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The Dynamics of Stochastic Predator-prey Models with Non-constant Mortality Rate and General Nonlinear Functional Response*

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Abstract In this paper, we investigate the dynamics of stochastic predator-prey models with non-constant mortality rate and general nonlinear functional response. For the stochastic system, we firstly prove the existence of the global unique positive solution. Secondly, we establish sufficient conditions for the extinction and persistence in the mean of autonomous stochastic model and obtain a critical value between them. Then by constructing a appropriate Lyapunov function, we prove that there exists a unique stationary distribution and it has ergodicity in the case of persistence. Finally, numerical simulations are introduced to illustrate our theoretical results.

Keywords Stochastic predator-prey model, Non-constant mortality rate, General nonlinear functional response, Extinction, Stationary distribution.

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1. Introduction

The dynamics of the predator-prey system has always been one of the hot topics of ecological science research. In the past decades, a lot of predator-prey models have been proposed and used to describe the food supply relationship between two species [1]. In [2], Cavani and Farkas considered the following predator-prey system with Holling type-II functional response

$$\begin{cases} \dot{N}(t) = \varepsilon N(t)\left(1 - \frac{N(t)}{K}\right) - \frac{aP(t)N(t)}{\beta + N(t)}, \\ \dot{P}(t) = P(t)\left(-M(P(t)) + \frac{bN(t)}{\beta + N(t)}\right), \end{cases} \tag{1.1}$$

where $N(t)$ and $P(t)$ are the densities of the prey and the predator at time t, respectively. All the parameters are positive constants. ε represents specific growth rate of the prey without predation and environmental constraints; K denotes the carrying capacity of the prey in the absence of predators; a, b and β are satiation coefficients or conversion rates; and here the function $M(P)$ is the specific mortality rate of predators in the absence of prey. At the same time, $M(P)$ could be constant or non-constant. If $M(P)$ is a constant (such as $M(P) = n$), the model (1.1) is the

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classic predator-prey model with Holling type-II functional response. In this paper, the mortality rate of predators

$$M(P) = \frac{\gamma + \delta P}{1 + P} = \delta + \frac{\gamma - \delta}{1 + P}, \quad (0 < \gamma < \delta)$$

is non-constant and it depends on the quantity of predator; here, γ is the mortality at low density, and δ is the maximal mortality with the natural assumption $\delta > \gamma$. Compared with the common models, the advantage of this model is that the predator mortality rate is neither a constant nor an unbounded function, and it increases with the increase of the quantity of predator. There have been many papers published about the model and its deformations, see [3–5].

In population dynamics, one significant component of the relationship between the predator and the prey is the functional response of the predator, which refers to the change in the density of prey attached by per unit time per predator as the density of prey changes. There have been several well-known types of nonlinear functional response: Holling type-II, type-III [6], Hassell-Varley type [7], Beddington-DeAngelis type [8–10], Crowley-Martin type [11], ratio-dependence type [12], etc. These important nonlinear response functions allow us to gain insight into the dynamic relationship between predators and preys. Therefore, it is reasonable to use a nonlinear functional response when we establish a predator-prey model. Based on model (1.1), we consider the following predator-prey model with general nonlinear functional response

$$\begin{cases} \dot{N}(t) = \varepsilon N(t)\left(1 - \frac{N(t)}{K}\right) - a\varphi(N(t))P(t), \\ \dot{P}(t) = P(t)\left(-\frac{\gamma + \delta P(t)}{1 + P(t)} + b\varphi(N(t))\right), \end{cases} \tag{1.2}$$

where $\varphi(N)$ is a general functional response.

However, environmental noise is ubiquitous in nature and population models are inevitably affected by environmental noise, which is an important part of reality [13,14]. Deterministic models have certain limitations in the mathematical modeling process of ecological models. For example,they are not easy to fit datas and not easy for us to accurately predict the future dynamics of the system. Therefore, it is necessary to consider stochastic fluctuations in the modeling process of the population model. In recent years, there have been many significant papers on the dynamics of stochastic population models [15, 16]. However, few papers have considered stochastic predator-prey models with non constant mortality rate and general nonlinear functional response in stochastic environments. In this paper, we will study the dynamics of the following stochastic predator-prey model

$$\begin{cases} dN = \left(\varepsilon N(t)\left(1 - \frac{N(t)}{K}\right) - a\varphi(N(t))P(t)\right)dt + \sigma_1 N(t)dB_1(t), \\ dP = P(t)\left(-\frac{\gamma + \delta P(t)}{1 + P(t)} + b\varphi(N(t))\right)dt + \sigma_2 P(t)dB_2(t). \end{cases} \tag{1.3}$$

where $B_1(t)$, $B_2(t)$ are mutually independent standard Brownian motions defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{F}_{t\geq 0}, \mathcal{P})$ with a σ-field filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions, and σ_i^2 represent the intensities of the white noise, $i = 1, 2$.

For the sake of biologically reality, we give the general assumption for $\varphi(N)$ above. Again, for the sake of clarity, we make two further assumptions for the generic nature

A_1 : $\varphi \in \mathcal{C}^2([0,+\infty),[0,+\infty))$ and $\varphi'(N) \le c_1$, $\varphi(N) \le c_2$ for any $N \in (0,+\infty)$, where c_1, c_2 are positive constants.

A_2 : $\varphi''(N) \ge c_3$, for any $N \in (0,+\infty)$. Where c_3 is a constant which requires no assumption on the sign.

For the predator-prey system, we consider the following prototypes of response functions that are often found in the literature. There are several special response functions:

(i) Lotka-Volterra function or Holling type-I function:

$$\varphi(N) = mN, \quad 0 \le N \le \frac{k}{m}; \qquad \varphi(N) = k, \quad N \ge \frac{k}{m} \tag{1.4}$$

where $m > 0$ denotes a constant and represents the maximal per capita consumption rate.

(ii) Michaelis-Menten function or Holling type-II function:

$$\varphi(N) = \frac{\alpha N}{\beta + N}, \tag{1.5}$$

where $\alpha > 0$ and $\beta > 0$ are constants. Here α is the maximal growth rate of the species and β is called the half-saturation (or Michaelis-Menten) constant.

