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Existence of Positive Solutions for a Nonlinear Second Order Periodic Boundary Value Problem*

Ping Liu[1], Yonghong Fan[1,†] and Linlin Wang[1]

Abstract By using the first eigenvalue corresponding to the relevant linear operator and the topological degree theorem, sufficient conditions for the existence of positive solutions for a nonlinear second order periodic boundary value problem are given. Our results improve and generalize some preliminary works.

Keywords Positive solutions, First positive eigenvalue, Green's function, Topological degree.

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1. Introduction

In recent years, due to the widespread applications in the field of physics and engineering, the study of the existence of the positive solutions for second-order differential equations has attracted the attention of many scholars [2, 9, 11].

In [12], Nieto studied the periodic boundary value problem for the second order differential equation

$$\begin{cases} -x'' = f(t, x(t)), \quad t \in [0, 2\pi], \\ x(0) = x(2\pi), \quad x'(0) = x'(2\pi), \end{cases}$$

where f satisfies Carath*é*odory conditions. Their main method is the upper and lower solutions.

In [13], by using the Krasnoselskii fixed point theorem, Torres obtained the existence of solutions to the following periodic boundary value problem

$$\begin{cases} x'' = f(t, x(t)), \quad t \in [0, T], \\ x(0) = x(T), \quad x'(0) = x'(T), \end{cases}$$

where f is also a function of L^1-Carath*é*odory type and T-periodic in t.

†the corresponding author.
Email address: 791821318@qq.com(P. Liu), fanyh_1993@sina.com(Y. Fan), wangll_1994@sina.com(L. Wang)
1School of Mathematics and Statistics Science, Ludong University, Yantai, Shandong 264025, China
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In [4], Jiang studied the existence of the positive solutions to the following equation

$$\begin{cases} x'' + Mx = f(t, x(t)), \quad t \in [0, 2\pi], \\ x(0) = x(2\pi), \quad x'(0) = x'(2\pi), \end{cases}$$

where $f \in C([0, 2\pi] \times \mathbb{R}^+, \mathbb{R}^+)$ and $M > 0$. The main method is Krasnoselskii fixed point theorem.

For the following periodic boundary value problem

$$\begin{cases} x'' + a(t)x = f(t, x(t)), \quad t \in [0, T], \\ x(0) = x(T), \quad x'(0) = x'(T), \end{cases} \tag{1.1}$$

when f is nonnegative, Li [8] obtained the existence of positive solutions for Eq.(1.1) by using the Krasnoselskii fixed point theorem, Li and Liang [7] also established the existence of the positive solutions for Eq.(1.1) by using the fixed point index theory on a cone. Moreover, in [10], the authors investigated the existence of the positive solutions for Eq.(1.1) under the condition that f may take negative values and the nonlinearity may be sign-changing.

Motivated by the above papers, in this paper, we study the existence of the positive solutions for the following second order periodic boundary value problem

$$\begin{cases} x'' + h\left(t\right)x' + a(t)x = g(t)f(t, x), \\ x(0) = x(T), \quad x'(0) = x'(T), \end{cases} \tag{1.2}$$

where $h \in C([0, T], \mathbb{R}^+)$, $a \in C([0, T], \mathbb{R}^+)$ and $a \not\equiv 0$, $g \in C((0, T), \mathbb{R}^+) \cap L[0, T]$ and $\int_0^T g(t)dt > 0$, $f \in C([0, T] \times \mathbb{R}, \mathbb{R})$, in which $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$. In particular, the function g may be singular at $t = 0$ or $t = T$, f may take negative values and the nonlinearity may be sign-changing. Moreover, when $h(t) \equiv 0$, $g(t) \equiv 1$, Eq.(1.2) becomes Eq.(1.1).

Three highlights should be pointed out. The damping term $h(t)x'$ has been added to generalize the previous equations, g may be singular at $t = 0$ or $t = T$ and f can take negative values and be sign-changing.

The paper is organized as follows. Some useful lemmas for the proof of the main results are given in Section 2. The main results will be given and proved in Section 3. Two examples are given to support our main results in Section 4.

