}r^7\cos^9\theta}\\
&-\sqrt{2\bar{h}} = 0,
\end{aligned}$$

because $0 < \bar{h} \ll 1$ and $0 < r \ll 1$.

Let $\rho = \sqrt{2\bar{h}}$ and

$$\begin{aligned}
&\mathbb{F}(r,\rho) = -\rho +\\
&r\sqrt{1+\frac{10}{3}r\cos^3\theta + \frac{11}{2}r^2\cos^4\theta + \frac{28}{5}r^3\cos^5\theta + \frac{56}{15}r^4\cos^6\theta + \frac{8}{5}r^5\cos^7\theta + \frac{2}{5}r^6\cos^8\theta + \frac{2}{45}r^7\cos^9\theta}.
\end{aligned}$$

Applying the Implicit Theorem to $\mathbb{F}(r,\rho)$ at $(r,\rho) = (0,0)$, we obtain that there exist a smooth function $r = \phi(\rho)$ and a small positive number δ, $0 < \delta \ll 1$ such that $\mathbb{F}(\phi(\rho),\rho) \equiv 0$ as $0 < \rho < \delta$. It can be checked that $\phi(\rho)$ has the following asymptotic expansion

$$\begin{aligned}
\phi(\rho) =&\rho - \frac{5}{3}\cos^3\theta\rho^2 + \Big(-\frac{11}{4}\cos^4\theta + \frac{125}{18}\cos^6\theta\Big)\rho^3\\
&+\Big(-\frac{14}{5}\cos^5\theta + \frac{55}{2}\cos^7\theta - \frac{1000}{27}\cos^9\theta\Big)\rho^4\\
&+\Big(-\frac{28}{15}\cos^6\theta + \frac{5677}{96}\cos^8\theta - \frac{1925}{8}\cos^{10}\theta + \frac{48125}{216}\cos^{12}\theta\Big)\rho^5 + O(\rho^6).
\end{aligned} \tag{3.8}$$

Let us compute Abelian integrals $I(\bar{h})$ in the coordinate (r,θ), where

$$I(\bar{h}) = \frac{1}{\sqrt{5}} \oint_{\Gamma_{\bar{h}}} \Big(\alpha_0 + \alpha_1(u+1) + \alpha_2(u+1)^2 + \alpha_3(u+1)^3\Big) v du.$$

From (3.8), we have

$$\begin{aligned} I(\bar{h}) =& \frac{1}{\sqrt{5}} \oint_{\Gamma_{\bar{h}}} \Big(\alpha_0 + \alpha_1(u+1) + \alpha_2(u+1)^2 + \alpha_3(u+1)^3\Big) v du \\ =& \frac{1}{\sqrt{5}} \iint\limits_{\mathrm{int}\Gamma_{\bar{h}}} \Big(\alpha_0 + \alpha_1(u+1) + \alpha_2(u+1)^2 + \alpha_3(u+1)^3\Big) du dv \\ =& \frac{1}{\sqrt{5}} \int_0^{2\pi} d\theta \int_0^{\phi(\rho)} \Big(\alpha_0 + \alpha_1(r\cos\theta+1) + \alpha_2(r\cos\theta+1)^2 + \alpha_3(r\cos\theta+1)^3\Big) r dr. \end{aligned} \tag{3.9}$$

Note that $\bar{h} = \frac{1}{2}\rho^2$. With the help of symbolic computation in (3.9), we obtain the asymptotic expansion of $I(\bar{h})$ as $\bar{h} \to 0^+$

$$I(l) = c_1\bar{h} + c_2\bar{h}^2 + c_3\bar{h}^3 + O(\bar{h}^4), \tag{3.10}$$

where

$$\begin{aligned} c_1 =& \frac{2\sqrt{5}}{5}\pi(\alpha_0+\alpha_1+\alpha_2+\alpha_3),\ c_2 = \frac{\sqrt{5}}{60}\pi(43\alpha_0 + 151\alpha_1 + 91\alpha_2 + 7\alpha_3), \\ c_3 =& \frac{\sqrt{5}}{4320}\pi(170401\alpha_0 + 3073\alpha_1 + 33529\alpha_2 + 86905\alpha_3). \end{aligned}$$