(iii) Holling type-III response function [17]:

$$\varphi(N) = \frac{mN^2}{\alpha N^2 + \beta N + 1} \tag{1.6}$$

where m and α are positive constants and β is a constant. When $\beta > -2\sqrt{\alpha}$ (so that $\alpha N^2 + \beta N + 1 > 0$ for all $x \ge 0$), the function $\varphi(N)$ is called the generalized Holling type-III functional response [18].

(iv) Ivlev functional response [19]:

$$\varphi(N) = h\Big(1 - e^{-cN}\Big) \tag{1.7}$$

where c and h are positive constants.

This paper is organized as follows: In section 2, we basically give a theorem concerning the existence and uniqueness of the global positive solution to model (1.3). In section 3, we investigate persistence in the mean and extinction of model (1.3) and furthermore, we try to obtain the critical value between them. In section 4, we show that the model exists a unique stationary distribution. In section 5, we use numerical simulations to illustrate our theoretical results. Finally, conclusion is given to end this paper.

2. Existence and uniqueness of the global positive solution

For simplicity, we introduce the following notations.

$\mathbb{R}^2_+ := \{x = (x_1, x_2) \in \mathbb{R}^2 : x_i > 0, i = 1, 2\}$.

$\langle f \rangle_t = \frac{1}{t}\int_0^t f(s)ds$.

If $f(t)$ is a continuous bounded function, define $f^l = \inf_{t\in[0,\infty)} f(t)$, $f^u = \sup_{t\in[0,\infty)} f(t)$.

Theorem 2.1. *For any initial value* $(N(0), P(0)) \in \mathbb{R}_+^2$*, there is a unique solution* $(N(t), P(t))$ *of system* (1.3) *on* $t \geq 0$*, and the solution will remain in* $\mathbb{R}_+^2$ *with probability* 1*.*

Proof. Obviously, the coefficients of model (1.3) are locally Lipschitz continuous, so there is a unique local solution $(N(t), P(t))$ on $t \in [0, \tau_e)$ for any initial value $(N(0), P(0)) \in \mathbb{R}_+^2$, where τ_e is the explosion time. If $\tau_e = \infty$ a.s., then this local solution is global. Let k_0 be sufficiently large for every component of $(N(0), P(0))$ lying within the interval $[1/k_0, k_0]$. For each integer $k \geq k_0$, define the stopping time

$$\tau_k = \inf\{t \in [0, \tau_e) | N(t) \notin (1/k, k) \textit{ or } P(t) \notin (1/k, k)\},$$

where throughout this paper we set $\inf \emptyset = \infty$ ($\emptyset$ denotes the empty set). Clearly, τ_k is increasing as $k \to \infty$. Set $\tau_\infty = \lim_{k\to\infty} \tau_k$, which implies $\tau_\infty < \tau_e$ a.s. If we show that $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ a.s. This means that $(N(t), P(t)) \in \mathbb{R}_+^2$ a.s. for all $t > 0$. If $\tau_e < \infty$a.s., then there is a pair of constants $T > 0$ and $\epsilon \in (0, 1)$ such that

$$\mathbb{P}\{\tau_\infty \leq T\} > \epsilon.$$

Hence there is an integer $k_1 \geq k_0$ such that

$$\mathbb{P}\{\tau_k \leq T\} > \epsilon \textit{ for all } k \geq k_1. \tag{2.1}$$

Define a C^2-function $V : \mathbb{R}_+^2 \to \mathbb{R}_+$ as follow:

$$\begin{aligned} V(N, P) =& \frac{4bc_1}{\delta}\left(N - \frac{\delta}{4bc_1} - \frac{\delta}{4bc_1}\log\frac{4bc_1 N}{\delta}\right) \\ &+ \frac{2ac_1}{\delta}\left(P - \frac{\delta}{2ac_1} - \frac{\delta}{2ac_1}\log\frac{2ac_1 P}{\delta}\right). \end{aligned}$$

Applying Itô's formula we have

$$\begin{aligned} dV(N, P) =& \mathcal{L}V(N, P)dt + \frac{4bc_1\sigma_1}{\delta}\left(N - \frac{\delta}{4bc_1}\right) dB_1(t) \\ &+ \frac{2ac_1\sigma_2}{\delta}\left(P - \frac{\delta}{2ac_1}\right) dB_2(t), \end{aligned}$$

where $\mathcal{L}V : \mathbb{R}_+^2 \to \mathbb{R}_+$ is defined by

$$\begin{aligned} \mathcal{L}V(N, P) =& \frac{4bc_1}{\delta}\left(N - \frac{\delta}{4bc_1}\right)\left(\varepsilon - \frac{\varepsilon N}{K} - \frac{a\varphi(N)P}{N}\right) + \frac{\sigma_1^2}{2} \\ &+ \frac{2ac_1}{\delta}\left(P - \frac{\delta}{2ac_1}\right)\left(-\delta - \frac{\gamma - \delta}{1 + P} + b\varphi(N)\right) + \frac{\sigma_2^2}{2} \\ =& \frac{4bc_1\varepsilon}{\delta K}N^2 + \left(\frac{4bc_1\varepsilon}{\delta} + \frac{\varepsilon}{K}\right) N - \frac{4abc_1}{\delta}\varphi(N)P + \frac{a\varphi(N)P}{N} \\ &- \varepsilon + \frac{\sigma_1^2}{2} - 2ac_1 P + \frac{2ac_1(\delta - \gamma)}{\delta}\frac{P}{1 + P} + \frac{2abc_1}{\delta}\varphi(N)P \\ &- \frac{\gamma - \delta}{1 + P} - b\varphi(N) + \delta + \frac{\sigma_2^2}{2} \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{4bc_1\varepsilon}{\delta K}N^2 + \left(\frac{4bc_1\varepsilon}{\delta} + \frac{\varepsilon}{K}\right)N - \frac{2abc_1}{\delta}\varphi(N)P \\
&\quad + ac_1P - 2ac_1P - \varepsilon + \frac{\sigma_1^2}{2} + \delta + \frac{\sigma_2^2}{2} + 2ac_1 \\
&\leq \frac{4bc_1\varepsilon}{\delta K}N^2 + \left(\frac{4bc_1\varepsilon}{\delta} + \frac{\varepsilon}{K}\right)N - \varepsilon + \frac{\sigma_1^2}{2} + \delta + \frac{\sigma_2^2}{2} + 2ac_1 \\
&\leq M,
\end{aligned}
$$

where M is a positive constant. We therefore obtain

$$
\mathbb{E}V(N(\tau_k \wedge T), P(\tau_k \wedge T)) \leq V(N(0), P(0)) + M\mathbb{E}(\tau_k \wedge T) \leq V(N(0), P(0)) + MT \tag{2.2}
$$

