Journal of Nonlinear Modeling and Analysis
Volume 2, Number 4, December 2020, 525–540
http://jnma.ca; http://jnma.ijournal.cn
DOI:10.12150/jnma.2020.525

Oscillation of Second Order Impulsive Differential Equations with Nonpositive Neutral Coefficients*

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Abstract In this work, sufficient conditions are established for a class of nonlinear second order neutral impulsive differential equations to have oscillatory solutions with nonpositive neutral coefficient. Our results extend and complement some of the known results in the literature. Examples are given to illustrate our results.

Keywords Oscillation, Nonoscillation, Neutral differential equation, Impulse, Nonlinear.

MSC(2010) 34K, 34K40, 34K45, 34K11.

1. Introduction

Consider the class of second order impulsive nonlinear neutral differential equations of the form:

$$(E)\begin{cases} [x(t)+p(t)x(t-\tau)]''+g(t,x(t),x(t-\sigma))=0,\ t\neq\theta_k,\ t\geq t_0, & (1.1)\\ x(\theta_k^+)=I_k(x(\theta_k)),\ k\in\mathbb{N}, & (1.2)\\ x'(\theta_k^+)=J_k(x'(\theta_k)),\ k\in\mathbb{N}, & (1.3)\end{cases}$$

where $\tau,\sigma\in\mathbb{N}$, $0\leq t_0<\theta_1<\cdots<\theta_k<\cdots$ with $\lim_{k\to\infty}\theta_k=\infty$ and $\theta_{k+1}-\theta_k>\rho=\max\{\tau,\sigma\}$. Throughout our work, we assume that the following hypotheses hold:

- (A_1) $g\in C([t_0-\rho,\infty)\times\mathbb{R}\times\mathbb{R},\mathbb{R})$, $ug(t,u,v)>0$ for $uv>0$, $\frac{g(t,u,v)}{h(v)}\geq q(t)$ for $v\neq 0$, where $q(t)\in C([t_0-\rho,\infty),\mathbb{R}_+)$ and $q(t)\not\equiv 0$ on all interval of the form $(\theta_k,\theta_{k+1}]$, $k\geq 1$, $xh(x)>0$ for all $x\neq 0$ and $h'(x)\geq\varepsilon>0$;
- (A_2) $I_k,J_k\in C(\mathbb{R},\mathbb{R})$, $I_k(0)=0=J_k(0)$ and there exist positive numbers c_k, c_k^*, d_k, d_k^*, such that $c_k^*\leq\frac{I_k(u)}{u}\leq c_k$, $d_k^*\leq\frac{J_k(u)}{u}\leq d_k$, $k\in\mathbb{N}$;
- (A_3) $p\in PC(\mathbb{R}_+,\mathbb{R})$ and $p(t)$, $p'(t)$ are left continuous on $(\theta_k,\theta_{k+1}]$, $k\geq 1$ such that $p(\theta_k^+)=d_kp(\theta_k)$, $p'(\theta_k^+)=d_kp'(\theta_k)$.

In the literature (see for e.g. [11]), the impulse operators are often treated as **under control**, that is, one may expect that either the impulse act as a control and

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*The authors were supported by Rajiv Gandhi National fellowship (UGC), New Delhi, India through the Letter No. F1-17.1/2017-18/RGNF-2017-18-SC-ORI-35849 dated 11th July, 2017.

cease the oscillation of the system, or operate to keep the system oscillating. In particular, impulse can make oscillating systems become nonoscillating and conversely by the imposition of suitable impulse control (see for e.g. [5]- [9], [13]- [17], [23], [27]).

One of the important application of second order differential equations with impulse is in impact theory. Billiard-type systems, models describing viscoelastic bodies colliding, systems with delay and impulse are more appropriate to apply (see for e.g. [10]). Of course, some extra conditions are required while we study impulsive equations (see for e.g. [2, 3, 21, 22, 26, 28]) to that of nonimpulsive equations. Furthermore, it is more challenging to study nonlinear neutral equations as we find a class of second order delay differential equations as special cases. In this respect, by using comparison technique, the second order impulsive neutral differential equations

$$(E^*)\begin{cases}[r(t)(v(t)+p(t)v(t-\tau))']'+q(t)v(t-\sigma)=0,\ t\neq\theta_k,\ t\geq t_0,\\ v(\theta_k^+)=(1+d_k)v(\theta_k),\ k\in\mathbb{N},\\ v'(\theta_k^+)=(1+d_k)v'(\theta_k),\ k\in\mathbb{N},\end{cases}$$

has been studied by Li et al. [13], where $\tau,\sigma\in\mathbb{N}$, $q(t)>0$, $r(t)>0$, $b_k>-1$ and $p(t)=p\geq 0$; they have extended and generalised the work of [6] to impulse equations.

By using the Riccati transfomation technique, Bonotto et al. [4] have considered the second order neutral differential equations with impulse of the form:

$$(E^*)\begin{cases}[r(t)(v(t)+p(t)v(t-\tau))']'+f(t,v(t),v(t-\sigma))=0,\ t\neq\theta_k,\ t\geq t_0,\\ v(\theta_k)=I_k(v(\theta_k^-)),\ k\in\mathbb{N},\\ v'(\theta_k)=J_k(v'(\theta_k^-)),\ k\in\mathbb{N},\end{cases}$$

where $\tau,\sigma\in\mathbb{N}$, $p\in PC([t_0,\infty),\mathbb{R}_+)$, $r(t)>0$, $\theta_{k+1}-\theta_k>\sigma=\max\{\tau,\sigma\}$ and $c_k^*\leq\frac{I_k(u)}{u}\leq c_k$, $J_k(u)=d_ku$, $k\in\mathbb{N}$, $c_k^*,\ c_k,\ d_k>0$ and $f(t,v(t),v(t-\sigma))\geq q(t)f(x(t-\sigma))$, and $f(x)=x$. In this work, the authors have extended and generalised the work of [12] to impulsive equations in the range $0\leq p(t)<1$.

However, it seems that there is no known results regarding the oscillation of second order impulsive neutral differential equations when the neutral coefficient $p(t)\leq 0$. More exactly, the existing literature does not provide any criteria which ensure oscillation of all solutions of (E) when $p(t)\leq 0$. In view of this motivation, our aim in this paper is to present sufficient conditions which ensure that all solutions of (E) are oscillatory.

