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Global Regularity of the Logarithmically Supercritical MHD System in Two-dimensional Space

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Abstract In this paper, we study the global regularity of logarithmically supercritical MHD equations in 2 dimensional, in which the dissipation terms are $-\mu\Lambda^{2\alpha}u$ and $-\nu\mathcal{L}^{2\beta}b$. We show that global regular solutions in the cases $0<\alpha<\frac{1}{2}, \beta>1, 3\alpha+2\beta>3$.

Keywords Logarithmically supercritical, MHD system, Global regularity.

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1. Introduction

We consider the two-dimensional logarithmically supercritical magnetohydrodynamics (MHD) system:

$$u_t + u\cdot\nabla u + \nabla\pi + \mu\Lambda^{2\alpha}u - b\cdot\nabla b = 0, \tag{1.1}$$
$$b_t + u\cdot\nabla b + \nu\mathcal{L}^{2\beta}b - b\cdot\nabla u = 0, \tag{1.2}$$
$$(u,b)(x,0) = (u_0,b_0)\ in\ \mathbb{R}^2, \tag{1.3}$$
$$\mathrm{div}u = \mathrm{div}b = 0. \tag{1.4}$$

where $u=u(x,t)\in\mathbb{R}^2$ is the unknown velocity field, $b=b(x,t)\in\mathbb{R}^2$ is the magnetic field, and $\pi=\pi(x,t)\in\mathbb{R}$ represents the pressure. $\alpha,\beta\geq 0$ are real parameters. $\Lambda=(-\Delta)^{1/2}$ is defined in terms of the Fourier transform $\widehat{\Lambda f}(\xi)=|\xi|\widehat{f}(\xi)$, and $\mathcal{L}^{2\beta}$ defined through a Fourier transform,

$$\widehat{\mathcal{L}^{2\beta}f}(\xi) = m(\xi)\hat{f}(\xi), m(\xi) = \frac{|\xi|^{2\beta}}{g^2(|\xi|)}, \beta\in\mathbb{R}^+.$$

with $g:\mathbb{R}^+\longrightarrow\mathbb{R}^+$ a radially symmetric, non-decreasing function such that $g\geq 1$.

When

$$\mathcal{L}^{2\beta} = \Lambda^{2\beta}.$$

For the system (1.1)-(1.4), We identify the case $\mu=\nu=0$ as the GMHD system with zero velocity and zero magnetic diffusion respectively (so called ideal MHD equations). The author in [1] studied the global existence of a weak solution when $\alpha\geq\frac{1}{2}+\frac{n}{4}$, $\alpha+\beta\geq 1+\frac{n}{2}, n\in\mathbb{R}^3$. In [2], the author showed that the GMHD

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equations exists a unique global smooth solution when $\alpha, \beta \geq \frac{1}{2} + \frac{n}{4}$, There are some results [3–8] about the existence of the strong solution.

We want to improve the lower bound on the power of the fractional Laplacian in the dissipative term of the generalized Navier-Stokes equations seems extremely difficult, the author introduced the notion of "logarithmic supercriticality" in [9,10], and also proved the global regularity of the solution. the author improved that the results [2] by using the notion of "logarithmic supercriticality" in [11], it were improved that the solution is globally regular in [12,13].

Tran, Yu and Zhai [14] proved that the solutions are globally regular in the following conditions:

$$(1)\alpha \geq \frac{1}{2}, \beta \geq 1; \quad (2) 0 \leq \alpha \leq \frac{1}{2}, 2\alpha + \beta > 2; \quad (3)\alpha \geq 2, \beta = 0.$$

it were improved that the solution is globally regular of the GMHD equations in [15–19], and there are some results [20–22] about logarithmic type.

Now we focus on our study. The authors in [16] got a global regular solution under the assumption that $0 \leq \alpha < \frac{1}{2}, \beta \geq 1, 3\alpha + 2\beta > 3$. In this paper, the dissipation term $-\nu\Lambda^{2\beta}b$ has been replaced by general negative-definite operator $-\nu\mathcal{L}^{2\beta}b$ by using the definition in [23], and in the proof, we will use the condition in [24] on g such that there exists an absolute constant $c \geq 0$ satisfying

$$g^2(\tau) \leq c \ln(e + \tau).$$

Theorem 1.1. *Let* $0 < \alpha < \frac{1}{2}, \beta > 1, 3\alpha + 2\beta > 3$, *Suppose* $u_0, b_0 \in H^s$ *with* $s \geq 2$ *and* $divu_0 = divb_0 = 0$ *in* $\mathbb{R}^2$. *Then the problem* (1.1)-(1.4) *exists the solution* (u, b) *satisfying*

$$u, b \in L^\infty(0, T; H^s), \ u \in L^2(0, T; H^{s+\alpha}), \ b \in L^2(0, T; H^{s+\beta'}). \tag{1.5}$$

for any $T > 0$ *and* $\beta > \beta' > 1$.

