Journal of Nonlinear Modeling and Analysis http://jnma.ca; http://jnma.ijournal.cn
Volume 2, Number 4, December 2020, 485–493 DOI:10.12150/jnma.2020.485

Eigenvalues of Fourth-order Singular Sturm-Liouville Boundary Value Problems*

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Abstract In this paper, by using Krasnoselskii's fixed-point theorem, some sufficient conditions of existence of positive solutions for the following fourth-order nonlinear Sturm-Liouville eigenvalue problem:

$$\begin{cases} \frac{1}{p(t)}(p(t)u''')'(t) + \lambda f(t,u) = 0, t \in (0,1), \\ u(0) = u(1) = 0, \\ \alpha u''(0) - \beta \lim_{t\to 0^+} p(t)u'''(t) = 0, \\ \gamma u''(1) + \delta \lim_{t\to 1^-} p(t)u'''(t) = 0, \end{cases}$$

are established, where $\alpha, \beta, \gamma, \delta \geq 0$, and $\beta\gamma + \alpha\gamma + \alpha\delta > 0$. The function p may be singular at $t = 0$ or 1, and f satisfies Carathéodory condition.

Keywords Sturm-Liouville problems, Eigenvalue, Krasnoselskii's fixed-point theorem.

MSC(2010) 34B15, 34B25.

1. Introduction

In this paper, we will study the existence of positive solutions for the following fourth-order nonlinear Sturm-Liouville eigenvalue problem:

$$\begin{cases} \frac{1}{p(t)}(p(t)u''')'(t) + \lambda f(t,u) = 0, \quad t \in (0,1), \\ u(0) = u(1) = 0, \\ \alpha u''(0) - \beta \lim_{t\to 0^+} p(t)u'''(t) = 0, \\ \gamma u''(1) + \delta \lim_{t\to 1^-} p(t)u'''(t) = 0, \end{cases} \tag{1.1}$$

where $\lambda > 0$ is a parameter, $\alpha, \beta, \gamma, \delta \geq 0$ are some constants satisfying $\beta\gamma + \alpha\gamma + \alpha\delta > 0$, $p \in C^1((0,1),(0,+\infty))$ satisfying $\int_0^1 \frac{ds}{p(s)} < +\infty$, and $f : [0,1] \times R^+ \to R^+$ satisfies Carathéodory condition. From the above conditions, the function p may be singular at $t = 0$ or 1.

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*The authors were supported by the Natural Science Foundation of China (11971145, 11775169), the Natural Science Foundation of Hebei Province (A2019205133, A2018208171).

Sturm-Liouville boundary problems have been widely investigated in various fields, such as mathematics, physics and meteorology. In recent decades, a vast amount of research was done on the existence of positive solutions of Sturm-Liouville boundary value problems. Within this development, they paid attention to the theory of eigenvalues and eigenfunctions of Sturm-Liouville problems [2-18]. In particular, many authors were interested in the nonlinear singular Sturm-Liouville problems [10-16]. In [10], Yao et al. proved that the BVP (1.1) has one or two positive solutions for some λ under the assumptions $f_0 = f_\infty = 0$ or $f_0 = f_\infty = \infty$. In [13], by a new comparison theorem, Zhang et al. proved that the BVP(1.1) has at least a positive solution for large enough λ under the assumptions:
(1) $p \in C^1((0,1),(0,+\infty))$ and $\int_0^1 \frac{ds}{p(s)} < +\infty$;
(2) $f(t,u) \in C((0,1)\times(0,+\infty),[0,+\infty))$ is decreasing in u ;
(3) For any $\mu > 0$, $f(t,\mu) \neq 0$ and $0 < \int_0^1 k(s)p(s)f(s,\mu s(1-s))ds < +\infty$;
(4) For any $u \in [0,+\infty)$, $\lim\limits_{\mu\to+\infty} \mu f(t,\mu u) = +\infty$ uniformly on $t \in (0,1)$.