Definition 1.1. A real valued continuous function $x(t)$ is said to be a solution of (E) satisfying the initial condition, if the following conditions are satisfied

1. $x(t)=\psi(t)$ for $t_0-\rho\leq t\leq t_0$, $x(t)\in C^2[t_0,\infty,\mathbb{R})$ and $t\neq\theta_k,k\in\mathbb{N}$;
2. $y(t)=x(t)+p(t)x(t-\tau)\in C^1([t_0,\infty),\mathbb{R})$ and $y'(t)\in C^1([t_0,\infty),\mathbb{R})$, $t\neq\theta_k,t\neq\theta_k+\tau,t\neq\theta_k+\sigma,k\in\mathbb{N}$ and satisfies (1.1);
3. $x(\theta_k^+),x(\theta_k^-),x'(\theta_k^+)$ and $x'(\theta_k^-)$ exist, $x(\theta_k^-)=x(\theta_k)$, $x'(\theta_k^-)=x'(\theta_k)$ and satisfies (1.2) and (1.3) respectively.

Definition 1.2. A nontrivial solution $x(t)$ of (E) is said to be nonoscillatory, if there exists a point $t_0\geq 0$ such that $x(t)$ has a constant sign for $t\geq t_0$. Otherwise, the solution $x(t)$ is said to be oscillatory. (E) is oscillatory, if all its solutions are oscillatory.

2. Some preliminaries

Throughout the paper, we use the following notations:

$$y(t) = x(t) + p(t)x(t-\tau),$$
$$\gamma_k = \max\{c_k, d_k\}, k \in \mathbb{N}.$$

PC($[t_0,\infty), \mathbb{R}_+$)= $\{\, x : [t_0,\infty) \to \mathbb{R}$; $x(t)$ and $x'(t)$ are continuously differentiable at $t \neq \theta_k$ and $x(\theta_k^-)$, $x(\theta_k^+)$, $x'(\theta_k^-)$, $x'(\theta_k^+)$ exist and $x(\theta_k^-) = x(\theta_k)$, $x'(\theta_k^-) = x'(\theta_k)\}$.

Proposition 2.1. *[24] A product $\prod_{k=1}^{\infty}(1+d_k)$ where all the terms d_k are positive is convergent if and only if the series $\sum_{k=1}^{\infty} d_k$ converges.*

Lemma 2.1. *[11] Suppose that*

(i) the sequence $\{\theta_k\}_{k\in\mathbb{N}}$ satisfies $0 \le t_0 < \theta_1 < \cdots < \theta_k < \cdots$ with $\lim_{k\to\infty}\theta_k = \infty$,

(ii) $v, v' \in PC(\mathbb{R}_+,\mathbb{R})$ and $v(t)$ is left continuous at θ_k, $k \in \mathbb{N}$.

(iii) $k \in \mathbb{N}$ and $t \ge t_0$, we have

$$v'(t) \le p(t)v(t) + q(t), t \neq \theta_k$$
$$v(\theta_k^+) \le a_k v(\theta_k) + b_k,$$

where $p, q \in C(\mathbb{R}_+,\mathbb{R})$, a_k and b_k are real constants with $a_k \ge 0$ hold. Then the following inequality holds

$$v(t) \le v(t_0)\prod_{t_0<\theta_k<t} a_k \exp\Big(\int_{t_0}^{t} p(s)ds\Big) + \int_{t_0}^{t}\prod_{s<\theta_k<t} a_k \exp\Big(\int_{s}^{t} p(\delta)d\delta\Big)q(s)ds$$
$$+ \sum_{t_0<\theta_k<t}\Big(\prod_{\theta_k<\theta_j<t} a_j \exp\Big(\int_{\theta_k}^{t} p(s)ds\Big)\Big)b_k.$$

Lemma 2.2. *[26] Let $x(t)$ be a solution of of (E) and $c_k^*, d_k \ge 1$ for $k \in \mathbb{N}$. Assume that there exists $T \ge t_0$ such that $x(t) > 0$ for $t \ge T$ and*

(A_4) $\int_T^{\infty}\prod_{T<\theta_k<s}\frac{d_k}{\gamma_k}\, ds = \infty$.

Then $y'(\theta_k^+) \ge 0$ and $y'(t) \ge 0$ for $t \in (\theta_k, \theta_{k+1}]$ and $\theta_k \ge T$.

Proof. Let $x(t)$ be a nonoscillatory of (E) for $t \ge t_0$. Without loss of generality, we may assume that $x(t) > 0$, $x(t-\tau) > 0$ and $x(t-\sigma) > 0$ for $t \ge t_1 > t_0 + \rho = \max\{\tau,\sigma\}$. Set

$$y(t) = x(t) + p(t)x(t-\tau).$$

Therefore, from (E) we have

$$y''(t) = -g(t, x(t), x(t-\sigma)) \le -q(t)h(x(t-\sigma)) \le 0$$

and hence $y'(t)$ is monotonically decreasing in all interval of the form $(\theta_k, \theta_{k+1}]$, $k \in \mathbb{N}$ and $\theta_k > t_2 > t_1 + \sigma$. We assert that $y'(\theta_k) \ge 0$, $\theta_k \ge t_2$, $k \in \mathbb{N}$. If not, then there exists $\theta_j \ge t_2$ such that $y'(\theta_j) < 0$. Let $y'(\theta_k) = -\alpha$, $\alpha > 0$. Since $\theta_{k+1} - \theta_k > \tau$, $\theta_{k+1} - \tau$ is not a impulsive point for all $k \in \mathbb{N}$. Therefore, from (E), we have

$$y(\theta_k^+) = x(\theta_k^+) + p(\theta_k^+)x(\theta_k^+ - \tau)$$

$$
\begin{aligned}
&= I_k(x(\theta_k)) + d_k p(\theta_k) x(\theta_k - \tau),\\
&\le c_k x(\theta_k) + d_k p(\theta_k) x(\theta_k - \tau),\\
&\le \gamma_k y(\theta_k),
\end{aligned}
$$