Remark 1.1. When $\alpha + \beta > 2, s > 2$, the author in [14] prove the global regularity.

2. Preliminaries

In this section, we will review some known facts and elementary inequalities that will be used frequently later.

Lemma 2.1. ***(ϵ-Young inequality)*** *If* a *and* b *are nonnegative real numbers and* p *and* q *are real numbers greater than* 1 *such that* $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$ab \leq \frac{\epsilon a^p}{p} + \epsilon^{-\frac{q}{p}} \frac{b^q}{q},$$

the equality holds if and only if $a^p = b^q$.

Lemma 2.2. ***(Gagliardo-Nirenberg inequality [25, 26])*** *Let* u *belong to* L^q *and its derivatives of order* m, $\Lambda^m u$, *belong to* L^r, $1 \leq q, r \leq \infty$. *For the derivatives* $\Lambda^j u$, $0 \leq j < m$, *the following inequalities hold*

$$\|\Lambda^j u\|_{L^p} \leq C \|\Lambda^m u\|_{L^r}^{\alpha} \|u\|_{L^q}^{1-\alpha}, \tag{2.1}$$

where

$$\frac{1}{p} = \frac{j}{n} + \alpha\left(\frac{1}{r} - \frac{m}{n}\right) + (1-\alpha)\frac{1}{q},$$

for all α *in the interval*

$$\frac{j}{m} \leq \alpha \leq 1.$$

(the constant depending only on n, m, j, q, r, α*), with the following exceptional cases*

1 If $j = 0$, $rm < n$, $q = \infty$ *then we make the additional assumption that either* u *tends to zero at infinity or* $u \in L^s$ *for some finite* $s > 0$*;*

2 If $1 < r < \infty$*, and* $m - j - \frac{n}{r}$ *is a non negative integer then* (2.1) *holds only for a satisfying* $\frac{j}{m} \leq \alpha \leq 1$*.*

Lemma 2.3. ***(Gronwall's Inequality** [27]**)***

(i) Let $\eta(\cdot)$ *be a nonnegative, absolutely continuous function on* $[0,T]$*, which satisfies for a.e.* t *the differential inequality*

$$\eta\prime(t) \leq \phi(t)\eta(t) + \psi(t),$$

where $\phi(t)$ *and* $\psi(t)$ *are nonnegative, summable function on* $[0,T]$*, Then*

$$\eta(t) \leq e^{\int_0^t \phi(s)ds}[\eta(0) + \int_0^t \psi(s)ds].$$

for all $0 \leq t \leq T$*;*

(ii) In particular, if

$$\eta\prime \leq \phi\eta \text{ on } [0,T] \text{ and } \eta(0) = 0,$$

then

$$\eta \equiv 0 \text{ on } [0,T].$$

3. A priori estimates

In the next section, without loss of generality, we assume $\mu = \nu = 1$.

Lemma 3.1. ***(Basic energy estimates)****It holds that for any* $T > 0$*,*

$$\sup_{0\leq\tau\leq T}(\|u\|_{L^2}^2 + \|b\|_{L^2}^2) + 2\int_0^T(\|\Lambda^\alpha u\|_{L^2}^2 + \|\mathcal{L}^\beta b\|_{L^2}^2)d\tau \leq \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2. \quad (3.1)$$

Proof.

