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Qualitative Analysis and Periodic Cusp Waves to a Class of Generalized Short Pulse Equations*

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Abstract In this paper, we qualitatively study periodic cusp waves to a class of generalized short pulse equations, which are of the general form of three special generalized short pulse equations, from the perspective of dynamical systems. We show the existence of smooth periodic waves, periodic cusp wave and compactons, obtain exact expression of periodic cusp wave and illustrate the limiting process of periodic cusp wave from smooth periodic waves.

Keywords Generalized short pulse equations, Periodic cusp waves, Periodic waves, Compactons, Bifurcation.

MSC(2010) 34A05, 35C07, 36C09, 35A24.

1. Introduction

In 2018, N. W. Hone, Novikov and Wang [4] obtained three generalized short pulse equations

$$u_{xt} = u + 2uu_{xx} + 2u_x^2, \tag{1.1}$$
$$u_{xt} = u + 2uu_{xx} + u_x^2, \tag{1.2}$$
$$u_{xt} = u + 4uu_{xx} + u_x^2, \tag{1.3}$$

which possess an infinite hierarchy of local higher symmetries, when they were classifying nonlinear partial differential equations of second order of the general form

$$u_{xt} = u + c_0u^2 + c_1uu_x + c_2uu_{xx} + c_3u_x^2 + d_0u^3 + d_1u^2u_x + d_2u^2u_{xx} + d_3uu_x^2. \tag{1.4}$$

Note that if we take $c_0 = c_1 = c_2 = c_3 = 0$, $d_0 = d_1 = 0$, $d_2 = 1$, and $d_3 = 2$, Eq.(1.4) becomes the following short pulse equation

$$u_{xt} = u + \frac{1}{3}(u^3)_{xx}, \tag{1.5}$$

which was derived by Schäfer and Wayne [10] as a model of ultra-short optical pulses in nonlinear media. In [9], the authors showed that the short pulse equation (1.5)

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is integrable, in the sense that it admits a Lax pair and a recursion operator that generates infinitely many commuting symmetries, and they also found a hodograph-type transformation which connects Eq.(1.5) with the sine-Gordon equation.

As suggested in [4], short pulses and their properties are a subject of current interest in nonlinear optics and electrodynamics, both theoretically and experimentally. For instance, a rigorous justification of the short pulse equation, starting from a quasilinear Klein-Gordon equation (a toy model for Maxwells equations) was given in [8]. Moreover, for electrons accelerated in short laser pulses, it was shown recently that, due to quantum effects, the radiation reaction can be quenched by suitably tuning the pulse length, although the lengths required are currently out of experimental reach [3].

In this paper, based on the forms of Eqs. (1.1), (1.2) and (1.3), we focus on a class of generalized short pulse equations, which have the following general form

$$u_{xt} = u + \alpha u u_{xx} + \beta u_x^2, \tag{1.6}$$

where we assume the parameters $\alpha > 0$ and $\beta > 0$, for conveniently. Obviously, taking $\alpha = 2, \beta = 2$, Eq.(1.6) becomes Eq.(1.1). Similarly, taking $\alpha = 2, \beta = 1$, Eq.(1.6) becomes Eq.(1.2), and taking $\alpha = 4, \beta = 1$, Eq.(1.6) becomes Eq.(1.3). We intend to study the solutions to the general form (1.6) qualitatively from the perspective of dynamical systems [2,6,7,11–14,17–24] and consequently, the results about three special forms (1.1), (1.2) and (1.3) follow immediately.