In this paper, we consider the existence of positive solutions of the BVP(1.1), under the following conditions:
(H_1) $p \in C^1((0,1),(0,+\infty))$ and $\int_0^1 \frac{ds}{p(s)} < +\infty$;
(H_2) $f : [0,1]\times R^+ \to R^+$ satisfies Carathéodory condition, that is $f(\cdot,u)$ is measurable for each fixed $u \in R^+$, and $f(t,\cdot)$ is continuous for a.e. $t \in [0,1]$;
(H_3) for any $r > 0$, there exists $h_r(t) \in L^1[0,1]$, such that $f(t,u) \leq h_r(t)$, a.e. $t \in [0,1]$, where $u \in [0,r]$, and $0 < \int_0^1 k(s)p(s)h_r(s) < +\infty$.

By Krasnoselskii's fixed-point theorem, two main results are obtained under $(H_1)-(H_3)$.

2. Preliminaries

In this section, we present some necessary definitions, theorems and lemmas.

Definition 2.1. A function u is called a solution of the BVP(1.1) if $u \in C^3([0,1],[0,+\infty))$ satisfies $p(t)u'''(t) \in C^1([0,1],[0,+\infty))$ and the BVP(1.1). Also, u is called a positive solution if $u(t) > 0$ for $t \in [0,1]$ and u is a solution of the BVP (1.1). For some λ, if the BVP (1.1) has a positive solution u, then λ is called an eigenvalue and u is called a corresponding eigenfunction of the BVP (1.1).

Theorem 2.1. ([1], [19]) *Let X be a real normal linear space, and let $P \subset X$ be a cone in X. Assume Ω_1, Ω_2 are relatively open subsets of X with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let $T : \overline{\Omega}_2 \to P$ be a completely continuous operator such that, either*
(1) $\| Tu \| \leq r_1, u \in \partial\Omega_1;$ $\| Tu \| \geq r_2, u \in \partial\Omega_2$ *or*
(2) $\| Tu \| \geq r_1, u \in \partial\Omega_1;$ $\| Tu \| \leq r_2, u \in \partial\Omega_2.$
Then T has a fixed point in $P \bigcap (\overline{\Omega}_2 \setminus \Omega_1)$.

In this paper, we always make the following assumption:
(H_1) $p \in C^1((0,1),(0,+\infty))$ and $\int_0^1 \frac{ds}{p(s)} < +\infty$.

Now we denote by $H(t,s)$ and $G(t,s)$, respectively, the Green's functions for the following boundary value problems:

$$\begin{cases} -u'' = 0, 0 < t < 1, \\ u(0) = u(1) = 0, \end{cases}$$

and

$$\begin{cases} -(p(t)u'(t))' = 0, 0 < t < 1, \\ \alpha u(0) - \beta \lim\limits_{t \to 0^+} p(t)u'(t) = 0, \\ \gamma u(1) + \delta \lim\limits_{t \to 1^-} p(t)u'(t) = 0. \end{cases}$$

It is well known that $H(t, s)$ and $G(t, s)$ can be written as

$$H(t, s) = \begin{cases} s(1-t), 0 \le s \le t \le 1, \\ t(1-s), 0 \le t \le s \le 1, \end{cases}$$

and

$$G(t, s) = \frac{1}{\rho} \begin{cases} (\beta + \alpha B(0, s))(\delta + \gamma B(t, 1)), & 0 \le s \le t \le 1, \\ (\beta + \alpha B(0, t))(\delta + \gamma B(s, 1)), & 0 \le t \le s \le 1, \end{cases}$$

where $B(t, s) = \int_t^s \frac{dv}{p(v)}$, $\rho = \alpha\delta + \alpha\gamma B(0, 1) + \beta\gamma > 0$ (see [13]).