2. Preliminaries

We say the linear system

$$x'' + h\left(t\right)x' + a(t)x = 0, \tag{2.1}$$

associated to periodic boundary conditions

$$x(0) = x(T), \quad x'(0) = x'(T) \tag{2.2}$$

is nonresonant when its unique solution is the trivial one. If (2.1)-(2.2) is nonresonant, as a consequence of Fredholm's alternative theorem, the nonhomogeneous equation

$$x'' + h(t)x' + a(t)x = l(t) \tag{2.3}$$

admits a unique solution which can be written as

$$x(t) = \int_0^T G(t,s)l(s)ds,$$

where $G(t,s)$ is the Green's function of (2.1)-(2.2).

Now we assume that:

$(A0)$ The Green's function $G(t,s)$, associated with (2.1)-(2.2), is positive for all $(t,s) \in [0,T] \times [0,T]$.

For the general case, it is difficult to verify that condition $(A0)$ holds. However, by the following definition, we can get that $G(t,s)$ is non-negative.

Definition 2.1. We say that (2.1)-(2.2) admits the anti-maximum principle if (2.3)-(2.2) has a unique solution for any $l \in C([0,T],\mathbb{R})$ and the unique solution x_l of (2.3)-(2.2) satisfies $x_l(t) > 0$ for all $t \in [0,T]$ if $l \geq 0$ and $l \not\equiv 0$.

We can apply the anti-maximum principle to prove the existence of a solution to an abstract nonlinear second order periodic boundary value problem. Moreover, we can apply an explicit criterion in [1] obtained by Chu, Fan and Torres to ensure that condition $(A0)$ holds, which is obtained by the anti-maximum principle established by Hakl and Torres (see [3]).

Define the functions

$$\sigma(h)(t) = \exp(\int_0^t h(s)ds), \qquad \sigma_1(h)(t) = \sigma(h)(T)\int_0^t \sigma(h)(s)ds + \int_t^T \sigma(h)(s)ds.$$

Lemma 2.1 (Corollary 2.6, [1])**.** *If $a \not\equiv 0$ and the following two inequalities*

$$\int_0^T a(s)\sigma(h)(s)\sigma_1(-h)(s)ds \geq 0, \tag{H1}$$

and

$$\sup_{0\leq t\leq T}\{\int_t^{t+T}\sigma(-h)(s)ds\int_t^{t+T}[a(s)]_+\sigma(h)(s)ds\} \leq 4. \tag{H2}$$

are satisfied, where $[a(s)]_+ = \max\{a(s),0\}$. Then $(A0)$ holds.

When $(A0)$ holds, we always denote

$$m = \min_{0\leq s,t\leq T} G(t,s), \quad M = \max_{0\leq s,t\leq T} G(t,s). \tag{2.4}$$

Obviously $M > m > 0$.

Lemma 2.2 (Theorem 20.10, [5])**.** *Let E be a real Banach space and $\Omega \subset E$ be a bounded open set with $0 \in \Omega$. Suppose that $A : \overline{\Omega} \to E$ is a completely continuous operator. If there is $y_0 \in E$ with $y_0 \neq 0$ such that $x \neq Ax + \lambda y_0$, for all $x \in \partial\Omega$ and $\lambda \geq 0$, then $\deg(I - A, \Omega, \theta) = 0$, where deg stands for the Leray-Schauder topological degree in E.*

Let

$$E = C([0,T],\ \mathbb{R}), \quad \| x \| = \max\{|x(t)| \,\big|\, x(t) \in E,\ t \in [0,T]\},$$

$$P = \{x(t) \in E : x(t) \geq 0,\ \forall t \in [0,T]\}.$$

Clearly, $(E, \| \cdot \|)$ is a real Banach space and P is a totally positive cone of E. Denote the dual space of E by E^* and the dual cone of P by P^*. Then

$E^* = \{y : y$ is right continuous on $[0,T)$ and is of bounded variation on $[0,T]$ with $y(0) = 0\}$,

$P^* = \{y \in E^* : y$ is nondecreasing on $[0,T]\}$.

Moreover, the bounded linear functional on E can be represented in the Riemann-Stieltjes integral

$$< y, x >= \int_0^T x(t)dy(t), \quad x \in E, \quad y \in E^*.$$

Define an operator A by

$$(Ax)(t) = \int_0^T G(t,s)g(s)f(s,x(s))ds, \quad x \in E. \tag{2.5}$$

Clearly, $A : E \to E$ is a completely continuous nonlinear operator, it is easy to verify that a positive solution of (1.2) is just a fixed point of the operator equation $x = Ax$.