Set $\Omega_k = \{\tau \leq T\}$ for $k \geq k_1$ and by (2.1), we have $\mathbb{P}(\Omega_k) \geq \epsilon$. Noting that for every $\omega \in \Omega_k$, there is at least one of $N(\tau_k, \omega), P(\tau_k, \omega)$ equals either k or $1/k$, therefore

$$
\begin{aligned}
V(N(\tau_k, \omega), P(\tau_k, \omega)) \geq & \frac{4bc_1}{\delta}\left(k - \frac{\delta}{4bc_1} - \frac{\delta}{4bc_1}\log\frac{4bc_1k}{\delta}\right) \\
& \wedge \frac{4bc_1}{\delta}\left(\frac{1}{k} - \frac{\delta}{4bc_1} - \frac{\delta}{4bc_1}\log\frac{4bc_1}{\delta k}\right) \\
& \wedge \frac{2ac_1}{\delta}\left(k - \frac{\delta}{2ac_1} - \frac{\delta}{2ac_1}\log\frac{2ac_1k}{\delta}\right) \\
& \wedge \frac{2ac_1}{\delta}\left(\frac{1}{k} - \frac{\delta}{2ac_1} - \frac{\delta}{2ac_1}\log\frac{2ac_1}{\delta k}\right).
\end{aligned}
$$

It then follows from (2.2) that

$$
\begin{aligned}
V(N(0), P(0)) + MT \geq & \mathbb{E}(I_{\Omega_k}V(N(\tau_k, \omega), P(\tau_k, \omega))) \\
\geq & \epsilon\Big\{\frac{4bc_1}{\delta}\Big(k - \frac{\delta}{4bc_1} - \frac{\delta}{4bc_1}\log\frac{4bc_1k}{\delta}\Big) \\
& \wedge \frac{4bc_1}{\delta}\Big(\frac{1}{k} - \frac{\delta}{4bc_1} - \frac{\delta}{4bc_1}\log\frac{4bc_1}{\delta k}\Big) \\
& \wedge \frac{2ac_1}{\delta}\Big(k - \frac{\delta}{2ac_1} - \frac{\delta}{2ac_1}\log\frac{2ac_1k}{\delta}\Big) \\
& \wedge \frac{2ac_1}{\delta}\Big(\frac{1}{k} - \frac{\delta}{2ac_1} - \frac{\delta}{2ac_1}\log\frac{2ac_1}{\delta k}\Big)\Big\}.
\end{aligned}
$$

Letting $k \longrightarrow \infty$ leads to the contradiction

$$
\infty > V(N(0), P(0)) + MT = \infty,
$$

so we must have $\tau_\infty = \infty$ *a.s.* The proof is completed. □

3. The persistence in mean and extinction

In the section, we investigate the persistence and extinction of stochastic predator-prey model (1.3) under a certain condition. In addition, by using the ergodic property of stochastic Logistic model, we try to give the critical value which determines the extinction and persistence of model (1.3). To this end, we first give the definition of the persistence and the extinction and lemmas.

Definition 3.1. [20]

(1) The model (1.3) is said to be extinct if $\lim_{t\to\infty} P(t) = 0$ *a.s.*

(2) The model (1.3) is said to be persistent in mean if $\liminf_{t\to\infty}\langle P\rangle_t > 0$ *a.s.*

Lemma 3.1. [20] Suppose that $Z(t) \in C(\Omega \times [0,\infty), \mathbb{R}_+)$.

(I) If there are two positive constants T and δ_0 such that

$$\ln Z(t) \le \delta t - \delta_0 \int_0^t Z(s)ds + \sum_{i=1}^n \alpha_i B(t) \ a.s.$$

for all $t > T$, where α_i, δ are constants, then

$$\begin{cases} \limsup_{t\to\infty}\langle Z\rangle_t \le \frac{\delta}{\delta_0} \ a.s., & \text{if } \delta \ge 0; \\ \lim_{t\to\infty} Z(t) = 0 \ a.s., & \text{if } \delta < 0. \end{cases}$$

(II) If there exist three positive constants T, δ, δ_0 such that

$$\ln Z(t) \ge \delta t - \delta_0 \int_0^t Z(s)ds + \sum_{i=1}^n \alpha_i B(t) \ a.s.$$

for all $t > T$, then $\liminf_{t\to\infty}\langle Z\rangle_t \ge \frac{\delta}{\delta_0}$ *a.s.*.

Lemma 3.2. [21] Consider the following one-dimensional stochastic Logistic model

$$dX(t) = \varepsilon X(t)\Big(1 - \frac{X(t)}{K}\Big)dt + \sigma_1 X(t)dB_1(t) \tag{3.1}$$

with $X(0) = N(0)$. If $\varepsilon - \frac{\sigma_1^2}{2} > 0$, model (3.1) has a unique ergodic stationary distribution $\nu(\cdot)$ with stationary density $\mu(x) = Cx^{\frac{2\varepsilon-\sigma_1^2}{\sigma_1^2}-1} e^{-\frac{2\varepsilon}{K\sigma_1^2}x}$, where $C = (2\varepsilon/K\sigma_1^2)^{(2\varepsilon-\sigma_1^2)/\sigma_1^2}/\Gamma((2\varepsilon-\sigma_1^2)/\sigma_1^2)$, and

$$\mathbb{P}\Big\{\lim_{t\to\infty}\frac{1}{t}\int_0^t f(X(s))ds = \int_{R_+} f(x)\mu(x)dx\Big\} = 1,$$

where f is a function integrable with respect to the measure ν.

Remark 3.1. From stochastic comparison theory it follows that $N(t) \le X(t)$ *a.s.* and

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \varphi(X(s))ds = \int_0^\infty \varphi(x)\mu(x)dx, \ a.s. \tag{3.2}$$

Theorem 3.1. *Assume that* $\varepsilon - \sigma_1^2/2 > 0$. *Let* $(N(t), P(t))$ *be a positive solution of model* (1.3) *with initial value* $(N(0), P(0)) \in \mathbb{R}_+^2$.