Since $\bar{h} = \frac{1}{5}h + \frac{1}{36}$ and $I(h) = \alpha_0 I_0(h) + \alpha_1 I_1(h) + \alpha_2 I_2(h) + \alpha_3 I_3(h)$, we get the conclusion from (3.5). $\square$

Since $\mathcal{J}_0(h)$ is the period of ovals Γ_h, $\mathcal{J}_0(h) \neq 0$ for $h \in (-\frac{5}{36}, 0)$. Set

$$P_i(h) = \frac{\mathcal{J}_i(h)}{\mathcal{J}_0(h)}, \quad i = 1,2,3. \tag{3.11}$$

Lemma 3.6. *For $i = 2,3$, $P_i(h)$ are monotonic on $(-\frac{5}{36}, 0)$.*

Proof. By direct calculation, we obtain

$$\begin{aligned} P_2'(h) &= \frac{d}{dh}\Big(\frac{\mathcal{J}_2(h)}{\mathcal{J}_0(h)}\Big) = \frac{\mathcal{J}_2'(h)\mathcal{J}_0(h) - \mathcal{J}_0'(h)\mathcal{J}_2(h)}{\mathcal{J}_0^2(h)}, \\ P_3'(h) &= \frac{d}{dh}\Big(\frac{\mathcal{J}_3(h)}{\mathcal{J}_0(h)}\Big) = \frac{\mathcal{J}_3'(h)\mathcal{J}_0(h) - \mathcal{J}_0'(h)\mathcal{J}_3(h)}{\mathcal{J}_0^2(h)}. \end{aligned}$$

From Lemmas 3.4 and 3.5, we have

$$\begin{aligned} & P_2(-\frac{5}{36}+) = 1, \qquad P_3(-\frac{5}{36}+) = 1, \\ & P_2(0-) = 0, \qquad P_3(0-) = 0, \\ & P_2'(-\frac{5}{36}+) = -\frac{9}{5}, \ \ P_3'(-\frac{5}{36}+) = -\frac{12}{5}, \\ & P_2'(0-) = -\infty, \qquad P_3'(0-) = -\infty. \end{aligned} \tag{3.12}$$

Now we assert that, for $i = 2, 3$, $P_i'(h)$ don't have isolated zeros on $(-\frac{5}{36}, 0)$. By reductio ad absurdum. We assume that $P_i'(h)$ $(i = 2, 3)$ have at least one zero on $(-\frac{5}{36}, 0)$. By (3.12) $P_i'(h)$ $(i = 2, 3)$ have even isolated zeros on $(-\frac{5}{36}, 0)$. Noting that $\alpha_0\mathcal{J}_0 + \alpha_i\mathcal{J}_i = \mathcal{J}_0\big(\alpha_0 + \alpha_i P_i(h)\big)$ $(i = 2, 3)$. Therefore, there exist values of α_0 and α_i such that $\alpha_0\mathcal{J}_0 + \alpha_i\mathcal{J}_i$ have at least two zeros on $(-\frac{5}{36}, 0)$, which contradicts that $\{\mathcal{J}_0(h), \mathcal{J}_i(h)\}$ $(i = 2, 3)$ are ECT-systems proved in Lemma 3.3. Therefore, $P_i(h)$ $(i = 2, 3)$ are monotonic on $(-\frac{5}{36}, 0)$. □

Theorem 3.1. *For $(\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^4$ with $\alpha_3 = 0$, $\mathcal{J}(h)$ has two zeros on $(-\frac{3}{36}, 0)$.*