Multiplying both sides of the equations of u and b in (1.1)-(1.2) by u and b, respectively, after integration by parts and taking the divergence free property into account, we have the following energy estimate

$$\frac{1}{2}\frac{d}{dt}(\|u\|_{L^2}^2 + \|b\|_{L^2}^2) + \|\Lambda^\alpha u\|_{L^2}^2 + \|\mathcal{L}^\beta b\|_{L^2}^2 = 0. \quad (3.2)$$

It implies that the inequality (3.1) holds and consequently completes the proof. □

Let $\omega = \nabla^{\perp} \cdot u = -\partial_2 u_1 + \partial_1 u_2$, $j = \nabla^{\perp} \cdot b = -\partial_2 b_1 + \partial_1 b_2$, then we can get the well-known equations for the vorticity ω and the current j:

$$\omega_t + u \cdot \nabla\omega + \Lambda^{2\alpha}\omega = b \cdot \nabla j, \tag{3.3}$$

$$j_t + u \cdot \nabla j + \mathcal{L}^{2\beta} j = b \cdot \nabla\omega + T(\nabla u, \nabla b). \tag{3.4}$$

with

$$T(\nabla u, \nabla b) = 2\partial_1 b_1(\partial_1 u_2 + \partial_2 u_1) + 2\partial_2 u_2(\partial_1 b_2 + \partial_2 b_1).$$

Now, we will give the H^1 estimation for (u, b).

Lemma 3.2. *Suppose that $\alpha > 0, \beta > 1$. Let $u_0, b_0 \in H^1$. For any $T > 0$, we have*

$$\|\omega\|_{L^2}^2(t) + \|j\|_{L^2}^2(t) + \int_0^t (2\|\Lambda^\alpha\omega\|_{L^2}^2 + \|\mathcal{L}^\beta j\|_{L^2}^2) d\tau \leq C(T). \tag{3.5}$$

Proof.

Multiplying (3.3)-(3.4) by ω and j, respectively, integrating over $\mathbb{R}^2$, and adding the resulting equations together, we can estimated like [15, p129], For the completeness of the article, it will provided in the appendix of this paper.

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|\omega\|_{L^2}^2 + \|j\|_{L^2}^2) &= \int_{\mathbb{R}^2} T(\nabla u, \nabla b) j dx - \|\Lambda^\alpha\omega\|_{L^2}^2 - \|\mathcal{L}^\beta j\|_{L^2}^2 \\ &\leq C\|\omega\|_{L^2}^2\|j\|_{L^2}^2 + \frac{1}{2\varepsilon}\|\nabla j\|_{L^2}^2 - \|\Lambda^\alpha\omega\|_{L^2}^2 - \|\mathcal{L}^\beta j\|_{L^2}^2,\end{aligned} \tag{3.6}$$

About this term $\|\nabla j\|_{L^2}^2$, we obtain

$$\begin{aligned}\|\nabla j\|_{L^2}^2 &= \int |\xi|^2 \widehat{j}\overline{\widehat{j}} d\xi \\ &= \int_{|\xi| \leq 1} |\xi|^2 \widehat{j}\overline{\widehat{j}} d\xi + \int_{|\xi|>1} |\xi|^2 \widehat{j}\overline{\widehat{j}} d\xi \\ &\leq \int_{|\xi| \leq 1} \widehat{j}\overline{\widehat{j}} d\xi + \int_{|\xi|>1} |\xi|^2 \widehat{j}\overline{\widehat{j}} d\xi,\end{aligned}$$

where

$$\begin{aligned}\int_{|\xi|>1} |\xi|^2 \widehat{j}\overline{\widehat{j}} d\xi &= \int_{|\xi|>1} \frac{|\xi|^{2\beta}}{g^2(|\xi|)} \widehat{j}\overline{\widehat{j}} \cdot |\xi|^2 \frac{g^2(|\xi|)}{|\xi|^{2\beta}} d\xi \\ &\leq \sup_{|\xi|>1} \frac{|\xi|^2}{|\xi|^{2\beta}} g^2(|\xi|) \int_{|\xi|>1} \frac{|\xi|^{2\beta}}{g^2(|\xi|)} \widehat{j}\overline{\widehat{j}} d\xi \\ &\leq \sup_{|\xi|>1} \frac{\ln(e + |\xi|)}{|\xi|^{2(\beta-1)}} \int_{|\xi|>1} \frac{|\xi|^{2\beta}}{g^2(|\xi|)} \widehat{j}\overline{\widehat{j}} d\xi \\ &\leq M\|\mathcal{L}^\beta j\|_{L^2}^2.\end{aligned}$$

when $\beta > 1$, the function is bounded. $g(|\xi|)$ such that there exists an absolute constant $c \geq 0$ satisfying

$$g^2(|\xi|) \leq c \ln(e + |\xi|).$$

so we can get

$$\|\nabla j\|_{L^2}^2 \leq \|j\|_{L^2}^2 + M\|\mathcal{L}^\beta j\|_{L^2}^2. \tag{3.7}$$