(i) If $\lambda := -\gamma - \frac{\sigma_2^2}{2} + b\int_0^\infty \varphi(x)\mu(x)dx < 0$, *then the predator populations go to extinction a.s.*

(ii) If $\lambda > 0$, *then system* (1.3) *will be persistent in the mean.*

Proof. (i). An application of Itô's formula to the second equation of (1.3) shows that

$$\begin{aligned} d\log P(t) &= \left(-\frac{\gamma+\delta P(t)}{1+P(t)} + b\varphi(N) - \frac{\sigma_2^2}{2}\right)dt + \sigma_2 dB_2(t) \\ &= \left(-\gamma - \frac{\sigma_2^2}{2} - \frac{(\delta-\gamma)P(t)}{1+P(t)} + b\varphi(N)\right)dt + \sigma_2 dB_2(t), \end{aligned} \tag{3.3}$$

Integrating above inequality from 0 to t and dividing t on both sides, we get

$$\begin{aligned} \frac{\log P(t) - \log P(0)}{t} &= -\gamma - \frac{\sigma_2^2}{2} - (\delta-\gamma)\langle\frac{P}{1+P}\rangle_t + \frac{b}{t}\int_0^t \varphi(N((s))ds + \frac{M_2(t)}{t} \\ &\le -\gamma - \frac{\sigma_2^2}{2} + \frac{b}{t}\int_0^t \varphi(N((s))ds + \frac{M_2(t)}{t} \\ &\le -\gamma - \frac{\sigma_2^2}{2} + \frac{b}{t}\int_0^t \varphi(X((s))ds + \frac{M_2(t)}{t}, \end{aligned}$$

where $M_i(t) = \int_0^t \sigma_i dB_i(t),\ i = 1,2$ are real-valued continuous local martingales. By strong law of large numbers [22], we have $\lim_{t\to\infty}\frac{M_i}{t} = 0\ a.s.,\ i = 1,2$. Applying (I) in Lemma 3.1 we obtain

$$\limsup_{t\to\infty}\frac{\log P(t)}{t} \le -\gamma - \frac{\sigma_2^2}{2} + b\int_0^\infty \varphi(x)\mu(x)dx.$$

Obviously, the predator populations $P(t)$ tends to zero a.s. when $\lambda < 0$.

(ii). Applying Itô's formula to the first equation of (1.3) and (3.1) respectively, we have

$$d\log N(t) = \left(\varepsilon\left(1 - \frac{N(t)}{K}\right) - a\frac{\varphi(N(t))P(t)}{N(t)} - \frac{\sigma_1^2}{2}\right)dt + \sigma_1 dB_1(t),$$

and

$$d\log X(t) = \left(\varepsilon\left(1 - \frac{X(t)}{K}\right) - \frac{\sigma_1^2}{2}\right)dt + \sigma_1 dB_1(t).$$

Then integrating above equality from 0 to t and dividing t on both sides, we get

$$\frac{\log N(t) - \log N(0)}{t} = \varepsilon - \frac{\sigma_1^2}{2} - \frac{1}{t}\int_0^t \frac{\varepsilon}{K}N(s)ds - \frac{a}{t}\int_0^t \frac{\varphi(N(s))P(s)}{N(s)}ds + \frac{M_1(t)}{t},$$

and

$$\frac{\log X(t) - \log X(0)}{t} = \varepsilon - \frac{\sigma_1^2}{2} - \frac{1}{t}\int_0^t \frac{\varepsilon}{K}X(s)ds + \frac{M_1(t)}{t}.$$

These imply that

$$\begin{aligned} 0 \ge \frac{\log N(t) - \log X(t)}{t} &= -\frac{1}{t}\int_0^t \frac{\varepsilon}{K}(N(s) - X(s))ds - \frac{a}{t}\int_0^t \frac{\varphi(N(s))P(s)}{N(s)}ds \\ &\ge -\frac{\varepsilon}{K}\langle N - X\rangle_t - a\langle\frac{\varphi(N)P}{N}\rangle_t \\ &\ge -\frac{\varepsilon}{K}\langle N - X\rangle_t - ac_1\langle P\rangle_t, \end{aligned}$$

that is to say,

$$\frac{\varepsilon}{K}\langle X-N\rangle_t \le ac_1\langle P\rangle_t. \tag{3.4}$$

From (6) we obtain

$$\begin{aligned}
d\log P(t) =& \left(-\gamma-\frac{\sigma_2^2}{2}-\frac{(\delta-\gamma)P(t)}{1+P(t)}+b\varphi(N(t))\right)dt+\sigma_2 dB_2(t)\\
=&\Big(-\gamma-\frac{\sigma_2^2}{2}-\frac{(\delta-\gamma)P(t)}{1+P(t)}+b\varphi(X(t))-b\Big(\varphi(X(t))-\varphi(N(t))\Big)\Big)dt\\
&+\sigma_2 dB_2(t)\\
\ge&\left(-\gamma-\frac{\sigma_2^2}{2}-(\delta-\gamma)P(t)+b\varphi(X(t))-b\Big(\varphi'(\xi(t))(X(t)-N(t))\Big)\right)dt\\
&+\sigma_2 dB_2(t)\\
\ge&\left(-\gamma-\frac{\sigma_2^2}{2}-(\delta-\gamma)P(t)+b\varphi(X(t))-bc_1\Big((X(t)-N(t))\Big)\right)dt\\
&+\sigma_2 dB_2(t),
\end{aligned} \tag{3.5}$$

where $\xi(t)\in(N(t),X(t)), t\in(0,\infty)$. Integrating (3.5) from 0 to t, combining (3.4), one can derive that

$$\begin{aligned}
\frac{\log P(t)-\log P(0)}{t} \ge& -\gamma-\frac{\sigma_2^2}{2}-(\delta-\gamma)\langle P\rangle_t+b\langle\varphi(X)\rangle_t-bc_1\langle X-N\rangle_t+\frac{M_2(t)}{t}\\
\ge& -\gamma-\frac{\sigma_2^2}{2}-(\delta-\gamma)\langle P\rangle_t+b\langle\varphi(X)\rangle_t-bc_1\frac{ac_1K}{\varepsilon}\langle P\rangle_t+\frac{M_2(t)}{t}\\
=& -\gamma-\frac{\sigma_2^2}{2}-(\delta-\gamma+\frac{abc_1^2K}{\varepsilon})\langle P\rangle_t+b\langle\varphi(X)\rangle_t+\frac{M_2(t)}{t},
\end{aligned}$$

for sufficiently large t. By virtue of arbitrariness of ϵ and (II) in Lemma 3.1, we derive that

$$\liminf_{t\to\infty}\langle P\rangle_t \ge \frac{\lambda}{\delta-\gamma+\frac{abc_1^2K}{\varepsilon}}>0,\ a.s.$$