Moreover, define an operator L by

$$(Lx)(t) = \int_0^T G(t,s)g(s)x(s)ds, \quad x \in E. \tag{2.6}$$

Clearly, $L : E \to E$ is a completely continuous linear operator, satisfying $L(P) \subset P$ and $L(P\backslash\{0\}) \subset \mathrm{int}P$. That is, L is a strongly positive, completely continuous, linear operator. Moreover, since $G(t,s)$ is positive, $g \in C((0,T), \mathbb{R}^+) \cap L[0,T]$ and $\int_0^T g(t)dt > 0$, the spectral radius $r(L)$ of the operator L is positive [14].

Let $L^* : E^* \to E^*$ be the dual operator of L, given by

$$(L^*y)(s) = \int_0^s \int_0^T G(t,\tau)g(\tau)dy(t)d\tau, \quad y \in E^*. \tag{2.7}$$

In order to obtain the properties of L and L^*, next we recall the Krein-Rutman Theorem [6].

Lemma 2.3 (Krein-Rutman theorem [6])**.** *Let P be a cone, and L is a completely continuous linear operator strongly positive with respect to P, then $r(L)$ is an eigenvalue of L and L^* with eigenvectors in $P\backslash\{0\}$ and $P^*\backslash\{0\}$.*

By Lemma 2.3, we have $p \in P\backslash\{0\}$ and $w \in P^*\backslash\{0\}$ such that

$$Lp = r(L)p \tag{2.8}$$

and

$$L^*w = r(L)w, \quad w(T) = 1. \tag{2.9}$$

From the definition of L^*, the continuity of G and the integrability of g, we have $w \in C^1[0,T]$. Denote $w'(t) = q(t)$, then $q \in P\backslash\{0\}$, and (2.9) can be written in the following equivalent form

$$r(L)q(s) = \int_0^T G(t,s)g(s)q(t)dt, \quad \int_0^T q(t)dt = 1. \tag{2.10}$$

Lemma 2.4. *Assume P_0 is the subcone of P, given by*

$$P_0 = \{x \in P : \int_0^T x(t)q(t)dt \geq \delta \parallel x \parallel\},$$

where $\delta = \frac{m}{M} \int_0^T q(t)dt = \frac{m}{M}$, *then* $L(P) \subset P_0$.

Proof. Since

$$\int_0^T (Lx)(t)q(t)dt = \int_0^T \int_0^T G(t,s)g(s)x(s)q(t)dsdt \geq m \int_0^T \int_0^T g(s)x(s)q(t)dsdt$$

and

$$\parallel Lx \parallel = \max_{0 \leq t \leq T} \mid \int_0^T G(t,s)g(s)x(s)ds \mid \leq M \int_0^T g(s)x(s)ds,$$

we have

$$\delta \parallel Lx \parallel \leq m \int_0^T \int_0^T g(s)x(s)q(t)dsdt \leq \int_0^T (Lx)(t)q(t)dt.$$

The proof is complete. □

3. Main results

Let $\lambda_1 = 1/r(L)$, then λ_1 is the first positive eigenvalue of the eigenvalue problem

$$\begin{cases} x'' + h\,(t)\,x' + a(t)x = \lambda g(t)x, \quad t \in [0,T], \\ x(0) = x(T), \quad x'(0) = x'(T). \end{cases}$$

We list the following assumptions on f:

$(A1)$ $\liminf_{x \to +\infty} \frac{f(t,x)}{x} > \lambda_1$, uniformly on $t \in [0,T]$; $\limsup_{x \to -\infty} \frac{f(t,x)}{x} < \lambda_1$, uniformly on $t \in [0,T]$.

$(A2)$ $\limsup_{x \to 0} \frac{|f(t,x)|}{|x|} < \lambda_1$, uniformly on $t \in [0,T]$.

$(A3)$ $\limsup_{x \to \infty} \frac{|f(t,x)|}{|x|} < \lambda_1$, uniformly on $t \in [0,T]$.