As well as

$$\begin{aligned}\|\mathcal{L}^\beta b\|_{L^2}^2 &= \int \frac{|\xi|^{2\beta}}{g^2(|\xi|)}\widehat{b}\bar{\widehat{b}}d\xi\\ &= \int |\xi|^2\widehat{b}\bar{\widehat{b}}\frac{|\xi|^{2\beta}}{|\xi|^2 g^2(|\xi|)}d\xi\\ &\le \sup\frac{|\xi|^{2\beta-2}}{g^2(|\xi|)}\int |\xi|^2\widehat{b}\bar{\widehat{b}}d\xi\\ &\le \sup\frac{|\xi|^{2(\beta-1)}}{\ln(e+|\xi|)}\int |\xi|^2\widehat{b}\bar{\widehat{b}}d\xi\\ &\le C\|j\|_{L^2}^2.\end{aligned}$$

when $\beta > 1$, if $\int_0^t \|\mathcal{L}^\beta b\|_{L^2}^2 d\tau$ is bounded, $\int_0^t \|j\|_{L^2}^2 d\tau$ is also bounded.

$$\int_0^t \|\mathcal{L}^\beta b\|_{L^2}^2 d\tau \le C \Longrightarrow \int_0^t \|j\|_{L^2}^2 d\tau \le C. \tag{3.8}$$

putting (3.7) into (3.6), we obtain

$$\begin{aligned}\frac{d}{dt}(\|\omega\|_{L^2}^2 + \|j\|_{L^2}^2) &\le C\|\omega\|_{L^2}^2\|j\|_{L^2}^2 + \frac{1}{\varepsilon}\|j\|_{L^2}^2\\ &\quad + \frac{M}{\varepsilon}\|\mathcal{L}^\beta j\|_{L^2}^2 - 2\|\Lambda^\alpha\omega\|_{L^2}^2 - 2\|\mathcal{L}^\beta j\|_{L^2}^2.\end{aligned}$$

taking ε small enough so that $\varepsilon = M$, and using Gronwall'inequality and (3.8), we obtain

$$\|\omega\|_{L^2}^2(t) + \|j\|_{L^2}^2(t) + \int_0^t (2\|\Lambda^\alpha\omega\|_{L^2}^2 + \|\mathcal{L}^\beta j\|_{L^2}^2)d\tau \le C(T).$$

The proof of the lemma is completed. □

Lemma 3.3. *(Lemma2.2, [16]) Suppose that* $0 < \alpha < \frac{1}{2}$, $\beta > \beta_1 > 1$, $r = \alpha + \beta_1 - 1 > 0$ *and* $k \ge \alpha + \beta$. *Let* $u_0, b_0 \in H^k$. *Then for any* $T > 0$, *we have*

$$\|\Lambda^r j\|_{L^2}^2 + \int_0^t \|\mathcal{L}^\beta \Lambda^r j\|_{L^2}^2 d\tau \le C(u_0, b_0, T). \tag{3.9}$$

Proof. Applying Λ^r on both sides of (3.4), and multiplying by $\Lambda^r j$, integrating over $\mathbb{R}^2$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\Lambda^r j\|_{L^2}^2 + \|\mathcal{L}^\beta\Lambda^r j\|_{L^2}^2\\ &= -\int_{\mathbb{R}^2}\Lambda^r(u\cdot\nabla j)\Lambda^r j dx + \int_{\mathbb{R}^2}\Lambda^r(b\cdot\nabla\omega)\Lambda^r j dx\\ &+ \int_{\mathbb{R}^2}\Lambda^r(T(\nabla u, \nabla b))\Lambda^r j dx\\ &= A_1 + A_2 + A_3.\end{aligned} \tag{3.10}$$

Now, we are ready to estimate the three terms.