We also have the conclusion [13] that $u(t)$ is a solution of the BVP(1.1) if and only if it is a solution of the integral equation $u(t) = \int_0^1 H(t, \xi) \int_0^1 G(\xi, s)p(s)f(s, u(s))dsd\xi$.

It is easy to verify the following properties of $H(t, s)$ and $G(t, s)$.

Lemma 2.1. (*Remark* 2.1, [13])

(i) For any $t, s \in [0, 1]$,

$$s(1-s)t(1-t) \le H(t, s) \le t(1-t) \quad (or\ s(1-s)).$$

(ii) For any $t, s \in [0, 1]$,

$$\omega k(t)k(s) \le G(t, s) \le \frac{k(t)}{\rho} \quad (or\ \frac{k(s)}{\rho}),$$

where $k(t) = (\beta + \alpha B(0, t))(\delta + \gamma B(t, 1)), \omega = \frac{\rho}{(\beta + \alpha B(0,1))(\delta + \gamma B(0,1))}$.

3. Main results

In this section, we will prove the existence of positive solutions for the BVP(1.1) by using the Krasnoselskii's fixed-point theorem.

Let the Banach space $X = C[0, 1]$ be equipped with the norm $\| u \| := \max\limits_{t \in [0,1]} | u(t) |$, and P be a cone of X defined by $P = \{u(t) \in X : u(t) \ge t(1-t) \| u \|\}$.

To obtain our results in this paper, We need the following lemma.

Lemma 3.1. *Assume that* $(H_1) - (H_3)$ *hold, and define the operator* $T_\lambda : P \to X$ *by*

$$(T_\lambda u)(t) = \lambda \int_0^1 H(t, \xi) \int_0^1 G(\xi, s)p(s)f(s, u(s))dsd\xi.$$

Then $T_\lambda : P \to P$ *is completely continuous.*

Proof. First, we prove that $T_\lambda : P \to P$. From lemma 2.1, for $u(t) \in P$, we have

$$(T_\lambda u)(t) = \lambda \int_0^1 H(t, \xi) \int_0^1 G(\xi, s)p(s)f(s, u(s))dsd\xi$$

$$\begin{aligned}
&\geq t(1-t)\lambda \int_0^1 \xi(1-\xi) \int_0^1 G(\xi,s)p(s)f(s,u(s))dsd\xi \\
&\geq t(1-t)\lambda \int_0^1 H(t',\xi) \int_0^1 G(\xi,s)p(s)f(s,u(s))dsd\xi \\
&= (T_\lambda u)(t')t(1-t).
\end{aligned}$$

By the arbitrariness of t', we can obtain $(T_\lambda u)(t) \geq t(1-t) \parallel T_\lambda u \parallel$, i.e. $T_\lambda(P) \subset P$.

According to the Lebesgue Dominated Convergence Theorem, we have $T_\lambda : P \to P$ is continuous.

Next, we show that T_λ is uniformly bounded.

Let $\overline{\Omega} = \{u(t) \in P : \parallel u \parallel \leq r\}$ and $\int_0^1 k(s)p(s)h_r(s)ds = M_r$. For any $u(t) \in \overline{\Omega}$, by (H_3), we have

$$\begin{aligned}
(T_\lambda u)(t) &= \lambda \int_0^1 H(t,\xi) \int_0^1 G(\xi,s)p(s)f(s,u(s))dsd\xi \\
&\leq \lambda \int_0^1 \xi(1-\xi)d\xi \int_0^1 \frac{k(s)}{\rho} p(s)h_r(s)ds \\
&\leq \frac{\lambda M_r}{4\rho}.
\end{aligned}$$

Hence T_λ is uniformly bounded.

Finally, we will show that T_λ is equicontinuous.

Since $H(t,s)$ is continuous in $[0,1] \times [0,1]$, it is uniformly continuous. Thus, for any $\varepsilon > 0$, there exists $\delta > 0$, such that for any fixed $s \in [0,1]$, when $\mid t_1 - t_2 \mid < \delta$, we have $\mid H(t_1,s) - H(t_2,s) \mid < \frac{\rho}{\lambda M_r}\varepsilon$.