that is,

$$y(\theta_k^+) \le \gamma_k y(\theta_k)$$

and

$$
\begin{aligned}
y'(\theta_k^+) &= x'(\theta_k^+) + p'(\theta_k^+) x(\theta_k^+ - \tau) + p(\theta_k^+) x'(\theta_k^+ - \tau),\\
&= J_k(x'(\theta_k)) + d_k p'(\theta_k) x(\theta_k - \tau) + d_k p(\theta_k) x'(\theta_k - \tau),\\
&\le d_k x'(\theta_k) + d_k p'(\theta_k) x(\theta_k - \tau) + d_k p(\theta_k) x'(\theta_k - \tau),\\
&= d_k y'(\theta_k),
\end{aligned}
$$

that is,

$$y'(\theta_k^+) \le d_k y'(\theta_k).$$

Since $y'(t)$ is monotonically decreasing for $t \in (\theta_{j+i-1}, \theta_{j+i}], i = 1, 2, 3 \cdots$, then for $t \in (\theta_j, \theta_{j+1}]$, we have

$$y'(\theta_{j+1}) \le y'(\theta_j^+) \le d_j y'(\theta_j) = -d_j \alpha < 0.$$

For $t \in (\theta_{j+1}, \theta_{j+2}]$, we have

$$
\begin{aligned}
y'(\theta_{j+2}) &\le y'(\theta_{j+1}^+)\\
&= x'(\theta_{j+1}^+) + p'(\theta_{j+1}^+) x(\theta_{j+1}^+ - \tau) + p(\theta_{j+1}^+) x'(\theta_{j+1}^+ - \tau),\\
&= J_{j+1}(x'(\theta_{j+1})) + d_{j+1} p'(\theta_{j+1}) x(\theta_{j+1} - \tau) + d_{j+1} p(\theta_{j+1}) x'(\theta_{j+1} - \tau),\\
&\le d_{j+1} x'(\theta_{j+1}) + d_{j+1} p'(\theta_{j+1}) x(\theta_{j+1} - \tau) + d_{j+1} p(\theta_{j+1}) x'(\theta_{j+1} - \tau),\\
&= d_{j+1} y'(\theta_{j+1})\\
&= -d_j d_{j+1} \alpha < 0.
\end{aligned}
$$

Consequently,

$$y'(\theta_{j+n}) \le -d_j d_{j+1} d_{j+2} \cdots d_{j+n-1} \alpha < 0.$$

Proceeding inductively, we obtain

$$y'(t) \le -d_j d_{j+1} d_{j+2} \cdots d_{j+n} \alpha < 0,$$

for $t \in (\theta_{j+n}, \theta_{j+n+1}]$. Consider the following impulsive differential inequalities

$$
\begin{aligned}
&y''(t) \le 0,\ t \ne \theta_k,\ t > \theta_j\\
&y'(\theta_k^+) \le d_k y'(\theta_k),\ k = j+1, j+2, \cdots .
\end{aligned}
$$

Therefore, by Lemma 2.1, we get

$$y'(t) \le y'(\theta_j^+) \prod_{\theta_j < \theta_k < t} d_k.$$

Again, consider the following impulsive differential inequalities

$$y'(t) \le -\alpha \prod_{\theta_j < \theta_k < t} d_k,\ t \ne \theta_k,\ t > \theta_j$$

$$y(\theta_k^+) \leq \gamma_k y(\theta_k),\ k = j+1, j+2, \cdots .$$

Therefore, by Lemma 2.1, we get

$$\begin{aligned} y(t) &\leq y(\theta_j^+) \prod_{\theta_j<\theta_k<t} \gamma_k - \alpha \int_{\theta_j}^{t} \prod_{s<\theta_k<t} \gamma_k \left[\prod_{\theta_j<\theta_k<t} d_k \right] ds \\ &\leq \prod_{\theta_j<\theta_k<t} \gamma_k \left[y(\theta_j^+) - \alpha \int_{\theta_j}^{t} \left(\prod_{\theta_j<\theta_k<s} \frac{d_k}{\gamma_k} \right) ds \right] \end{aligned}$$

implies that

$$y(t) \leq \prod_{\theta_j<\theta_k<t} \gamma_k \left[y(\theta_j^+) - \alpha \int_{\theta_j}^{t} \left(\prod_{\theta_j<\theta_k<s} \frac{d_k}{\gamma_k} \right) ds \right]. \tag{2.1}$$

(2.1) is not possible, if $y(t) > 0$ due to (A_4). Indeed, $y(t) > 0$ when $p(t) \geq 0$. Let $-1 < p \leq p(t) \leq 0$. We claim that $y(t) > 0$ for $t \geq t_2$. If not, then (2.1) implies $y(t) \to -\infty$ as $t \to \infty$. Hence, there exists a $t_3 > t_2 > \theta_j$ and $C > 0$ such that $y(t) \leq -C$ for $t \geq t_3$. We arise two possible cases:

Case 1. If $x(t)$ is unbounded, then there exists a sequence $\{s_n\}$ such that $s_n \to \infty$, $x(s_n^+) \to \infty$ as $n \to \infty$ and $x(s_n^+) = \max\{x(t) : t_3 \leq t \leq s_n\}$ (if s_n is not impulsive point, then $x(s_n^+) = x(s_n)$). Since $t - \tau < t$, then

$$x(s_n^+ - \tau) = \max\{x(t) : t_3 \leq t \leq s_n - \tau\} \leq \max\{x(t) : t_3 \leq t \leq s_n\} = x(s_n^+).$$

Therefore, for all large n

$$0 > y(s_n^+) = x(s_n^+) + p(s_n^+)x(s_n^+ - \tau) \geq (1 + p(s_n^+))x(s_n^+) > 0,$$

a contradiction.

Case 2. If $x(t)$ is bounded, then $y(t)$ is bounded, a contradiction to $y(t) \to -\infty$ as $t \to \infty$.