That is to say model (1.3) will be persistent in the mean when $\lambda>0$. The proof is complete. □

4. Existence of stationary distribution

Consider the stochastic equation:

$$dY(t)=f(Y(t))dt+\sum_{r=1}^{k}\sigma_r(Y)dB_r(t), \tag{4.1}$$

where $Y(t)$ is a homogeneous Markov process in l-dimension Euclidean space R^l. The diffusion matrix $A(Y)=(a_{ij}(Y))$, $a_{ij}(Y)=\sum_{r=1}^{k}\sigma_r^i(Y)\sigma_r^j(Y)$.

In the section, we will give a lemma which illustrates a criteria for the existence of a unique stationary distribution, (see [23]). For the convenience, we give the definition of stationary distribution.

Definition 4.1. [23] Let $\mathbb{P}(t, Y, \cdot)$ be the probability measure induced by $Y(t) = (N(t), P(t))$ with initial value $(N(0), P(0))$. That is,

$$\mathbb{P}(t, Y, \cdot) = \mathbb{P}(Y(t) \in B | Y(0) = (N(0), P(0))), \textit{for any Borel set } B \subset \mathbb{R}_+^2$$

If there exists a probability measure $\mu(\cdot)$ such that $\lim_{t\to\infty} \mathbb{P}(t, Y, \cdot) = \mu(B)$ for all $Y(t) \in \mathbb{R}_+^2$, then we say that $Eq.(4.1)$ has a stationary distribution $\mu(\cdot)$.

Lemma 4.1. [23] The Markov process $Y(t)$ has a unique ergodic stationary distribution $\mu(\cdot)$ if there exists a bounded domain $D \subset E_l$ with regular boundary Γ and

$B_1:$ there is a positive number H such that $\sum\limits_{i,j=1}^{l} a_{i,j}(y)\xi_i\xi_j \geq H|\xi|^2$, $x \in D, \xi \in \mathbb{R}^l$.

$B_2:$ there exists a nonnegative C^2-function V such that $\mathcal{L}V$ is negative for any $E_l \backslash D$.

Then

$$\mathbb{P}\Big\{ \lim_{T\to\infty} \frac{1}{T} \int_0^T f(Y(s))ds = \int_{R^l} f(y)\mu(dy) \Big\} = 1$$

for all $y \in E_l$, where $f(\cdot)$ is function integrable with respect to the measure μ.

Theorem 4.1. *Assume that* $k_1 = \frac{b\varphi'(K)}{\varepsilon}$, $k_2 = \max\left\{0, -\frac{bKc_3}{2\varepsilon} + k_1\right\}$ *and* $A = -\gamma - \frac{\sigma_2^2}{2} + b\varphi(K) - \frac{Kk_2}{2}\sigma_1^2 > 0$, *then the model (1.3) admits a unique stationary distribution and has the ergodic property.*

Proof. It follows from Theorem 2.1 that for any initial value $(N(0), P(0)) \in \mathbb{R}_+^2$, there exists a unique global positive solution $(N(t), P(t))$. In the following analysis, for the simplification, we denote $N(t)$, $P(t)$ as N, P respectively.

Define a C^2-function

$$\begin{aligned} V(t, N, P) =& M\Big(- \log P - k_1 N + k_2 \Big(N - K - K \log \frac{N}{K} \Big) + \frac{aKc_1k_2 + \delta - \gamma}{\gamma} P \Big) \\ &+ \frac{\Big(N + \frac{a}{b} P \Big)^{\theta+1}}{\theta + 1} \\ =& MV_1(N, P) + V_2(N, P), \end{aligned}$$

here $\theta \in (0, 1)$, M are positive constants satisfying the following conditions respectively

$$\frac{\theta}{2}\sigma_2^2 < (\delta - \gamma), \tag{4.2}$$

$$- AM + f^u + g^u \leq -2, \tag{4.3}$$

and positive constant A and functions $f(x), g(x)$ will be determined later. Applying

Itô's formula, we obtain

$$
\begin{aligned}
\mathcal{L}(-\log P) =& \frac{\gamma+\delta P}{1+P} - b\varphi(N) + \frac{\sigma_2^2}{2} \\
=& \gamma + \frac{\sigma_2^2}{2} + \frac{(\delta-\gamma)P}{1+P} - b\varphi(N) \\
\leq& \gamma + \frac{\sigma_2^2}{2} + (\delta-\gamma)P - b\varphi(N), \\
\mathcal{L}(-N) =& -\varepsilon N\Big(1-\frac{N}{K}\Big) + a\varphi(N)P, \\
\mathcal{L}(P) =& -\frac{\gamma+\delta P}{1+P}P + b\varphi(N)P,
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{L}\left(N-K-K\log\frac{N}{K}\right) &= \left(1-\frac{K}{N}\right)\left(\varepsilon N\Big(1-\frac{N}{K}\Big) - a\varphi(N)P\right) + \frac{K}{2}\sigma_1^2 \\
&= -\frac{\varepsilon}{K}(N-K)^2 - a\varphi(N)P + aK\frac{\varphi(N)}{N}P + \frac{K}{2}\sigma_1^2.
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}V_1(N,P) =& \gamma + \frac{\sigma_2^2}{2} + \frac{(\delta-\gamma)P}{1+P} - b\varphi(N) - k_1\varepsilon N\Big(1-\frac{N}{K}\Big) + ak_1\varphi(N)P \\
&+ k_2\left(-\frac{\varepsilon}{K}(N-K)^2 - a\varphi(N)P + aK\frac{\varphi(N)}{N}P + \frac{K}{2}\sigma_1^2\right) \\
&+ \frac{aKc_1k_2+\delta-\gamma}{\gamma}\Big(-\frac{\gamma+\delta P}{1+P}P + b\varphi(N)P\Big) \\
\leq& \gamma + \frac{\sigma_2^2}{2} + (\delta-\gamma)P - b\varphi(N) - \frac{k_1\varepsilon}{K}N(K-N) + ak_1\varphi(N)P \\
&+ k_2\Big(-\frac{\varepsilon}{K}(N-K)^2 + aKc_1P + \frac{K}{2}\sigma_1^2\Big) \\
&+ \frac{aKc_1k_2+\delta-\gamma}{\gamma}(-\gamma P + b\varphi(N)P) \\
\leq& \gamma + \frac{\sigma_2^2}{2} - b\varphi(K) + \frac{Kk_2}{2}\sigma_1^2 \\
&+ \Big(-b\varphi(N) + b\varphi(K) - \frac{k_1\varepsilon}{K}N(K-N) - \frac{k_2\varepsilon}{K}(N-K)^2\Big) \\
&+ \Big(ak_1 + \frac{aKc_1k_2+\delta-\gamma}{\gamma}b\Big)\varphi(N)P \\
=& -A + F(N) + \Big(ak_1 + \frac{aKc_1k_2+\delta-\gamma}{\gamma}b\Big)\varphi(N)P, \qquad (4.4)
\end{aligned}
$$