$(A4)$ $\liminf_{x \to 0^+} \frac{f(t,x)}{x} > \lambda_1$, uniformly on $t \in [0,T]$; $\limsup_{x \to 0^-} \frac{f(t,x)}{x} < \lambda_1$, uniformly on $t \in [0,T]$.

Theorem 3.1. *If* $(A0) - (A2)$ *hold, then* (1.2) *has at least one positive solution.*

Proof. $(A1)$ implies that there are $\varepsilon \in (0, \lambda_1)$ and $C_1 > 0$ such that

$$f(t,x) \geq (\lambda_1 + \varepsilon)x - C_1, \quad \forall x \geq 0, \quad t \in [0,T], \tag{3.1}$$

and

$$f(t,x) \geq (\lambda_1 - \varepsilon)x - C_1, \quad \forall x \leq 0, \quad t \in [0,T]. \tag{3.2}$$

The above inequalities imply that

$$f(t,x) \geq (\lambda_1 + \varepsilon)x - C_1 \geq (\lambda_1 - \varepsilon)x - C_1,$$

if $(t,x) \in [0,T] \times [0,+\infty)$, and

$$f(t,x) \geq (\lambda_1 - \varepsilon)x - C_1 \geq (\lambda_1 + \varepsilon)x - C_1,$$

if $(t,x)\in[0,T]\times(-\infty,0]$. Thus, we have

$$f(t,x)\geq(\lambda_1+\varepsilon)x-C_1,\quad \forall x\in\mathbb{R},\quad t\in[0,T]. \tag{3.3}$$

and

$$f(t,x)\geq(\lambda_1-\varepsilon)x-C_1,\quad \forall x\in\mathbb{R},\quad t\in[0,T]. \tag{3.4}$$

Let

$$M_1=\{x\in E: \text{there exists some } \sigma\geq 0 \text{ such that } x=Ax+\sigma p\},$$

where $p\in P\backslash\{0\}$ is given by (2.8). Next we prove that M_1 is bounded on E. From the definition of M_1, if $x_0\in M_1$, there exists $\sigma_0\geq 0$, such that

$$x_0(t)=(Ax_0)(t)+\sigma_0 p(t)=\int_0^T G(t,s)g(s)f(s,x_0(s))ds+\sigma_0 p(t). \tag{3.5}$$

Combining (3.5) with (3.3), we have

$$x_0(t)\geq(\lambda_1+\varepsilon)(Lx_0)(t)-C_1(L\mathbf{1})(t), \tag{3.6}$$

where $\mathbf{1}$ refers to the constant function $\mathbf{1}(t)\equiv 1$, $\forall t\in[0,T]$. Multiply $q(t)$ on both sides of (3.6) and integrate over $[0,T]$, also in view of (2.10), we can obtain

$$\int_0^T x_0(t)q(t)dt\geq[1+\varepsilon r(L)]\int_0^T x_0(t)q(t)dt-C_1 r(L),$$

where $q(t)$ is given by (2.10), thus

$$\int_0^T x_0(t)q(t)dt\leq\frac{C_1}{\varepsilon}. \tag{3.7}$$

Moreover, (3.5) is equivalent to

$$\begin{aligned}x_0(t)-(\lambda_1-\varepsilon)(Lx_0)(t)+C_1(L\mathbf{1})(t)=&(L[Fx_0-(\lambda_1-\varepsilon)x_0+C_1\mathbf{1}])(t)+\sigma_0 p(t)\\=&(L[Fx_0-(\lambda_1-\varepsilon)x_0+C_1\mathbf{1}])(t)+\sigma_0\lambda_1(Lp)(t)\\=&(L[Fx_0-(\lambda_1-\varepsilon)x_0+C_1\mathbf{1}+\sigma_0\lambda_1 p])(t),\end{aligned}$$

where $Fx(t)=f(t,x(t))$. Since (3.4) holds, $(Fx_0-(\lambda_1-\varepsilon)x_0+C_1)(t)\in P$, that is, $(Fx_0-(\lambda_1-\varepsilon)x_0+C_1\mathbf{1}+\sigma_0\lambda_1 p)(t)\in P$. By Lemma 2.4, we obtain