We have noticed that $y'(\theta_k) > 0$ for any $\theta_k > t_2$. Since $y''(t) < 0$ for any $t \in (\theta_{k+i-1}, \theta_{k+i}]$, then $y'(t) > y'(\theta_k^+)$, that is,

$$\begin{aligned} y'(t) \geq y'(\theta_k^+) &= x'(\theta_k^+) + p'(\theta_k^+)x(\theta_k^+ - \tau) + p(\theta_k^+)x'(\theta_k^+ - \tau) \\ &= I_k(x'(\theta_k)) + d_k p'(\theta_k)x(\theta_k - \tau) + d_k p(\theta_k)x'(\theta_k - \tau), \\ &\geq d_k^* x(\theta_k) + d_k p'(\theta_k)x(\theta_k - \tau) + d_k p(\theta_k)x'(\theta_k - \tau) \geq d^* y'(\theta_k) > 0. \end{aligned}$$

Therefore, $y'(\theta_k^+) > 0$ and $y'(t) > 0$ for $t \in [\theta_{k+i-1}, \theta_{k+i})$, $t \geq t_2$. This completes the proof of the lemma. □

Remark 2.1. Let $x(t)$ be an eventually negative solution of (E). Then using (A_4), it is easy to prove that $y'(\theta_k^+) \leq 0$ *and* $y'(t) \leq 0$ for $t \in (\theta_k, \theta_{k+1}]$, $\theta_k \geq T$.

3. Impulsive conditions for oscillation

Theorem 3.1. *Assume that* $-1 < p \leq p(t) \leq 0$, $c_k^* \geq 1, k \in \mathbb{N}$ *and* (A_4) *hold. Furthermore, assume that there exists a function* $f(t) \in PC([t_0, \infty), \mathbb{R}_+)$ *such that*

(A_5) $\quad \lim_{t\to\infty} \int_{t_0}^{t} \prod_{t_0<\theta_k<s} \frac{1}{d_k} \left(q(l) - \frac{\varepsilon f^2(l)}{4} \right) exp \left(\int_{t_2}^{l} \varepsilon f(s)ds \right) dl = \infty$

hold, then every unbounded solution of (E) *oscillates.*

Proof. On the contrary, let $x(t)$ be an unbounded nonoscillatory solution of (E). Without loss of generality, we may assume that $x(t) > 0$, $x(t-\tau) > 0$, $x(t-\sigma) > 0$ for $t \geq t_0 > \rho$. From Lemma 2.2, it follows that $y'(t) > 0$ and $y'(\theta_k^+) > 0$ for $t \in (\theta_k, \theta_{k+1}]$, $k \in \mathbb{N}$, $t \geq t_1$ and hence $y'(t-\sigma) > 0$ for $t \geq t_1 + \sigma$. Here, we consider two cases namely, $y(t) < 0$ and $y(t) > 0$.

Case 1. Since $x(t)$ is unbounded for $t \in (\theta_k, \theta_{k+1}]$, then proceeding as in the proof of Lemma 2.2 (Case 1), we get a contradiction to $y(t) < 0$ for all $t(\theta_k, \theta_{k+1}]$, $k \in \mathbb{N}$.

Case 2. For $t \in (\theta_{k+i-1}, \theta_{k+i}]$, and for any θ_k, $k \in \mathbb{N}$

$$\begin{aligned} y(\theta_k^+) &= x(\theta_k^+) + p(\theta_k^+)x(\theta_k^+ - \tau) = I_k(x(\theta_k)) + d_k p'(\theta_k)x(\theta_k - \tau), \\ &\geq c_k^* x(\theta_k) + d_k p(\theta_k) x(\theta_k - \tau), \\ &\geq \min\{c_k^*, d_k\} y(\theta_k) \geq y(\theta_k) > 0. \end{aligned}$$

Clearly $-1 < p \leq p(t) \leq 0$ and $y(t) > 0$ implies that $y(t) \leq x(t)$. We may note that

$$g(t, x(t), x(t-\sigma)) \geq q(t)h(x(t-\sigma)) \geq q(t)h(y(t-\sigma))$$

for $t \neq \theta_k$, $t \geq t_2 > t_1 + \sigma$. Therefore, (1.1) can be written as

$$y''(t) + q(t)h(y(t-\sigma)) \leq 0,\ t \neq \theta_k,\ t \geq t_2.$$

For $t \geq t_2$, define

$$w(t) = \frac{y'(t)}{h(y(t-\sigma))}. \tag{3.1}$$

Then, $w(\theta_k^+) \geq 0$ and $w(t) \geq 0$ for $\theta_k \geq t_2$. Differentiating (3.1), it follows that

$$\begin{aligned} w'(t) &= \frac{y''(t)h(y(t-\sigma)) - y'(t)h'(y(t-\sigma))y'(t-\sigma)}{h^2(y(t-\sigma))} \\ &\leq \frac{-q(t)h(y(t-\sigma))}{h(y(t-\sigma))} - \frac{y'(t)h'(y(t-\sigma))y'(t-\sigma)}{h^2(y(t-\sigma))} \\ &\leq -q(t) - \frac{(y'(t))^2}{h^2(y(t-\sigma))}h'(y(t-\sigma)) \\ &\leq -q(t) - \varepsilon w^2(t) \\ &= -\left(q(t) - \frac{\varepsilon f^2(t)}{4}\right) - \left(\varepsilon w^2(t) + \frac{\varepsilon f^2(t)}{4}\right) \end{aligned}$$

for $t \neq \theta_k$. Since $a^2 + b^2 \geq 2ab$, then the last inequality can be written as

$$w'(t) \leq -\left(q(t) - \frac{\varepsilon f^2(t)}{4}\right) - \varepsilon w(t)f(t),\ \text{for } t \neq \theta_k,\ t \geq t_2.$$

For $t = \theta_k$,

$$w(\theta_k^+) = \frac{y'(\theta_k^+)}{h(y(\theta_k^+ - \sigma))} \leq \frac{d_k y'(\theta_k)}{h(c_k^* y(\theta_k - \sigma))} \leq \frac{d_k y'(\theta_k)}{h(y(\theta_k - \sigma))} = d_k w(\theta_k).$$

Consider the impulsive inequalities:

$$w'(t) \leq -\left(q(t) - \frac{\varepsilon f^2(t)}{4}\right) - \varepsilon w(t)f(t),\ t \neq \theta_k, t \geq t_2$$

$$w(\theta_k^+) \le d_k w(\theta_k),\ k \in \mathbb{N}$$

and applying Lemma 2.1, we get

$$\begin{aligned} w(t) &\le w(t_2^+) \prod_{t_2<\theta_k<t} d_k exp\left(\int_{t_2}^t -\varepsilon f(s)ds\right) - \int_{t_2}^t \prod_{s<\theta_k<t} d_k exp\left(\int_s^t -\varepsilon f(\delta)d\delta\right) \\ &\times \left(q(s) - \frac{\varepsilon f^2(s)}{4}\right) ds \\ &\le \prod_{t_2<\theta_k<t} d_k exp\left(\int_{t_2}^t -\varepsilon f(s)ds\right)\left[w(t_2^+) - \int_{t_2}^t \prod_{t_2<\theta_k<s} \frac{1}{d_k} Q_1(s)ds\right], \end{aligned}$$

where $Q_1(t) = \left(q(t) - \frac{\varepsilon f^2(t)}{4}\right) exp\left(\int_{t_2}^t \varepsilon f(s)ds\right)$. Letting $t \to \infty$ and using (A_5), we obtain $w(t) < 0$ which is a contradiction. Hence, the theorem is proved. □