where

$$
-A = \gamma + \frac{\sigma_2^2}{2} - b\varphi(K) + \frac{Kk_2}{2}\sigma_1^2,
$$

and

$$
F(N) = -b\varphi(N) + b\varphi(K) - \frac{k_1\varepsilon}{K}N(K-N) - \frac{k_2\varepsilon}{K}(N-K)^2.
$$

Then let us calculate

$$
F'(N) = -b\varphi'(N) + \frac{k_1\varepsilon}{K}(2N-K) - \frac{2k_2\varepsilon}{K}(N-K),
$$

and

$$F''(N) = -b\varphi''(N) + \frac{2k_1\varepsilon}{K} - \frac{2k_2\varepsilon}{K}.$$

Let $k_1 = \frac{b\varphi'(K)}{\varepsilon}$ and $k_2 = \max\left\{0, -\frac{bKc_3}{2\varepsilon} + k_1\right\}$. So we get

$$F'(K) = 0,$$

$$F''(N) \le -bc_3 + \frac{2k_1\varepsilon}{K} - \frac{2k_2\varepsilon}{K} < 0.$$

Thus, we can have

$$F(N) \le F(K) = 0.$$

This, together with (4.2), implies that

$$\begin{aligned} \mathcal{L}V_1(N,P) \le& - A + \Big(ak_1 + \frac{aKc_1k_2 + \delta - \gamma}{\gamma}b\Big)\varphi(N)P \\ \le& - A + k_3\varphi(N)P \\ \le& - A + k_3c_2P. \end{aligned} \tag{4.5}$$

where k_3 is a positive constant and $k_3 \ge ak_1 + \frac{aKc_1k_2+\delta-\gamma}{\gamma}b$. Also

$$\begin{aligned} \mathcal{L}V_2(N,P) =& \Big(N + \frac{a}{b}P\Big)^{\theta}\Big(\varepsilon N\Big(1 - \frac{N}{K}\Big) - a\varphi(N)P + \frac{a}{b}P\Big(- \frac{\gamma + \delta P}{1+P} + b\varphi(N)\Big)\Big) \\ &+ \frac{\theta}{2}\Big(N + \frac{a}{b}P\Big)^{\theta-1}\Big(\sigma_1^2N^2 + \frac{a^2}{b^2}\sigma_2^2P^2\Big) \\ \le& \Big(N + \frac{a}{b}P\Big)^{\theta}\Big(\varepsilon N - \frac{\varepsilon}{K}N^2 - \frac{a}{b}\frac{(\delta-\gamma)P^2}{1+P}\Big) \\ &+ \frac{\theta}{2}\Big(N + \frac{a}{b}P\Big)^{\theta-1}\Big(\sigma_1^2N^2 + \frac{a^2}{b^2}\sigma_2^2P^2\Big) \\ \le& \Big(N + \frac{a}{b}P\Big)^{\theta}\varepsilon N - \Big(N + \frac{a}{b}P\Big)^{\theta}\frac{\varepsilon}{K}N^2 - \Big(N + \frac{a}{b}P\Big)^{\theta}\frac{a}{b}\frac{(\delta-\gamma)P^2}{1+P} \\ &+ \frac{\theta}{2}\Big(N + \frac{a}{b}P\Big)^{\theta-1}\Big(\sigma_1^2N^2 + \frac{a^2}{b^2}\sigma_2^2P^2\Big) \\ \le& 2^{\theta}\varepsilon N\left(N^{\theta} + \Big(\frac{a}{b}P\Big)^{\theta}\right) - \frac{\varepsilon}{K}N^{2+\theta} - \Big(\frac{a}{b}\Big)^{\theta+1}(\delta-\gamma)\frac{P^{2+\theta}}{1+P} \\ &+ \frac{\theta}{2}\sigma_1^2N^{1+\theta} + \frac{\theta}{2}\Big(\frac{a}{b}\Big)^{\theta+1}\sigma_2^2P^{1+\theta} \\ \le& 2^{\theta}\varepsilon N^{1+\theta} + 2^{\theta-1}\varepsilon a^{\theta}N^2 - \frac{\varepsilon}{K}N^{2+\theta} + \frac{\theta}{2}\sigma_1^2N^{1+\theta} \\ &+ 2^{\theta-1}\varepsilon\Big(\frac{a}{b}\Big)^{\theta}P^{2\theta} + \frac{\theta}{2}\Big(\frac{a}{b}\Big)^{\theta+1}\sigma_2^2P^{1+\theta} - \Big(\frac{a}{b}\Big)^{\theta+1}(\delta-\gamma)\frac{P^{2+\theta}}{1+P} \\ =&: f(N) + g(P). \end{aligned} \tag{4.6}$$