Proof. For $\alpha_3 = 0$, we have

$$\begin{aligned}\mathcal{J}(h) =&\alpha_0\mathcal{J}_0(h) + \alpha_1\mathcal{J}_1(h) + \alpha_2\mathcal{J}_2(h)\\ =&\mathcal{J}_0(h)\big(\alpha_0 + \alpha_1 P_1(h) + \alpha_2 P_2(h)\big).\end{aligned}$$

It is easy to obtain that $P_2'(h) < 0$ for $h \in (-\frac{5}{36}, 0)$ by (3.12). Let $P = P_2(h)$, we get $h = P_2^{-1}(P)$, and then we define the curve

$$\Sigma_1 = \{(P, P_1)(h) | P_1(P) = P_1(P_2^{-1}),\ h \in (-\frac{5}{36}, 0)\}.$$

So the number of zeros of $\mathcal{J}(h)$ is the number of intersection points of the straight line

$$L : \alpha_0 + \alpha_2 P + \alpha_1 P_1 = 0$$

and the curve Σ_1. By direct calculation, we have

$$\frac{d^2P_1}{dP_2^2} = \frac{P_1''(h)P_2'(h) - P_2''(h)P_1'(h)}{P_2^2(h)}.$$

It follows from Lemmas 3.4 and 3.5 that $\frac{d^2P_1}{dP_2^2}(0-) = -\infty$ and

$$\frac{d^2P_1}{dP_2^2}(-\frac{5}{36}+) = \frac{14701243}{24300000}\pi^2 - \frac{15506491\sqrt{5}}{9720000}\pi \approx -5.23580712800000026 < 0.$$

Hence, Σ_1 is strictly concave for $0 < h + \frac{5}{36} \ll 1$ and $0 < -h \ll 1$. We assert that Σ_1 is globally concave for $h \in (-\frac{5}{36}, 0)$. In fact, if Σ_1 has at least one inflection point, then it will have even number of inflection points and this number will be at least 2. Therefore, there exists $(\alpha_0^*, \alpha_1^*, \alpha_2^*)$ such that L and Σ_1 with $(\alpha_0, \alpha_1, \alpha_2) = (\alpha_0^*, \alpha_1^*, \alpha_2^*)$ have at least 4 intersection points (counting the multiplicity), which yields that $\alpha_0^*\mathcal{J}_0(h) + \alpha_1^*\mathcal{J}_1(h) + \alpha_2^*\mathcal{J}_2(h)$ has at least 4 zeros in $(-\frac{5}{36}, 0)$ (counting the multiplicity). But this contradicts the fact that $\{\mathcal{J}_1(h), \mathcal{J}_2(h), \mathcal{J}_0(h)\}$ is an ECT-system with accuracy 1 proved in Lemma 3.2. Therefore, Σ_1 has no inflection point and is globally concave on $(-\frac{5}{36}, 0)$, which yields that there exists $(\alpha_0^*, \alpha_1^*, \alpha_2^*)$ such that L and Σ_1 have exactly 2 intersection points (counting the multiplicity). □

Theorem 3.2. *For $(\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^4$ with $\alpha_2 = 0$, $\mathcal{J}(h)$ has two zeros on $(-\frac{5}{36}, 0)$.*

Proof. For $\alpha_2 = 0$, we have

$$\begin{aligned}\mathcal{J}(h) =&\alpha_0\mathcal{J}_0(h) + \alpha_1\mathcal{J}_1(h) + \alpha_3\mathcal{J}_3(h)\\ =&\mathcal{J}_0(h)\big(\alpha_0 + \alpha_1 P_1(h) + \alpha_3 P_3(h)\big).\end{aligned}$$