Remark 3.1. Let $I_k = I$ and $J_k = J$, where I and J are identity function, then (E) reduces to

$$(E')\ \ [x(t) + p(t)x(t-\tau)]'' + q(t)h(x(t-\sigma)) = 0,\ t \ge t_0.$$

By [1, Theorem 3.4.3], every unbounded solution of (E') oscillates. We may note that Theorem 3.1 improves or generalizes the known result [1, Theorem 3.4.3].

Example 3.1. Consider the impulsive differential system

$$\begin{cases} [x(t) - \frac{1}{e^\pi}x(t-\pi)]'' + 2e^{2t}x(t-2\pi) = 0, t \ne \theta_k, t > 2\pi, \\ x(\theta_k^+) = \frac{k}{k+1}x(\theta_k), \quad k \in \mathbb{N}, \\ x'(\theta_k^+) = \frac{1}{k+1}x'(\theta_k), \quad k \in \mathbb{N}, \end{cases} \tag{3.2}$$

where $\tau = \pi$, $\sigma = 2\pi$, $p(t) = -\frac{1}{e^\pi}$, $q(t) = 4e^{2t} \ge 0$, $h(u) = u$, and $f(t) = 0$, $c_k^* = c_k = \frac{k}{k+1}$, $d_k^* = d_k = \frac{1}{k+1}$, $\theta_k = 3k\pi$, $\theta_{k+1} - \theta_k = 3\pi > 2\pi, k \in \mathbb{N}$. Here

$$\begin{aligned} \int_T^\infty \prod_{T<\theta_k<s} \frac{d_k}{\gamma_k}\, ds &= \int_2^\infty \prod_{2<\theta_k<s} \frac{1}{k} ds \\ &= \int_2^{\theta_1} \prod_{2<\theta_k<s} \frac{1}{k} ds + \int_{\theta_1^+}^{\theta_2} \prod_{2<\theta_k<s} \frac{1}{k} ds + \int_{\theta_2^+}^{\theta_3} \prod_{2<\theta_k<s} \frac{1}{k} ds + \cdots \\ &= \frac{1}{2}(\theta_1 - 2) + \frac{1}{2} \times \frac{2}{3}(\theta_2 - \theta_1) + \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4}(\theta_3 - \theta_2) + \cdots \\ &= \frac{1}{2} \times (3\pi - 2) + \frac{1}{3} \times 3\pi + \frac{1}{4} \times 3\pi + \frac{1}{5} \times 3\pi + \cdots \\ &= \frac{1}{2} \times (3\pi - 2) + \pi + \frac{3}{4} \times \pi + \frac{3}{5} \times \pi + \cdots \\ &\ge \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots = \sum_{i=2}^\infty \frac{1}{i} = \infty \end{aligned}$$

and

$$\int_2^\infty \prod_{2<\theta_k<s} \frac{1}{d_k} q(s)ds = \int_2^\infty \prod_{2<\theta_k<s} (k+1)4e^{2s}ds = \infty.$$

By Theorem 3.1, (3.2) is oscillatory. Clearly, if (3.2) is without impulse, then $x(t) = e^t \cos t$ is an unbounded oscillatory solution of (3.2).

Theorem 3.2. *Let* (A_4) *hold and* $-1 < p \le p(t) \le 0$. *Assume that there exists a positive integer* k_0 *such that* $c_k^* \ge 1, d_k \ge 1$ *for* $k \ge k_0$ *and*

(A_6) $\sum_{k=1}^{\infty} |d_k - 1| < \infty$,

(A_7) h *satisfies* $\int_{\pm\alpha}^{\pm\infty} \frac{du}{h(u)} < \infty,\ \alpha > 0$,

(A_8) $\sum_{k=1}^{\infty} \int_{\theta_k}^{\theta_{k+1}} \Big(\int_{t_0}^{\infty} \prod_{t_0<\theta_k<v} \frac{1}{d_k} q(v)dv \Big) ds = \infty$

hold. Then every unbounded solution of (E) *oscillates.*

Proof. Let's assume that $x(t)$ be an unbounded nonoscillatory solution of (E). By Lemma 2.2, we get $y'(t) > 0$ and $y'(\theta_k^+) > 0$ for $t \in (\theta_k, \theta_{k+1}], k \in \mathbb{N}$, $t \ge t_1$ and hence we have

$$\begin{aligned} y(\theta_k^+) &= x(\theta_k^+) + p(\theta_k^+)x(\theta_k^+ - \tau) \\ &\ge c_k^* x(\theta_k) + d_k p(\theta_k) x(\theta_k - \tau) \\ &\ge x(\theta_k) + p(\theta_k)x(\theta_k - \tau) \\ &\ge y(\theta_k), \end{aligned}$$

that is, $y(t)$ is nondecreasing for $t \in (\theta_k, \theta_{k+1}], k \in \mathbb{N}$. Especially,

$$y(t_1^+) \le y(\theta_1) \le y(\theta_1^+) \le y(\theta_2) \le \cdots \tag{3.3}$$

represents that $y(t)$ is monotonically nondecreasing for $t \in [t_1, \infty)$. From (E), we get

$$\begin{aligned} &y''(t) \le -q(t)h(y(t-\sigma)), t \ne \theta_k, t \ge t_1 \\ &y'(\theta_k^+) \le d_k y'(\theta_k), k \in \mathbb{N}. \end{aligned}$$

Let $z(t) = y'(t)$, then the last impulsive inequality can be written as

$$\begin{aligned} &z'(t) \le -q(t)h(y(t-\sigma)), t \ne \theta_k, t \ge t_1 \\ &z(\theta_k^+) \le d_k z(\theta_k), k \in \mathbb{N}. \end{aligned}$$