For all $u(t) \in \overline{\Omega}$, $t_1, t_2 \in [0,1]$, $\mid t_1 - t_2 \mid < \delta$, we obtain

$$\begin{aligned}
\mid (T_\lambda u)(t_1) - (T_\lambda u)(t_2) \mid &\leq \lambda \int_0^1 \mid H(t_1,\xi) - H(t_2,\xi) \mid \int_0^1 G(\xi,s)p(s)f(s,u(s))dsd\xi \\
&\leq \frac{\lambda}{\rho} \int_0^1 \mid H(t_1,\xi) - H(t_2,\xi) \mid d\xi \int_0^1 k(s)p(s)h_r(s)ds \\
&\leq \frac{\lambda M_r}{\rho} \int_0^1 \mid H(t_1,\xi) - H(t_2,\xi) \mid d\xi \\
&\leq \frac{\lambda M_r}{\rho} \frac{\rho}{\lambda M_r}\varepsilon \\
&= \varepsilon.
\end{aligned}$$

This implies that T_λ is equicontinuous.

By the Arzela-Ascoli theorem, $T_\lambda : P \to P$ is completely continuous. □

For the convenience, we introduce the following notations:

$$\liminf_{u \to 0^+} \inf_{s \in [0,1] \setminus E_0} \frac{f(s,u)}{u} = f_0,$$

$$\limsup_{u \to 0^+} \sup_{s \in [0,1] \setminus E_0} \frac{f(s,u)}{u} = f^0,$$

$$\liminf_{u \to +\infty} \inf_{s \in [0,1] \setminus E_0} \frac{f(s,u)}{u} = f_\infty,$$

where $E_0 \subset [0,1]$ and $m(E_0) = 0$.

Theorem 3.1. *Assume that* $(H_1)-(H_3)$ *hold,* $f_0>0$, *and suppose that there exist* $R_1>0$ *and* $h_{R_1}(t)$, *such that*

$$\omega f_0\int_0^1\xi(1-\xi)k(\xi)d\xi\int_0^1 k(s)p(s)s(1-s)ds>\frac{\int_0^1 k(s)p(s)h_{R_1}(s)ds}{\rho R_1},$$

where $h_{R_1}(t)$ *is defined by* (H_3). *Then for each* λ *satisfying*

$$\frac{4}{\omega f_0\int_0^1\xi(1-\xi)k(\xi)d\xi\int_0^1 k(s)p(s)s(1-s)ds}<\lambda<\frac{4\rho R_1}{\int_0^1 k(s)p(s)h_{R_1}(s)ds},$$

the BVP(1.1) has at least one positive solution.

Proof. Let $\Omega_1=\{u(t)\in P:\| u \|< R_1\}$. Then for any $u(t)\in\partial\Omega_1$, by Lemma 2.1 and (H_3), we have

$$\begin{aligned}(T_\lambda u)(t)&=\lambda\int_0^1 H(t,\xi)\int_0^1 G(\xi,s)p(s)f(s,u(s))dsd\xi\\&\le\lambda\int_0^1\xi(1-\xi)d\xi\int_0^1\frac{k(s)}{\rho}p(s)h_{R_1}(s)ds\\&\le\frac{\lambda}{4\rho}\int_0^1 k(s)p(s)h_{R_1}(s)ds.\end{aligned}$$

Thus, $\| T_\lambda u \|\le R_1=\| u \|$, if $\lambda<\frac{4\rho R_1}{\int_0^1 k(s)p(s)h_{R_1}(s)ds}$.