$$x_0(t)-(\lambda_1-\varepsilon)(Lx_0)(t)+C_1(L\mathbf{1})(t)\in P_0.$$

Thus,

$$\begin{aligned}&\parallel x_0(t)-(\lambda_1-\varepsilon)(Lx_0)(t)+C_1(L\mathbf{1})(t)\parallel\\&\quad\leq\frac{1}{\delta}\int_0^T[x_0(t)-(\lambda_1-\varepsilon)(Lx_0)(t)+C_1(L\mathbf{1})(t)]q(t)dt\\&\quad=\frac{r(L)\varepsilon}{\delta}\int_0^T x_0(t)q(t)dt+\frac{C_1 r(L)}{\delta}\\&\quad\leq\frac{2C_1 r(L)}{\delta}.\end{aligned}$$

Since $(\lambda_1-\varepsilon)r(L)<1$, the operator $I-(\lambda_1-\varepsilon)L$ has the bounded inverse operator $(I-(\lambda_1-\varepsilon)L)^{-1}$. Thus, there exists $Q>0$ such that $\parallel x\parallel\leq Q$, for all $x\in M_1$. Therefore, M_1 is bounded. For each $R>\sup\limits_{x\in M_1}\parallel x\parallel$, we have

$$x\neq Ax+\sigma p,\quad \forall x\in\partial B_R,\quad \forall\sigma\geq 0,$$

where $B_R = \{x \in E : \| x \| < R\}$. By Lemma 2.2, we obtain

$$\deg(I - A, B_R, \theta) = 0. \tag{3.8}$$

On the other hand, $(A2)$ shows that there are $\rho \in (0, \lambda_1)$ and $r > 0$, such that

$$|f(t,x)| \leq (\lambda_1 - \rho)|x|, \quad \forall |x| \leq r, \quad t \in [0,T].$$

Moreover, we may choose $r > 0$ such that $r < \sup\limits_{x \in M_1} \| x \|$. Therefore

$$|(Ax)(t)| \leq (\lambda_1 - \rho)(L|x|)(t), \quad \forall x \in \overline{B_r}, \quad t \in [0,T].$$

Next, we need to prove that

$$x \neq \mu Ax, \quad \forall x \in \partial B_r, \quad \mu \in [0,1]. \tag{3.9}$$

If not, then there are $x_0 \in \partial B_r$ and $\mu_0 \in [0,1]$ such that $x_0 = \mu_0 Ax_0$. Let $v(t) = |x_0(t)|$, then $v \in P$ and $v \leq \mu_0(\lambda_1 - \rho)Lv \leq (\lambda_1 - \rho)Lv$. Obviously L is a linear increasing operator, thus, $Lv \leq (\lambda_1 - \rho)L^2 v$, so $v \leq (\lambda_1 - \rho)^2 L^2 v$, thence, the nth iteration of this inequality implies that $v \leq (\lambda_1 - \rho)^n L^n v (n = 2, 3, ...)$, therefore, $\| v \| \leq (\lambda_1 - \rho)^n \| L^n \| \| v \|$, where $\| L \| = \sup\limits_{x \neq 0, x \in E} \frac{\|Lx\|}{\|x\|}$. Thus, we have $1 \leq (\lambda_1 - \rho)^n \| L^n \|$, which means

$$(\lambda_1 - \rho)r(L) = (\lambda_1 - \rho)\lim_{n \to \infty} \sqrt[n]{\| L^n \|} \geq 1,$$

however, $(\lambda_1 - \rho)r(L) = 1 - \rho r(L) < 1$. Therefore, (3.9) holds. So I and $I - A$ are homotopic on ∂B_r. From the homotopy invariance of Leray-Schauder degree, we obtain

$$\deg(I - A, B_r, \theta) = 1.$$

Combine this with (3.8), we have

$$\deg(I - A, B_R \setminus \overline{B_r}, \theta) = \deg(I - A, B_R, \theta) - \deg(I - A, B_r, \theta) = -1.$$

Thus the operator A has at least one fixed point in $B_R \setminus \overline{B_r}$. That is, (1.2) has at least one positive solution. This proves the theorem. □