Clearly

$$f(N) \to -\infty,\ as\ N \to +\infty.$$

Applying inequalities $0 < \theta < 1$ and (4.2) yields

$$g(P) \to -\infty,\ as\ P \to +\infty.$$

From (4.5) and (4.6), we obtain

$$\mathcal{L}V(N,P) \le M(-A + k_3c_2P) + f(N) + g(P),$$

where M satisfy

$$-AM + f^u + g^u < -2.$$

To confirm condition B_2 in Lemma 4.1, we consider the following bounded subset

$$D_{\epsilon_1} = \Big\{\epsilon_1 \le N \le \frac{1}{\epsilon_1},\ \epsilon_1 \le P \le \frac{1}{\epsilon_1}\Big\}$$

where $0 < \epsilon_1 < 1$ is a sufficiently small constant. In the set $\mathbb{R}_+^2 \backslash D_{\epsilon_1}$, we can choose ϵ_1 sufficiently small such that the following conditions hold

$$-MA + Mk_3c_2\epsilon_1 + f^u + g^u \le -1, \tag{4.7}$$

$$-MA + (Mk_3c_2P + g(P))^u + f^u \le -1, \tag{4.8}$$

$$-MA + f^u + B - \Big(\frac{a}{b}\Big)^{\theta+1}\frac{\eta}{2}\frac{1}{\epsilon_1^{1+\theta}} \le -1, \tag{4.9}$$

$$-MA + (Mk_3c_2P + g(P))^u + C - \frac{\varepsilon}{2K}\frac{1}{\epsilon_1^{2+\theta}} < -1, \tag{4.10}$$

where inequality (4.7) can be derived from (4.3), the constants η, B and C will be determined later. Then

$$\mathbb{R}_+^2 \backslash D_{\epsilon_1} = D_1 \cup D_2 \cup D_3 \cup D_4,$$

with

$$\begin{aligned} D_1 =& \Big\{(N,P) \in \mathbb{R}_+^2 \mid 0 < P < \epsilon_1\Big\},\ D_2 = \Big\{(N,P) \in \mathbb{R}_+^2 \mid 0 < N < \epsilon_1\Big\},\\ D_3 =& \Big\{(N,P) \in \mathbb{R}_+^2 \mid P > \frac{1}{\epsilon_1}\Big\},\ D_4 = \Big\{(N,P) \in \mathbb{R}_+^2 \mid N > \frac{1}{\epsilon_1}\Big\}. \end{aligned}$$

Case 1. If $(N,P) \in D_1$, (4.7) implies that

$$\mathcal{L}V \le M(-A + k_3c_2P) + f(N) + g(P) \le -MA + Mk_3c_2\epsilon_1 + f^u + g^u \le -1.$$

Case 2. If $(N,P) \in D_2$, we obtain that

$$\mathcal{L}V \le M(-A + k_3c_2P) + f(N) + g(P) \le -MA + (Mk_3c_2P + g(P))^u + f^u \le -1.$$

Case 3. If $(N,P) \in D_3$, we have

$$\mathcal{L}V \le -MA + f^u + B - \eta\Big(\frac{a}{b}\Big)^{\theta+1}\frac{P^{2+\theta}}{1+P} \le -MA + f^u + B - \Big(\frac{a}{b}\Big)^{\theta+1}\frac{\eta}{2}\frac{1}{\epsilon_1^{1+\theta}} \le -1,$$

which follow from (4.9), where η and B satisfy $\frac{\theta}{2}\sigma_2^2 < (\delta - \gamma) - \eta$ and

$$B = \sup_{P \in (0,\infty)} \Big\{Mk_3c_2P + 2^{\theta-1}\varepsilon\Big(\frac{a}{b}\Big)^{\theta}P^{2\theta} + \frac{\theta}{2}\Big(\frac{a}{b}\Big)^{\theta+1}\sigma_2^2P^{1+\theta} - \Big(\frac{a}{b}\Big)^{\theta+1}(\delta - \gamma - \eta)\frac{P^{2+\theta}}{1+P}\Big\} < \infty.$$

Case 4. If $(N,P) \in D_4$, we have by (4.10)

$$\mathcal{L}V \leq -MA+Mk_3c_2P+g(P)+C-\frac{\varepsilon}{2K}N^{2+\theta} \leq -MA+(Mk_3c_2P+g(P))^u+C-\frac{\varepsilon}{2K}\frac{1}{\epsilon_1^{2+\theta}} < -1,$$

where

$$C = \sup_{N\in(0,\infty)}\left\{2^\theta\varepsilon N^{1+\theta}+2^{\theta-1}\varepsilon a^\theta N^2-\frac{\varepsilon}{K}\frac{N^{2+\theta}}{2}+\frac{\theta}{2}\sigma_1^2N^{1+\theta}\right\} < \infty.$$

From the above discussion it follows that

$$\mathcal{L}V \leq -1,\ (N,P) \in \mathbb{R}_+^2\backslash D_{\epsilon_1}.$$

Thus, the condition B_2 in Lemma 4.1 satisfied. We take $\overline{D}_\sigma$ to be a neighborhood of D_{ϵ_1} with $\overline{D}_\sigma \subset \mathbb{R}_+^2$. It is obvious that there is $H = \min_{(N,P)\in\overline{D}_\sigma}\{\sigma_1^2N^2, \sigma_2^2P^2\} > 0$, such that

$$\sum_{i,j=1}^2 e_{i,j}(y)\xi_i\xi_j = \sigma_1^2N^2\xi_1^2+\sigma_2^2P^2\xi_2^2 \geq H|\xi|^2,\ (N,P) \in \overline{D}_\sigma,\ \xi \in \mathbb{R}_+^2.$$

Then the condition B_1 in Lemma 4.1 holds. According to Lemma 4.1, we know that the model (1.3) has a unique stationary distribution. The proof is completed. □

5. Numerical simulations

In this section, in order to verify the above results, we will numerically simulate the solution of system (1.3) based on the Milstein's Higher Order Method proposed by Higham [24]. At the same time, we will select a functional response for numerical simulation. we choose the function (1.6), so the model (1.3) will be transformed into the following form:

$$\begin{cases} dN = \left(\varepsilon N(t)\left(1-\frac{N(t)}{K}\right)-\frac{amP(t)N(t)^2}{\alpha N(t)^2+\beta N(t)+1}\right)dt+\sigma_1N(t)dB_1(t), \\ dP = P(t)\left(-\frac{\gamma+\delta P(t)}{1+P(t)}+\frac{bmN(t)^2}{\alpha N(t)^2+\beta N(t)+1}\right)dt+\sigma_2P(t)dB_2(t). \end{cases} \tag{5.1}$$

Example 5.1. In autonomous stochastic model (5.1), let the parameters be

$$\varepsilon = 0.08,\ K = 100,\ a = 1,\ \alpha = 2, \quad m = 1,\ \beta = 0,\ \gamma = 0.2,\ \delta = 0.4,\ b = 0.9.$$

and the initial value $(N(0),P(0)) = (0.9, 0.6)$.