2. Phase portrait

Employing the traveling wave transformation $u(x,t) = \varphi(\xi)$, $\xi = x - ct$, where $c > 0$ is the wave speed, we can convert Eq.(1.6) into the following ordinary differential equation

$$(\alpha\varphi + c)\varphi'' + \varphi + \beta(\varphi')^2 = 0. \tag{2.1}$$

Introducing $y = \varphi'$, we obtain a planar system

$$\begin{cases} \frac{d\varphi}{d\xi} = y, \\ \frac{dy}{d\xi} = \frac{-\varphi - \beta y^2}{\alpha\varphi + c}, \end{cases} \tag{2.2}$$

with first integral

$$\begin{aligned} H(\varphi, y) =& (\alpha\varphi + c)^{\frac{2\beta}{\alpha}} y^2 + \frac{2}{\alpha(\alpha + 2\beta)}(\alpha\varphi + c)^{\frac{2\beta}{\alpha}+1} - \frac{c}{\alpha\beta}(\alpha\varphi + c)^{\frac{2\beta}{\alpha}}, \\ & \text{for } \varphi \geq -\frac{c}{\alpha}, \\ H(\varphi, y) =& (-\alpha\varphi - c)^{\frac{2\beta}{\alpha}} y^2 - \frac{2}{\alpha(\alpha + 2\beta)}(-\alpha\varphi - c)^{\frac{2\beta}{\alpha}+1} - \frac{c}{\alpha\beta}(-\alpha\varphi - c)^{\frac{2\beta}{\alpha}}, \\ & \text{for } \varphi < -\frac{c}{\alpha}. \end{aligned} \tag{2.3}$$

Transformed by $d\xi = (\alpha\varphi + c)d\tau$, system (2.2) becomes a Hamiltonian system

$$\begin{cases} \frac{d\varphi}{d\tau} = (\alpha\varphi + c)\ y, \\ \frac{dy}{d\tau} = -\varphi - \beta y^2. \end{cases} \tag{2.4}$$

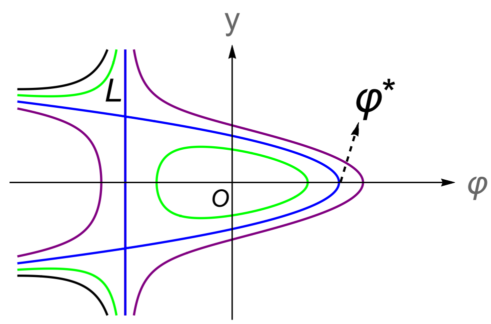

Figure 1. The phase portrait of system (2.2).

Since the first integral of system (2.2) is the same as that of the Hamiltonian system (2.4), we can analyze the phase portrait of system (2.2) from that of system (2.4).

We can easily know that system (2.4) has a center $O:(0,0)$ on the φ–axis, and two saddles $S_{\pm}:\left(-\frac{c}{\alpha},\pm\sqrt{\frac{c}{\alpha\beta}}\right)$ on the singular line $L:\varphi=-\frac{c}{\alpha}$. From the qualitative theories of differential equations and the bifurcation theories of dynamical systems, we easily derive the phase portrait of system (2.2) in Figure 1.

3. Main results

To state conveniently, suppose that $\varphi^*=\frac{c}{2\beta}$, $h_0=H(0,0)$ and $h_1=H\left(\frac{-c}{\alpha},\pm\sqrt{\frac{c}{\alpha\beta}}\right)$.

Theorem 3.1. *Eq.(1.6) has a family of smooth periodic waves, a periodic cusp wave*

$$u(\xi)=\frac{c}{2\beta}-\left(\sqrt{\frac{c(\alpha+2\beta)}{2\alpha\beta}}-\frac{1}{\sqrt{2(\alpha+2\beta)}}|\xi-2iT_0|\right)^2, \tag{3.1}$$

where $i=0,\pm1,\pm2,\cdots,\xi=x-ct\in[(2i-1)T_0,(2i+1)T_0]$, and $T_0=(\alpha+2\beta)\sqrt{\frac{c}{\alpha\beta}}$, and a family of compactons. Moreover, the smooth periodic waves converge to the periodic cusp wave (3.1) when $h\to h_1-0$.