From Theorem 3.1 and Lemma 2.1, we have

Corollary 3.1. *If $a \not\equiv 0$, $(H1)$ and $(H2)$ hold, $f(t,x)$ satisfies $(A1)$ and $(A2)$, then Theorem 3.1 is still true. In particular, when $h(t) \equiv 0$, $(H1)$ and $(H2)$ can be changed to*

$$\int_0^T a(s)ds \geq 0, \tag{H3}$$

and

$$\int_0^T a(s)ds \leq \frac{4}{T}. \tag{H4}$$

Theorem 3.2. *If $(A0), (A3)$ and $(A4)$ hold, then (1.2) has at least one positive solution.*

Proof. $(A4)$ implies that there are $\varepsilon \in (0, \lambda_1)$ and $r > 0$ such that

$$f(t, x) \geq (\lambda_1 + \varepsilon)x, \quad \forall x \in [0, r], \quad t \in [0, T], \tag{3.10}$$

and

$$f(t, x) \geq (\lambda_1 - \varepsilon)x, \quad \forall x \in [-r, 0], \quad t \in [0, T]. \tag{3.11}$$

From the above inequalities, we have

$$f(t, x) \geq (\lambda_1 + \varepsilon)x \geq (\lambda_1 - \varepsilon)x,$$

if $(t, x) \in [0, T] \times [0, r]$, and

$$f(t, x) \geq (\lambda_1 - \varepsilon)x \geq (\lambda_1 + \varepsilon)x,$$

if $(t, x) \in [0, T] \times [-r, 0]$. Thus, we have

$$f(t, x) \geq (\lambda_1 + \varepsilon)x, \quad \forall x \in [-r, r], \quad t \in [0, T], \tag{3.12}$$

and

$$f(t, x) \geq (\lambda_1 - \varepsilon)x, \quad \forall x \in [-r, r], \quad t \in [0, T]. \tag{3.13}$$

Next we prove that

$$x \neq Ax + \sigma p, \quad \forall x \in \partial B_r, \quad \forall \sigma \geq 0, \tag{3.14}$$

where $p \in P \setminus \{0\}$ is given by (2.8). If not, then there are $x_0 \in \partial B_r$ and $\sigma_0 \geq 0$ such that

$$x_0(t) = (Ax_0)(t) + \sigma_0 p(t). \tag{3.15}$$

By (3.10), we obtain $(Ax_0)(t) \geq (\lambda_1 + \varepsilon)(Lx_0)(t)$, therefore,

$$x_0(t) \geq (\lambda_1 + \varepsilon)(Lx_0)(t). \tag{3.16}$$

Multiply (3.16) by $q(t)$ on both sides and integrate over $[0, T]$ and use (2.10), we can obtain

$$\int_0^T x_0(t)q(t)dt \geq (\lambda_1 + \varepsilon) \int_0^T (Lx_0)(t)q(t)dt = (1 + r(L)\varepsilon) \int_0^T x_0(t)q(t)dt,$$

where $q(t)$ is given by (2.10), thus

$$\int_0^T x_0(t)q(t)dt \leq 0. \tag{3.17}$$

Moreover, from (3.15), we have

$$\begin{aligned} x_0(t) - (\lambda_1 - \varepsilon)(Lx_0)(t) =& (Ax_0)(t) - (\lambda_1 - \varepsilon)(Lx_0)(t) + \sigma_0 p(t) \\ =& (L[Fx_0 - (\lambda_1 - \varepsilon)x_0])(t) + \sigma_0 p(t). \end{aligned}$$

By (3.13) and Lemma 2.4, we have $x_0(t) - (\lambda_1 - \varepsilon)(Lx_0)(t) \in P_0$, thus

$$\| x_0(t) - (\lambda_1 - \varepsilon)(Lx_0)(t) \| \leq \frac{1}{\delta} \int_0^T [x_0(t) - (\lambda_1 - \varepsilon)(Lx_0)(t)]q(t)dt$$

$$= \frac{\varepsilon r(L)}{\delta} \int_0^T x_0(t)q(t)dt \leq 0.$$

Then we have $x_0(t) - (\lambda_1 - \varepsilon)(Lx_0)(t) = (I - (\lambda_1 - \varepsilon)L)x_0(t) = 0$, where I is the identity operator. Moreover, since $(\lambda_1 - \varepsilon)r(L) < 1$, the operator $I - (\lambda_1 - \varepsilon)L$ has the bounded linear inverse operator $(I - (\lambda_1 - \varepsilon)L)^{-1}$. Thus, $x_0(t) = (I - (\lambda_1 - \varepsilon)L)^{-1}0 = 0$, which contradicts $x_0 \in \partial B_r$, that is, (3.14) holds. By Lemma 2.2, we have