On the other hand, if $\lambda>\frac{4}{\omega f_0\int_0^1\xi(1-\xi)k(\xi)d\xi\int_0^1 k(s)p(s)s(1-s)ds}$, there exists $\eta_1>0$ small enough, such that $f_0-\eta_1>0$ and

$$\lambda>\frac{4}{\omega(f_0-\eta_1)\int_0^1\xi(1-\xi)k(\xi)d\xi\int_0^1 k(s)p(s)s(1-s)ds}.$$

From the definition of f_0, there exists $r_1>0$ such that $\frac{f(s,u)}{u}>f_0-\eta_1$ for $0<u\le r_1$. Let $\Omega_2=\{u(t)\in P:\| u \|< R_2\}$, where $R_2<\min\{R_1,r_1\}$. For any $u\in\partial\Omega_2$, we obtain that

$$\begin{aligned}(T_\lambda u)(t)&=\lambda\int_0^1 H(t,\xi)\int_0^1 G(\xi,s)p(s)f(s,u(s))dsd\xi\\&\ge\omega\lambda t(1-t)\int_0^1\xi(1-\xi)k(\xi)d\xi\int_0^1 k(s)p(s)(f_0-\eta_1)u(s)ds\\&\ge\omega\lambda(f_0-\eta_1)t(1-t)\int_0^1\xi(1-\xi)k(\xi)d\xi\int_0^1 k(s)p(s)s(1-s)\| u \| ds\\&\ge\omega\lambda(f_0-\eta_1)t(1-t)\int_0^1\xi(1-\xi)k(\xi)d\xi\int_0^1 k(s)p(s)s(1-s)ds\| u \|.\end{aligned}$$

Hence

$$|(T_\lambda u)(\frac{1}{2})|\ge\lambda\frac{\omega(f_0-\eta_1)}{4}(\int_0^1\xi(1-\xi)k(\xi)d\xi\int_0^1 k(s)p(s)s(1-s)ds)\| u \|.$$

From the definition of norm, we have $\| T_\lambda u \| = \max_{t\in[0,1]} \mid (T_\lambda u)(t) \mid \geq \mid (T_\lambda u)(\frac{1}{2}) \mid$. Hence

$$\| T_\lambda u \| \geq \lambda \frac{\omega(f_0-\eta_1)}{4}\left(\int_0^1 \xi(1-\xi)k(\xi)d\xi \int_0^1 k(s)p(s)s(1-s)ds\right) \| u \| .$$

Then, for each

$$\lambda > \frac{4}{\omega f_0 \int_0^1 \xi(1-\xi)k(\xi)d\xi \int_0^1 k(s)p(s)s(1-s)ds},$$

$\| T_\lambda u \| \geq \| u \| = R_2$.

In summary, for each λ with

$$\frac{4}{\omega f_0 \int_0^1 \xi(1-\xi)k(\xi)d\xi \int_0^1 k(s)p(s)s(1-s)ds} < \lambda < \frac{4\rho R_1}{\int_0^1 k(s)p(s)h_{R_1}(s)ds},$$

T_λ has a fixed point in $P\bigcap(\overline{\Omega}_2 \setminus \Omega_1)$, i.e. the BVP(1.1) has a positive solution $u(t)$ such that $R_1 \leq \| u \| \leq R_2$. □

Theorem 3.2. *Assume that* $(H_1)-(H_3)$ *hold,* $f^0 > 0$, $f_\infty > 0$, *and suppose that there exist* $0 < \theta_1 < \theta_2 < 1$ *such that*

$$f_\infty > f^0 \frac{\int_0^1 k(s)p(s)ds}{\rho\omega\delta(\theta_1,\theta_2)\int_{\theta_1}^{\theta_2} k(s)p(s)ds \int_0^1 \xi(1-\xi)k(\xi)d\xi},$$

where $\delta(\theta_1,\theta_2) = \min_{\theta_1\leq t\leq\theta_2}\{t(1-t)\}$. *Then for each* λ *satisfying*

$$\frac{4}{\delta(\theta_1,\theta_2)\omega f_\infty \int_0^1 \xi(1-\xi)k(\xi)d\xi \int_{\theta_1}^{\theta_2} k(s)p(s)ds} < \lambda < \frac{4\rho}{f^0 \int_0^1 k(s)p(s)ds},$$

the BVP(1.1) has at least one positive solution.