Using Lemma 2.1, we get

$$z(t) \le z(u) \prod_{u<\theta_k<t} d_k - \int_u^t \prod_{s<\theta_k<t} d_k q(s) h(y(s-\sigma))ds,\ u \ge t_1$$

implies that

$$y'(t) \le y'(u) \prod_{u<\theta_k<t} d_k - \int_u^t \prod_{s<\theta_k<t} d_k q(s) h(y(s-\sigma))ds,\ u \ge t_1, \tag{3.4}$$

that is,

$$y'(u) \ge \int_u^t \prod_{u<\theta_k<s} d_k^{-1} q(s) h(y(s-\sigma))ds.$$

Therefore,

$$\frac{y'(u)}{h(y(u-\sigma))} \geq \int_u^t \prod_{u<\theta_k<s} d_k^{-1} q(s) \frac{h(y(s-\sigma))}{h(y(u-\sigma))} ds,$$
$$\geq \int_u^t \prod_{u<\theta_k<s} d_k^{-1} q(s) ds.$$

We notice from (3.4) that

$$y'(t) \leq y'(u) \prod_{u<\theta_k<t} d_k,\ u \geq t_1$$

which then implies that

$$y'(u) \leq y'(u-\sigma) \prod_{u-\sigma<\theta_k<u} d_k,\ u \geq t_1+\sigma. \tag{3.5}$$

Let $u \in (\theta_k, \theta_{k+1}]$. Using (3.5), we get

$$\int_{\theta_k}^{\theta_{k+1}} \frac{y'(u)}{h(y(u-\sigma))} ds \leq \int_{\theta_k}^{\theta_{k+1}} \prod_{u-\sigma<\theta_k<u} d_k \frac{y'(u-\sigma)}{h(y(u-\sigma))} du.$$

Due to Proposition 2.1 and (A_6), there exists $t_2 \geq t_1+\sigma$ and a constant $K>0$ such that $\prod_{u-\sigma<\theta_k<u} d_k \leq K$. Therefore, the preceding inequality reduces to

$$\int_{\theta_k}^{\theta_{k+1}} \frac{y'(u)}{h(y(u-\sigma))} ds \leq K \int_{\theta_k}^{\theta_{k+1}} \frac{y'(u-\sigma)}{h(y(u-\sigma))} du,\ u \geq t_2$$
$$\leq K \int_{y(\theta_k-\sigma)}^{y(\theta_{k+1}-\sigma)} \frac{dv}{h(v)}$$

and hence

$$\int_{\theta_k}^{\theta_{k+1}} \left(\int_s^t \prod_{s<\theta_k<u} d_k^{-1} q(u) du \right) ds \leq K \int_{y(\theta_k-\sigma)}^{y(\theta_{k+1}-\sigma)} \frac{dv}{h(v)}.$$

Since (3.3) holds, then the above inequality becomes

$$\sum_{k=1}^{\infty} \int_{\theta_k}^{\theta_{k+1}} \left(\int_s^t \prod_{s<\theta_k<u} d_k^{-1} q(u) du \right) ds \leq K \int_{y(\theta_1-\sigma)}^{\infty} \frac{dv}{h(v)} < \infty$$

due to (A_7), a contradiction to (A_8). Hence, the theorem is proved. □

Remark 3.2. Let $I_k = I$ and $J_k = J$, where I and J are identity function, then (E) reduces to

$$(E')\ \ [x(t)+p(t)x(t-\tau)]'' + q(t)h(x(t-\sigma)) = 0,\ t \geq t_0.$$

We may note that, by [1, Theorem 3.4.4], every unbounded solution of (E') oscillates. Therefore, Theorem 3.2 improves the existing result [1, Theorem 3.4.4].

Theorem 3.3. *Let $-1 < p \leq p(t) \leq 0$ and $c_k^* \geq 1$. Assume that (A_4) and*

(A_9) *there exists $\beta > 0$ such that $|h(u)| \geq \beta|u|$,*

(A_{10}) $\limsup_{k\to\infty} \int_{\theta_k}^{\theta_k+\sigma} \left(\frac{t-\sigma}{2}\right) q(t)dt > \frac{d_k}{\beta}$, $t > \sigma$

hold. Then every unbounded solution of (E) oscillates.

Proof. Let $x(t)$ be an unbounded nonoscillatory solution of (E) and we assume that $x(t) > 0$ for $t \geq t_0$. By Theorem 3.1, we get $y'(t)$ is nonincreasing for $t \in (\theta_k, \theta_{k+1}]$, $k \in \mathbb{N}$. From (1.1) and (A_9), we have

$$y''(t) \leq -q(t)h(y(t-\sigma)) \leq -\beta q(t)y(t-\sigma) \leq 0, t \geq t_1. \tag{3.6}$$

We note that

$$\begin{aligned}\int_{t_1}^{t} y'(s)ds &= \int_{t_1}^{\theta_1} y'(s)ds + \int_{\theta_1}^{\theta_2} y'(s)ds + \int_{\theta_2}^{\theta_3} y'(s)ds + \cdots + \int_{\theta_k}^{t} y'(s)ds \\ &= y(\theta_1) - y(t_1^+) + y(\theta_2) - y(\theta_1^+) + y(\theta_3) - y(\theta_2^+) + \cdots + y(t) - y(\theta_k^+) \\ &= y(t) - y(t_1^+) + \sum_{t_1<\theta_k<t} [y(\theta_k) - I_k(y(\theta_k))],\end{aligned}$$

and since $c_k^* \leq \frac{I_k(y(\theta_k))}{y(\theta_k)} \leq c_k$, then the last integral can be viewed as

$$\begin{aligned}\int_{t_1}^{t} y'(s)ds &\leq y(t) - y(t_1^+) + \sum_{t_1<\theta_k<t} [y(\theta_k) - c_k^* y(\theta_k)] \\ &= y(t) - y(t_1^+) + \sum_{t_1<\theta_k<t} (1 - c_k^*)y(\theta_k) \\ &\leq y(t) - y(t_1^+).\end{aligned}$$