$$\deg(I - A, B_r, \theta) = 0. \tag{3.18}$$

On the other hand, $(A3)$ shows that there are $\varepsilon \in (0, \lambda_1)$ and $C_2 > 0$ such that

$$|f(t,x)| \leq (\lambda_1 - \varepsilon)|x| + C_2, \quad x \in \mathbb{R}, \quad t \in [0,T]. \tag{3.19}$$

Let

$$M_2 = \{x \in E : x = \sigma Ax, \text{ for some } \sigma \in [0,1]\}.$$

Next we prove that M_2 is bounded on E. From the definition of M_2, if $x_0 \in M_2$, there is $\sigma_0 \in [0,1]$, such that $x_0 = \sigma_0 Ax_0$. By (3.19), we have

$$|x_0| \leq (\lambda_1 - \varepsilon)L|x_0| + C_2 L\mathbf{1}.$$

Let $v_0 = |x_0|$ and $u_0 = C_2 L\mathbf{1}$, then $v_0 \in P$, $u_0 \in P$, and

$$v_0 \leq (\lambda_1 - \varepsilon)Lv_0 + u_0.$$

Obviously L is a linear increasing operator, thus, $Lv_0 \leq (\lambda_1 - \varepsilon)L^2 v_0 + Lu_0$, so $v_0 \leq (\lambda_1 - \varepsilon)^2 L^2 v_0 + (\lambda_1 - \varepsilon)Lu_0 + u_0$, therefore, the iteration of this inequality has the following form

$$v_0 \leq \sum_{i=0}^{n} (\lambda_1 - \varepsilon)^i L^i u_0 + (\lambda_1 - \varepsilon)^{n+1} L^{n+1} v_0, \quad (n = 1, 2, ...).$$

Since $(\lambda_1 - \varepsilon)r(L) = 1 - \varepsilon r(L) < 1$, we have

$$\lim_{n\to\infty} \sum_{i=0}^{n} (\lambda_1 - \varepsilon)^i L^i u_0 = (I - (\lambda_1 - \varepsilon)L)^{-1} u_0, \quad \lim_{n\to\infty} (\lambda_1 - \varepsilon)^{n+1} L^{n+1} v_0 = 0.$$

Then $v_0 \leq (I - (\lambda_1 - \varepsilon)L)^{-1} u_0$, therefore, M_2 is bounded.

Choose $R > \max\{\sup_{x \in M_2} \| x \|, r\}$, then

$$x \neq \sigma Ax, \quad \forall x \in \partial B_R, \quad \forall 0 \leq \sigma \leq 1.$$

From the homotopy invariance of Leray-Schauder degree, we obtain

$$\deg(I - A, B_R, \theta) = 1.$$

Combine this and (3.18), we have

$$\deg(I - A, B_R \backslash \overline{B_r}, \theta) = 1 - 0 = 1.$$

Thus the operator A has at least one fixed point in $B_R \setminus \overline{B_r}$. That is, (1.2) has at least one positive solution. The proof is finished. □

From Theorem 3.2 and Lemma 2.1, we also have

Corollary 3.2. *If $a \not\equiv 0$, $(H1)$ and $(H2)$ hold, $f(t,x)$ satisfies $(A3)$ and $(A4)$, then Theorem 3.2 is still true. In particular, when $h(t) \equiv 0$, $(H1)$ and $(H2)$ can be changed to*

$$\int_0^T a(s)ds \geq 0, \tag{H3}$$

and

$$\int_0^T a(s)ds \leq \frac{4}{T}. \tag{H4}$$

Moreover, if we set
$(A1)'$ $\limsup\limits_{x\to+\infty}\frac{f(t,x)}{x} < \lambda_1$, uniformly on $t \in [0,T]$; $\liminf\limits_{x\to-\infty}\frac{f(t,x)}{x} > \lambda_1$, uniformly on $t \in [0,T]$.
$(A4)'$ $\limsup\limits_{x\to0^+}\frac{f(t,x)}{x} < \lambda_1$, uniformly on $t \in [0,T]$; $\liminf\limits_{x\to0^-}\frac{f(t,x)}{x} > \lambda_1$, uniformly on $t \in [0,T]$, then, we have the following theorems.