From Theorem 3.1 and the relations among Eqs.(1.1), (1.2), (1.3) and (1.6), we immediately have the following results.

Corollary 3.1. (i) *Eq.(1.1) has periodic cusp wave*

$$u(\xi)=\frac{c}{4}-\frac{1}{12}\left(3\sqrt{c}-|\xi-2iT_1|\right)^2,$$

where $i=0,\pm1,\pm2,\cdots,\xi=x-ct\in[(2i-1)T_1,(2i+1)T_1]$, and $T_1=3\sqrt{c}$.

(ii) *Eq.(1.2) has periodic cusp wave*

$$u(\xi)=\frac{c}{2}-\left(\sqrt{c}-\frac{\sqrt{2}}{4}|\xi-2iT_2|\right)^2,$$

where $i=0,\pm1,\pm2,\cdots,\xi=x-ct\in[(2i-1)T_2,(2i+1)T_2]$, and $T_2=2\sqrt{2c}$.

(iii) *Eq.*(1.3) *has periodic cusp wave*

$$u(\xi) = \frac{c}{2} - \frac{1}{12}\left(3\sqrt{c} - |\xi - 2iT_3|\right)^2,$$

where $i = 0, \pm 1, \pm 2, \cdots, \xi = x - ct \in [(2i-1)T_3, (2i+1)T_3]$, *and* $T_3 = 3\sqrt{c}$.

4. The derivation to Theorem 3.1

In this section, we show the procedure of deriving Theorem 3.1.

Corresponding to the family of periodic orbits passing through $(\varphi_0, 0)$ with $\varphi_0 \in (-\frac{c}{\alpha}, 0)$ defined by $H(\varphi, y) = h, h \in (h_0, h_1)$ in Figure 1, Eq.(1.6) has a family of smooth periodic waves. Further, when $h \to h_1$, the periodic orbits become periodic cusp orbit [1], which can be expressed as, from (2.3),

$$y = \pm\sqrt{\frac{2}{\alpha + 2\beta}}\sqrt{\varphi^* - \varphi}, \text{ for } -\frac{c}{\alpha} \leq \varphi \leq \varphi^*, \tag{4.1}$$

and

$$\varphi = -\frac{c}{\alpha}, \text{ for } |y| \leq \sqrt{\frac{c}{\alpha\beta}}. \tag{4.2}$$

Substituting (4.1) into the first equation of system (2.2) and integrating along the periodic cusp orbit, we have

$$\int_{-\frac{c}{\alpha}}^{\varphi} \frac{1}{\sqrt{\varphi^* - s}} ds = \sqrt{\frac{2}{\alpha + 2\beta}}|\xi|, \tag{4.3}$$

which give rise to the periodic cusp wave (3.1).

We illustrate the limiting process of periodic cusp wave from smooth periodic waves in Figure 2 by taking $\alpha = 2, \beta = 2, c = 4$ and the initial point (a) $\varphi_0 = -1.5, y_0 = 0$; (b) $\varphi_0 = -1.8, y_0 = 0$; (c) $\varphi_0 = -1.9, y_0 = 0$; (d) $\varphi_0 = -1.999, y_0 = 0$, respectively.

Additionally, according to the theory of singular nonlinear waves [5, 15, 16] corresponding to the family of orbits passing through $(\varphi_0, 0)$ with $\varphi_0 \in (\varphi^*, +\infty)$ defined by $H(\varphi, y) = h, h \in (h_1, +\infty)$ (bounded by the singular line L) in Figure 1, Eq.(1.6) has a family of compactons, which are illustrated in Figure 3 by taking $\alpha = 2, \beta = 2, c = 4$ and the initial point (a) $\varphi_0 = 1.01, y_0 = 0$; (b) $\varphi_0 = 1.2, y_0 = 0$; (c) $\varphi_0 = 1.5, y_0 = 0$; (d) $\varphi_0 = 2, y_0 = 0$, respectively.