Therefore,

$$y(t) \geq y(t_1^+) + \int_{t_1}^{t} y'(s)ds \geq y'(t)(t-t_1),\ t \geq t_1.$$

that is,

$$y(t) \geq \frac{t}{2} y'(t),\ t \geq t_2 > 2t_1.$$

For $\theta_{k+1} - \theta_k > \rho = \sigma$,

$$y(t-\sigma) \geq \left(\frac{t-\sigma}{2}\right) y'(t-\sigma),\ t \geq t_3 > t_2 + \sigma$$

and hence (3.6) becomes

$$y''(t) + \beta\left(\frac{t-\sigma}{2}\right) q(t)y'(t-\sigma) \leq 0. \tag{3.7}$$

Integrating (3.7) from θ_k to $\theta_k + \sigma$, we get

$$y'(\theta_k+\sigma) - y'(\theta_k^+) + \beta \int_{\theta_k}^{\theta_k+\sigma} \left(\frac{t-\sigma}{2}\right) q(t)y'(t-\sigma)dt \leq 0,$$

that is,

$$y'(\theta_k+\sigma)-y'(\theta_k^+)+\beta y'(\theta_k)\int_{\theta_k}^{\theta_k+\sigma}\left(\frac{t-\sigma}{2}\right)q(t)dt\le 0.$$

Consequently,

$$y'(\theta_k+\sigma)+y'(\theta_k^+)\left[\frac{\beta}{d_k}\int_{\theta_k}^{\theta_k+\sigma}\left(\frac{t-\sigma}{2}\right)q(t)dt-1\right]\le 0$$

which is not possible due to (A_{10}). This completes the proof of the theorem. □

Remark 3.3. The prototype of h satisfying (A_9) could be of the form

$$h(u)=u(\beta+|u|^\gamma),\ u\in\mathbb{R},\ \gamma>0.$$

Example 3.2. Consider the impulsive differential system

$$\begin{cases}[x(t)-\frac{1}{t}x(t-1)]''+(t+1)[x(t-2)+x^3(t-2)]=0,\ t\neq\theta_k, t>2,\\ x(\theta_k^+)=\frac{k-1}{k}x(\theta_k), & k\in\mathbb{N},k>k_0,\\ x'(\theta_k^+)=\frac{1}{k}x'(\theta_k), & k\in\mathbb{N},k>k_0,\end{cases}\tag{3.8}$$

where $\tau=1$, $\sigma=2$, $p(t)=-\frac{1}{t}$, $q(t)=(t+1)$, $c_k^*=c_k=\frac{k-1}{k}$, $d_k^*=d_k=\frac{1}{k}$, $\theta_k=2^k$, $\theta_{k+1}-\theta_k=2^k>1, k\in\mathbb{N}, k>k_0=1$, $h(u)=u(1+u^2)$. Clearly,

$$\begin{aligned}&\int_T^\infty\prod_{T<\theta_k<s}\frac{d_k}{\gamma_k}\,ds\\ &=\int_1^\infty\prod_{1<\theta_k<s}\frac{1}{k-1}ds\\ &=\int_1^{\theta_2}\prod_{1<\theta_k<s}\frac{1}{k-1}ds+\int_{\theta_2^+}^{\theta_3}\prod_{1<\theta_k<s}\frac{1}{k-1}ds+\int_{\theta_3^+}^{\theta_4}\prod_{1<\theta_k<s}\frac{1}{k-1}ds+\cdots\\ &=(\theta_2-1)+\frac{1}{2}\times(\theta_3-\theta_2)+\frac{1}{2}\times\frac{1}{3}\times(\theta_4-\theta_3)+\cdots\\ &=2+\frac{1}{2}\times 2^2+\frac{1}{2}\times\frac{1}{3}\times 2^3+\frac{1}{2}\times\frac{1}{3}\times\frac{1}{4}\times 2^4+\cdots\\ &\ge 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots=1+\sum_{i=2}^\infty\frac{1}{i}=\infty\end{aligned}$$

and let's choose $\sigma>2h+1$ for $0<h<\frac{1}{4}$. Then $\theta_k=2k$ and $\theta_k^+=2k+2h$,

$$\limsup_{k\to\infty}\Big(\int_{\theta_k}^{\theta_k+\sigma}(t+1)\left(\frac{t-\sigma}{2}\right)dt\Big)\ge\limsup_{k\to\infty}\Big(\int_{\theta_k}^{\theta_k+\sigma}\left(\frac{t-2}{2}\right)dt\Big)>1.$$

By Theorem 3.3, (3.8) is oscillatory.

Next, we show the Kamenev-type oscillation criteria for (E) when $f(t)=0$.

Theorem 3.4. *Let* $-1\le p\le p(t)\le 0$. *If* (A_4) *and*
(A_{11}) $\limsup_{k\to\infty}\frac{1}{t^m}\int_{\theta_k}^{\theta_{k+1}}(t-s)^m q(s)ds=\infty$ *for some* $m>1$
hold, then every unbounded solution of (E) *oscillates.*

Proof. We proceed as in the proof of Theorem 3.1 to obtain

$$w'(t) \leq -q(t) \text{ for } t \neq \theta_k, \ \ t \geq t_2.$$

Clearly, $w(t)$ is nonincreasing and positive for $t \geq t_2$. Multiplying $(t-s)^m$ $(t > s)$ for some $m > 1$ to both sides of the above inequality and then integrating from θ_k to θ_{k+1}, we get

$$\int_{\theta_k}^{\theta_{k+1}} (t-s)^m w'(s) ds \leq -\int_{\theta_k}^{\theta_{k+1}} (t-s)^m q(s) ds.$$

Indeed,

$$\begin{aligned}
&\int_{\theta_k}^{\theta_{k+1}} (t-s)^m w'(s) ds \\
&= \int_{\theta_k}^{\theta_{k+1}} m(t-s)^{m-1} w(s) ds + (t-\theta_{k+1})^m w(\theta_{k+1}) - (t-\theta_k)^m w(\theta_k^+).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_{\theta_k}^{\theta_{k+1}} (t-s)^m q(s) ds \\
&\leq -\int_{\theta_k}^{\theta_{k+1}} m(t-s)^{m-1} w(s) ds - (t-\theta_{k+1})^m w(\theta_{k+1}) + (t-\theta_k)^m w(\theta_k^+) \\
&\leq (t-\theta_k)^m w(\theta_k^+) \\
&\leq d_k (t-\theta_k)^m w(\theta_k),
\end{aligned}$$

that is,

$$\frac{1}{t^m} \int_{\theta_k}^{\theta_{k+1}} (t-s)^m q(s) ds \leq d_k \left(\frac{t-\theta_k}{t}\right)^m w(\theta_k).$$