Theorem 3.3. *If $(A0), (A1)'$ and $(A2)$ hold, then (1.2) has at least one positive solution.*

Theorem 3.4. *If $(A0), (A3)$ and $(A4)'$ hold, then (1.2) has at least one positive solution.*

The proofs of the two theorems are similar to that of Theorem 3.1 and Theorem 3.2 respectively, so they are omitted.

4. Examples

We give two examples to verify the validity of our results.

Example 4.1. Consider the following equation:

$$\begin{cases} x'' + x' + 2x = b_1x + x^2, \\ x(0) = x(1), \quad x'(0) = x'(1). \end{cases} \tag{4.1}$$

where $0 < b_1 < \lambda_1$ and λ_1 is the first positive eigenvalue of the following equation

$$\begin{cases} x'' + x' + 2x = \lambda x, \\ x(0) = x(1), \quad x'(0) = x'(1). \end{cases}$$

Through some calculations, we obtain $\lambda_1 = 2$, so $0 < b_1 < 2$. Moreover, the conditions of Lemma 2.1 are satisfied, that is, $(A0)$ holds. Let $g(t) = 1$, $f(t,x) = b_1x + x^2$, then

$\liminf\limits_{x\to+\infty}\frac{f(t,x)}{x} = \liminf\limits_{x\to+\infty}(b_1 + x) = +\infty > \lambda_1 = 2$, uniformly on $t \in [0,1]$,

$\limsup\limits_{x\to-\infty}\frac{f(t,x)}{x} = \limsup\limits_{x\to-\infty}(b_1 + x) = -\infty < \lambda_1 = 2$, uniformly on $t \in [0,1]$,

$\limsup\limits_{x\to0}\frac{|f(t,x)|}{|x|} = \limsup\limits_{x\to0}|b_1 + x| = b_1 < \lambda_1 = 2$, uniformly on $t \in [0,1]$.

Thus, $(A1)$ and $(A2)$ are satisfied. By Theorem 3.1, we obtain that (4.1) has at least one positive solution.

Example 4.2. .Consider the following equation:

$$\begin{cases} x'' + x' + (1+\cos 4t)x = \frac{1}{\sqrt{t}}(b_2 x + x^{\frac{1}{2}}), \\ x(0) = x(\frac{\pi}{2}), \quad x'(0) = x'(\frac{\pi}{2}). \end{cases} \tag{4.2}$$

where $0 < b_2 < \lambda_1$ and λ_1 is the first positive eigenvalue of the following equation

$$\begin{cases} x'' + x' + (1+\cos 4t)x = \frac{1}{\sqrt{t}}\lambda x, \\ x(0) = x(\frac{\pi}{2}), \quad x'(0) = x'(\frac{\pi}{2}). \end{cases}$$

Through some calculations, the conditions of Lemma 2.1 are satisfied, that is, $(A0)$ holds. Let $g(t) = \frac{1}{\sqrt{t}}$, $f(t,x) = b_2 x + x^{\frac{1}{2}}$. Moreover, notice that $g(t)$ is singular at $t = 0$, then

$\limsup\limits_{x\to\infty} \frac{|f(t,x)|}{|x|} = \limsup\limits_{x\to\infty} |b_2 + x^{-\frac{1}{2}}| = b_2 < \lambda_1$, uniformly on $t \in [0, \frac{\pi}{2}]$,

$\liminf\limits_{x\to 0^+} \frac{f(t,x)}{x} = \liminf\limits_{x\to 0^+} (b_2 + x^{-\frac{1}{2}}) = +\infty > \lambda_1$, uniformly on $t \in [0, \frac{\pi}{2}]$,

$\limsup\limits_{x\to 0^-} \frac{f(t,x)}{x} = \limsup\limits_{x\to 0^-} (b_2 + x^{-\frac{1}{2}}) = -\infty < \lambda_1$, uniformly on $t \in [0, \frac{\pi}{2}]$.

Thus, $(A3)$ and $(A4)$ are satisfied. By Theorem 3.2, we obtain that (4.2) has at least one positive solution.

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