5. Conclusions

In this paper, we qualitatively study a class of generalized short pulse equations (1.6), which can be viewed as the general form of Eqs.(1.1), (1.2) and (1.3). From the phase portrait and Eq.(2.3), we show the existence of smooth periodic waves, periodic cusp wave and compactons, obtain exact expression of periodic cusp wave and illustrated the limiting process of periodic cusp wave from smooth periodic waves. It is worth noting that the so-called loop solutions are not found in Eq.(1.6), compared to the original short pulse equation (1.5) [25].

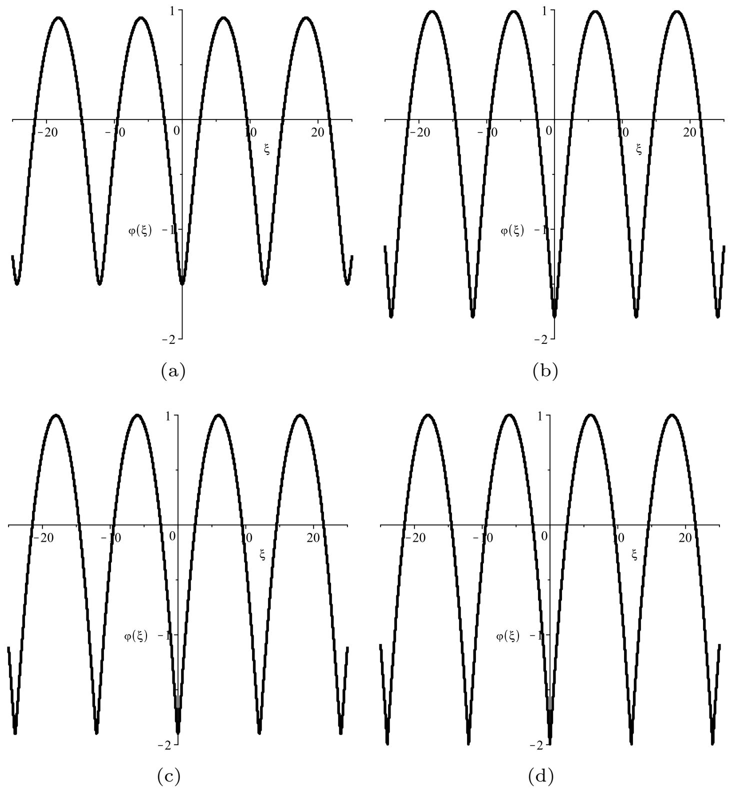

Figure 2. The limiting process of periodic cusp waves from smooth periodic waves by taking $\alpha = 2, \beta = 2, c = 4$ and the initial point (a) $\varphi_0 = -1.5, y_0 = 0$; (b) $\varphi_0 = -1.8, y_0 = 0$; (c) $\varphi_0 = -1.9, y_0 = 0$; (d) $\varphi_0 = -1.999, y_0 = 0$.

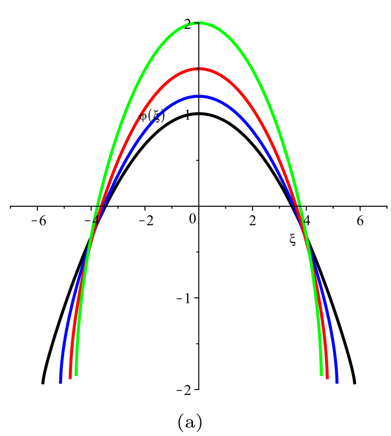

Figure 3. The illustration of the family of compactons by taking $\alpha = 2, \beta = 2, c = 4$ and the initial point $\varphi_0 = 1.01, y_0 = 0$, $\varphi_0 = 1.2, y_0 = 0$, $\varphi_0 = 1.5, y_0 = 0$, $\varphi_0 = 2, y_0 = 0$, respectively..

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