Proof. If $\lambda < \frac{4\rho}{f^0\int_0^1 k(s)p(s)ds}$, there exists $\eta_2 > 0$ small enough, such that $\lambda < \frac{4\rho}{(f^0+\eta_2)\int_0^1 k(s)p(s)ds}$. By the definition of f^0, there exists $R_3 > 0$ such that $\frac{f(s,u)}{u} < f^0 + \eta_2$ for $0 < u \leq R_3$. Let $\Omega_3 = \{u(t) \in P : \| u \| < R_3\}$. For any $u \in \partial\Omega_3$, we have

$$\begin{aligned}(T_\lambda u)(t) &= \lambda \int_0^1 H(t,\xi)\int_0^1 G(\xi,s)p(s)f(s,u(s))dsd\xi \\ &\leq \frac{\lambda}{\rho}\int_0^1 \xi(1-\xi)d\xi \int_0^1 k(s)p(s)(f^0+\eta_3)u(s)ds \\ &\leq \frac{\lambda}{4\rho}(f^0+\eta_3)\int_0^1 k(s)p(s)ds \| u \| .\end{aligned}$$

Thus, $\| T_\lambda u \| \leq \| u \| = R_3$, if $\lambda < \frac{4\rho}{f^0\int_0^1 k(s)p(s)ds}$.

On the other hand, if $\lambda > \frac{4}{\delta(\theta_1,\theta_2)\omega f_\infty \int_0^1 \xi(1-\xi)k(\xi)d\xi \int_{\theta_1}^{\theta_2} k(s)p(s)ds}$, there exists $\eta_3 > 0$ small enough, such that $f_\infty - \eta_3 > 0$ and

$$\lambda > \frac{4}{\delta(\theta_1,\theta_2)\omega(f_\infty-\eta_3)\int_0^1 \xi(1-\xi)k(\xi)d\xi \int_{\theta_1}^{\theta_2} k(s)p(s)ds}.$$

By the definition of f_∞, there exists $r_3 > 0$ such that $\frac{f(s,u)}{u} > f_\infty - \eta_3$ for $u \geq r_3$. From the definition of P, we have $u(t) \geq \delta(\theta_1, \theta_2) \parallel u \parallel$, for any $u(t) \in P$, $t \in [\theta_1, \theta_2]$. Let $\Omega_4 = \{u(t) \in P : \parallel u \parallel < R_4\}$, where $R_4 = \max\{R_3 + 1, \frac{r_3}{\delta(\theta_1,\theta_2)}\}$. For any $u \in \partial\Omega_4$, we have

$$
\begin{aligned}
(T_\lambda u)(t) &= \lambda \int_0^1 H(t,\xi) \int_0^1 G(\xi,s)p(s)f(s,u(s))dsd\xi \\
&\geq \lambda \int_0^1 H(t,\xi) \int_{\theta_1}^{\theta_2} G(\xi,s)p(s)f(s,u(s))ds \\
&\geq \omega\lambda t(1-t) \int_0^1 \xi(1-\xi)k(\xi)d\xi \int_{\theta_1}^{\theta_2} k(s)p(s)(f_\infty - \eta_3)u(s)ds \\
&\geq \omega\lambda(f_\infty - \eta_3)t(1-t) \int_0^1 \xi(1-\xi)k(\xi)d\xi \int_{\theta_1}^{\theta_2} k(s)p(s)\delta(\theta_1,\theta_2) \parallel u \parallel ds \\
&\geq \omega\lambda(f_\infty - \eta_3)t(1-t)\delta(\theta_1,\theta_2) \int_0^1 \xi(1-\xi)k(\xi)d\xi \int_{\theta_1}^{\theta_2} k(s)p(s)ds \parallel u \parallel .
\end{aligned}
$$