As a result

$$\limsup_{k\to\infty} \frac{1}{t^m} \int_{\theta_k}^{\theta_{k+1}} (t-s)^m q(s) ds \leq d_k \left(1 - \frac{\theta_k}{t}\right)^m w(\theta_k) < \infty,$$

which contradicts (A_{11}). Hence, the theorem is proved. □

Theorem 3.5. *Let $0 \leq p(t) \leq p_2 < \infty$ and $\sigma > 2\tau$. If (A_4), (A_9) and*
(A_{12}) $\limsup_{k\to\infty} \frac{1}{d_k} \int_{\theta_k}^{\theta_k+\tau} \left(\frac{t-\sigma}{2}\right) Q(t) dt > \frac{1+p_2}{\beta}$
hold, then every solution of (E) oscillates, where $Q(t) = \min\{q(t), q(t-\tau)\}, t \geq \tau$.

Proof. Suppose on the contrary that $x(t)$ is a nonoscillatory solution of (E). Proceeding as in the proof of Theorem 3.1, we get $y'(t)$ is nonincreasing for $t \in (\theta_k, \theta_{k+1}]$, $k \in \mathbb{N}$. From (1.1), it is easy to see that

$$\begin{aligned}
0 &\geq y''(t) + \beta q(t) x(t-\sigma) + p_2 y''(t-\tau) + p_2 \beta q(t-\tau) x(t-\tau-\sigma) \\
&\geq y''(t) + p_2 y''(t-\tau) + \beta Q(t)[x(t-\tau) + p_2 x(t-\tau-\sigma)] \\
&\geq y''(t) + p_2 y''(t-\tau) + \beta Q(t) y(t-\sigma),
\end{aligned}$$

that is,

$$y''(t) + p_2 y''(t-\tau) + \beta Q(t) y(t-\sigma) \leq 0. \tag{3.9}$$

From Theorem 3.3, it follows that

$$y(t-\sigma) \geq \left(\frac{t-\sigma}{2}\right) y'(t-\sigma),\ t > t_3.$$

Therefore, (3.9) becomes

$$y''(t) + p_2 y''(t-\tau) + \beta \left(\frac{t-\sigma}{2}\right) Q(t) y'(t-\sigma) \leq 0.$$

Integrating the above inequality from θ_k to $\theta_k + \tau$, we get

$$y'(\theta_k+\tau) - y'(\theta_k^+) + p_2 y'(\theta_k) - p_2 y'(\theta_k^+ - \tau) + \beta \int_{\theta_k}^{\theta_k+\tau} \left(\frac{t-\sigma}{2}\right) Q(t) y'(t-\sigma) dt \leq 0.$$

Using $y'(\theta_k+\tau) \leq y'(\theta_k)$ and $y'(\theta_k^+) \leq y'(\theta_k^+ - \tau)$ in the above inequality, we find

$$y'(\theta_k+\tau) + p_2 y'(\theta_k+\tau) - y'(\theta_k^+ - \tau) - p_2 y'(\theta_k^+ - \tau) + \beta \int_{\theta_k}^{\theta_k+\tau} \left(\frac{t-\sigma}{2}\right) Q(t) y'(t-\sigma) dt \leq 0,$$

that is,

$$(1+p_2) y'(\theta_k+\tau) - (1+p_2) y'(\theta_k^+ - \tau) + \beta y'(\theta_k + \tau - \sigma) \int_{\theta_k}^{\theta_k+\tau} \left(\frac{t-\sigma}{2}\right) Q(t) dt \leq 0.$$

Since $\sigma \geq 2\tau$, then the above relation reduces to

$$(1+p_2) y'(\theta_k+\tau) - (1+p_2) y'(\theta_k^+ - \tau) + \beta y'(\theta_k - \tau) \int_{\theta_k}^{\theta_k+\tau} \left(\frac{t-\sigma}{2}\right) Q(t) dt \leq 0,$$

that is,

$$(1+p_2) y'(\theta_k+\tau) + y'(\theta_k^+ - \tau) \Big[\frac{\beta}{d_k} \int_{\theta_k}^{\theta_k+\tau} \left(\frac{t-\sigma}{2}\right) Q(t) dt - (1+p_2)\Big] \leq 0,$$

which is not possible due to (A_{12}). Thus, the theorem is proved. □

Remark 3.4. We may note that, Theorem 3.5 improves the known result [1, Theorem 3.4.8].

Remark 3.5. Theorem 3.5 extends the result of [25, Theorem 3.2] when $\mathbb{T} = \mathbb{R}$, $\gamma = 1$ and $r(t) = 1$. In fact, when $c_k^* = c_k = d_k^* = d_k = 1$ for $k \in \mathbb{N}$, (E) is no more a impulsive differential system. Therefore, (A_{12}) reduces to

$$\limsup_{t\to\infty} \int_t^{t+\tau} \left(\frac{t-\sigma}{2}\right) Q(t) dt > \frac{1+p_2}{\beta}$$

which is same as in [25, Theorem 3.2].

4. Conclusion

Remark 4.1. We may note that, Theorem 3.1- Theorem 3.4 gives a partial answer to the open problem raised by Bonotto et al. [4], that is, every unbounded solution of (E) are oscillatory when the neutral coefficient $-1 < p \leq p(t) \leq 0$.

Remark 4.2. In [26], the authors have studied the impulsive system (E) and established the sufficient conditions for oscillation in the range $0 \leq p(t) < 1$. In this work also, we have made an effort to establish sufficient conditions for oscillation when $-1 < p(t) \leq 0$ and $0 \leq p(t) < \infty$.

On the basis of Remark 4.1 and 4.2, two interesting problems for future research can be formulated as follows:

- Is it possible to suggest a different method to study (E) and obtain some sufficient conditions which ensure that all solutions of (E) are oscillatory?
- Is it possible to establish oscillation criteria for (E) for the range $-\infty < p(t) < -1$?

Acknowledgements

The authors are thankful to the editor and anonymous referees for their careful reading of the manuscript and insightful comments, which help to improve the quality of the paper.

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