It is easy to get that $P_3'(h)<0$ for $h\in(-\frac{5}{36},0)$ from (3.12). Let $P=P_3(h)$, we get $h=P_3^{-1}(P)$, and then we define the curve

$$\Sigma_2=\{(P,P_1)(h)|P_1(P)=P_1(P_3^{-1}),\ h\in(-\frac{5}{36},0)\}.$$

So the number of zeros of $\mathcal{J}(h)$ is the number of intersection points of the straight line

$$L:\alpha_0+\alpha_3P+\alpha_1P_1=0$$

and the curve Σ_2. From Lemmas 3.4 and 3.5 that $\frac{d^2P_1}{dP_3^2}(0-)=-\infty$ and

$$\frac{d^2P_1}{dP_3^2}(-\frac{5}{36}+)=\frac{13522747}{32400000}\pi^2-\frac{15506491\sqrt{5}}{12960000}\pi\approx-4.28584575899999986<0.$$

Using the similar arguments as in the proof of Theorem 3.1, the conclusion can be concluded. □

Theorem 3.3. *For $(\alpha_0,\alpha_1,\alpha_2,\alpha_3)\in\mathbb{R}^4$ with $\alpha_1=0$, $\mathcal{J}(h)$ has two zeros on $(-\frac{5}{36},0)$.*

Proof. For $\alpha_1=0$, we have

$$\begin{aligned}\mathcal{J}(h)=&\alpha_0\mathcal{J}_0(h)+\alpha_2\mathcal{J}_2(h)+\alpha_3\mathcal{J}_3(h)\\=&\mathcal{J}_0(h)\big(\alpha_0+\alpha_2P_2(h)+\alpha_3P_3(h)\big).\end{aligned}$$

It is easy to obtain that $P_2'(h)<0$ for $h\in(-\frac{5}{36},0)$ from (3.12). Let $P=P_2(h)$, we get $h=P_2^{-1}(P)$, and then we define the curve

$$\Sigma_3=\{(P,P_3)(h)|P_3(P)=P_3(P_2^{-1}),\ h\in(-\frac{5}{36},0)\}.$$

So the number of zeros of $\mathcal{J}(h)$ is the number of intersection points of the straight line

$$L:\alpha_0+\alpha_2P+\alpha_3P_3=0$$

and the curve Σ_3. From Lemmas 3.4 and 3.5 that $\frac{d^2P_3}{dP_2^2}(0-)=+\infty$ and

$$\frac{d^2P_3}{dP_2^2}(-\frac{5}{36}+)=\frac{2830951}{24300000}\pi^2-\frac{1192807\sqrt{5}}{9720000}\pi\approx0.287746749999999996>0.$$

Hence, Σ_3 has strictly convex for $0<h+\frac{5}{36}\ll1$ and $0<-h\ll1$. We assert that Σ_3 is globally convex for $h\in(-\frac{5}{36},0)$. In fact, if Σ_3 has at least one inflection point, then it will have even number of inflection points and this number will be at least 2. Therefore, there exists $(\alpha_0^*,\alpha_2^*,\alpha_3^*)$ such that L and Σ_3 with $(\alpha_0,\alpha_2,\alpha_3)=(\alpha_0^*,\alpha_2^*,\alpha_3^*)$ have at least 4 intersection points (counting the multiplicity), which yields that $\alpha_0^*\mathcal{J}_0(h)+\alpha_2^*\mathcal{J}_2(h)+\alpha_3^*\mathcal{J}_3(h)$ has at least 4 zeros in $(-\frac{5}{36},0)$ (counting the multiplicity). But this contradicts Lemma 3.2. Therefore, Σ_3 has no inflection point and is globally convex on $(-\frac{5}{36},0)$, which yields that there exists $(\alpha_0^*,\alpha_2^*,\alpha_3^*)$ such that L and Σ_3 have exactly 2 intersection points (counting the multiplicity). That is $\mathcal{J}(h)$ has two zeros on $(-\frac{5}{36},0)$. □

Remark 3.1. Theorem 1.1 follows from Remark 2.1, Lemma 3.1 and Theorems 3.1-3.3.

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