Hence,

$$
\mid (T_\lambda u)(\frac{1}{2}) \mid \geq \frac{\omega\lambda(f_\infty - \eta_3)\delta(\theta_1,\theta_2)}{4} \int_0^1 \xi(1-\xi)k(\xi)d\xi \int_{\theta_1}^{\theta_2} k(s)p(s)ds \parallel u \parallel .
$$

From the definition of norm, we have $\parallel T_\lambda u \parallel = \max_{t\in[0,1]} \mid (T_\lambda u)(t) \mid \geq \mid (T_\lambda u)(\frac{1}{2}) \mid$. Thus

$$
\parallel T_\lambda u \parallel \geq \frac{\omega\lambda(f_\infty - \eta_3)\delta(\theta_1,\theta_2)}{4} \int_0^1 \xi(1-\xi)k(\xi)d\xi \int_{\theta_1}^{\theta_2} k(s)p(s)ds \parallel u \parallel .
$$

So we have that $\parallel T_\lambda u \parallel \geq \parallel u \parallel = R_4$, if $\lambda > \frac{4}{\delta(\theta_1,\theta_2)\omega f_\infty \int_0^1 \xi(1-\xi)k(\xi)d\xi \int_{\theta_1}^{\theta_2} k(s)p(s)ds}$.

From Theorem 2.1, for each λ satisfying

$$
\frac{4}{\delta(\theta_1,\theta_2)\omega f_\infty \int_0^1 \xi(1-\xi)k(\xi)d\xi \int_{\theta_1}^{\theta_2} k(s)p(s)ds} < \lambda < \frac{4\rho}{f^0 \int_0^1 k(s)p(s)ds},
$$

T_λ has a fixed point in $P \bigcap (\overline{\Omega}_4 \setminus \Omega_3)$, i.e. the BVP(1.1) has a positive solution $u(t)$ such that $R_3 \leq \parallel u \parallel \leq R_4$. □

4. Examples

To illustrate the usefulness of the results, we give some examples in this section.

Example 4.1. Let $p(x) = 1$, and

$$
f(t,u) = \begin{cases} \cos^2 u + t, t \in [0,1] \text{ and } t \in R \setminus Q, \\ 0, \qquad\qquad t \in [0,1] \text{ and } t \in Q. \end{cases}
$$

For given $u \in R^+$, we have $f(t,u) \leq 1 + t$. Let $E = \{t : t \in [0,1] \text{ and } t \in Q\}$. Then we have $m(E) = 0$. Clearly, $(H_1) - (H_3)$ are satisfied, and $f_0 = +\infty$. From Theorem 3.1, for each $0 < \lambda < \frac{4(\beta\gamma+\alpha\gamma+\alpha\delta)R_1}{\frac{3}{2}\beta\delta+\frac{2}{3}\beta\gamma+\frac{5}{6}\alpha\delta+\frac{1}{4}\alpha\gamma}$, the BVP(1.1) has at least a positive solution for R_1 enough large.

Example 4.2. Let $p(x) = 1$, and

$$f(t,u) = \begin{cases} e^u + \sqrt{t} - 2, t \in [0,1] \text{ and } t \in R \setminus Q, \\ 0, \qquad\qquad\quad t \in [0,1] \text{ and } t \in Q. \end{cases}$$

Let $E = \{t : t \in [0,1] \text{ and } t \in Q\}$. Then we have $m(E) = 0$. Clearly, $(H_1) - (H_3)$ are satisfied, $f^0 = 1$, and $f_\infty = +\infty$. From Theorem 3.2, for each $0 < \lambda < \frac{4(\beta\gamma+\alpha\gamma+\alpha\delta)}{\beta\delta+\frac{\beta\gamma}{2}+\frac{\alpha\gamma}{6}+\frac{\alpha\delta}{2}}$, the BVP(1.1) has at least a positive solution.

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