Case 1. Let the environmental noise intensities be $\sigma_1 = \sigma_2 = 0.1$. Then $\varepsilon > \sigma_1^2/2$ and we will use Theorem 3.1 to verify.

$$\lambda = -\gamma-\frac{\sigma_2^2}{2}+b\int_0^\infty \varphi(x)\mu(x)dx = 0.2450 > 0.$$

It can be seen from Theorem 3.1 that the stochastic model (5.1) is persistent in the mean. As shown in Figure 1.

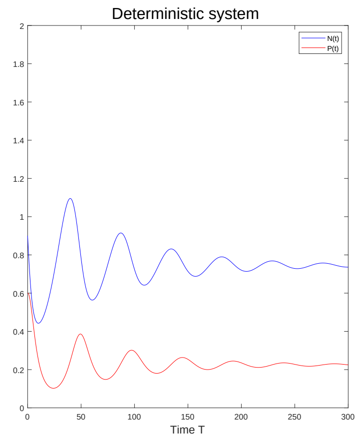

Figure 1. The left figure is the solution $(N(t), P(t))$ of deterministic model (1.2). The right figure is the solution of autonomous stochastic model (5.1) with $\sigma_1 = \sigma_2 = 0.1$.

Case 2. We choose environment noise $\sigma_1 = 0.1$, $\sigma_2 = 0.8$. Then $\varepsilon > \sigma_1^2/2$ and

$$\lambda = -\gamma - \frac{\sigma_2^2}{2} + b\int_0^\infty \varphi(x)\mu(x)dx = -0.07003 < 0.$$

According to the result of Theorem 3.1, we can know that the predator populations go to extinction and the prey is persistent in the mean. As shown in Figure 2. Hence large environmental noise can make population species extinct.

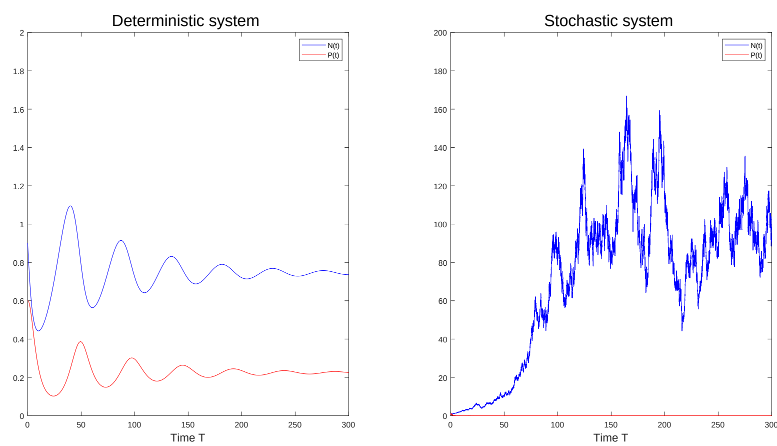

Figure 2. The left figure is the solution $(N(t), P(t))$ of deterministic model (1.2). The right figure is the solution of autonomous stochastic model (5.1)with $\sigma_1 = 0.1$ and $\sigma_2 = 0.8$.

Case 3. Let the environmental noise intensities be $\sigma_1 = \sigma_2 = 0.05$. Then

$$A = -\gamma - \frac{\sigma_2^2}{2} + b\varphi(K) - \frac{Kk_2}{2}\sigma_1^2 = 0.2487 > 0$$

Theorem 4.1 means that stochastic system (5.1) admits a unique stationary distribution, Figure 3 confirms this.

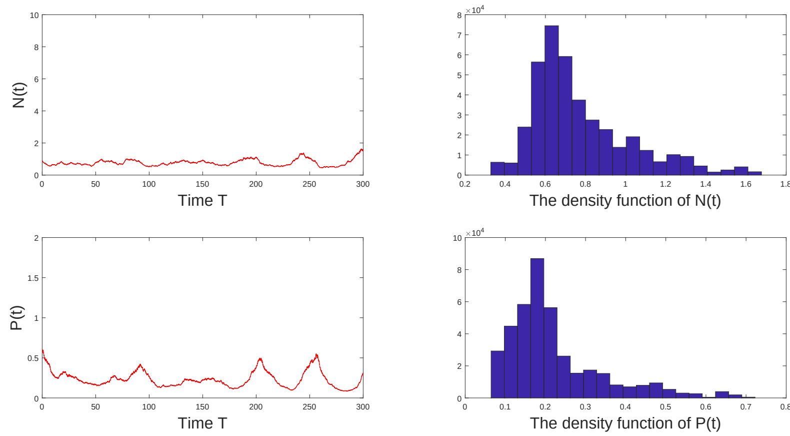

Figure 3. The left figure is the solution $(N(t), P(t))$ of autonomous stochastic model (1.2). The right figure is density function diagrams of $(N(t), P(t))$ with $\sigma_1 = \sigma_2 = 0.05$.

6. Conclusion

The present paper is concerned with the dynamics of stochastic predator-prey models with non-constant mortality rate and general nonlinear functional response. By constructing suitable stochastic Lyapunov functions, we establish sufficient conditions for persistence in the mean and extinction of system (1.3). In addition, we also establish sufficient condition for the existence of ergodic stationary distribution to the stochastic system (1.3). In addition, the predator-prey models may be disturbed by the coloured noise, that is, the telegraph noise which can make the system switch from one environmental regime to another. This noise means a random switching between two or more environmental regimes will distinguished by factors such as nutrition and rainfall. For this problem, we can construct a new model. We will leave these investigations for future work.

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