For A_1, we can estimate like [16, p480],

$$\begin{aligned} A_1 \leq & \epsilon\|\Lambda^{\beta_1+r}j\|_{L^2}^2 + C\|u\|_{L^2}^{1-2\alpha}\|\omega\|_{L^2}^{1+2\alpha}\|j\|_{L^2}^{2-\frac{1}{\beta_1}}\|\Lambda^{\beta_1}j\|_{L^2}^{\frac{1}{\beta_1}} \\ & + C\|u\|_{L^2}\|\omega\|_{L^2}\|j\|_{L^2}^{\frac{2\beta_1-2\alpha-1}{\beta_1}}\|\Lambda^{\beta_1}j\|_{L^2}^{\frac{1+2\alpha}{\beta_1}}. \end{aligned} \tag{3.11}$$

About this term $\|\Lambda^{\beta_1+r}j\|_{L^2}^2$, we have

$$\begin{aligned} \|\Lambda^{\beta_1+r}j\|_{L^2}^2 &= \int |\xi|^{2(\beta_1+r)}\widehat{j}\overline{\widehat{j}}d\xi \\ &= \int_{|\xi|\leq 1} |\xi|^{2(\beta_1+r)}\widehat{j}\overline{\widehat{j}}d\xi + \int_{|\xi|>1} |\xi|^{2(\beta_1+r)}\widehat{j}\overline{\widehat{j}}d\xi \\ &\leq \int_{|\xi|\leq 1} \widehat{j}\overline{\widehat{j}}d\xi + \int_{|\xi|>1} |\xi|^{2(\beta_1+r)}\widehat{j}\overline{\widehat{j}}d\xi, \end{aligned}$$

where

$$\begin{aligned} \int_{|\xi|>1} |\xi|^{2(\beta_1+r)}\widehat{j}\overline{\widehat{j}}d\xi &= \int_{|\xi|>1} |\xi|^{2r}\frac{|\xi|^{2\beta}}{g^2(|\xi|)}\widehat{j}\overline{\widehat{j}} \cdot |\xi|^{2\beta_1}\frac{g^2(|\xi|)}{|\xi|^{2\beta}}d\xi \\ &\leq \sup_{|\xi|>1}\frac{g^2(|\xi|)}{|\xi|^{2(\beta-\beta_1)}}\int_{|\xi|>1} |\xi|^{2r}\frac{|\xi|^{2\beta}}{g^2(|\xi|)}\widehat{j}\overline{\widehat{j}}d\xi \\ &\leq \sup_{|\xi|>1}\frac{\ln(e+|\xi|)}{|\xi|^{2(\beta-\beta_1)}}\int_{|\xi|>1} |\xi|^{2r}\frac{|\xi|^{2\beta}}{g^2(|\xi|)}\widehat{j}\overline{\widehat{j}}d\xi \\ &\leq M\|\mathcal{L}^{\beta}\Lambda^r j\|_{L^2}^2. \end{aligned}$$

when $\beta > \beta_1 > 1$, the function is bounded. $g(|\xi|)$ such that there exists an absolute constant $c \geq 0$ satisfying

$$g^2(|\xi|) \leq c\ln(e+|\xi|).$$

so we can get

$$\|\Lambda^{\beta_1+r}j\|_{L^2}^2 \leq \|j\|_{L^2}^2 + M\|\mathcal{L}^{\beta}\Lambda^r j\|_{L^2}^2. \tag{3.12}$$

putting (3.12) into (3.11), we have

$$\begin{aligned} A_1 \leq & M\epsilon\|\mathcal{L}^{\beta}\Lambda^r j\|_{L^2}^2 + \epsilon\|j\|_{L^2}^2 + C\|u\|_{L^2}^{1-2\alpha}\|\omega\|_{L^2}^{1+2\alpha}\|j\|_{L^2}^{2-\frac{1}{\beta_1}}\|\Lambda^{\beta_1}j\|_{L^2}^{\frac{1}{\beta_1}} \\ & + C\|u\|_{L^2}\|\omega\|_{L^2}\|j\|_{L^2}^{\frac{2\beta_1-2\alpha-1}{\beta_1}}\|\Lambda^{\beta_1}j\|_{L^2}^{\frac{1+2\alpha}{\beta_1}}, \end{aligned}$$

for $A_2 - A_3$, we can estimate like [16, p480], and using (3.12), we get

$$\begin{aligned} A_2 \leq & \epsilon\|\Lambda^{\beta_1+r}j\|_{L^2}^2 + C\|b\|_{L^2}^{\frac{2(\beta_1-\alpha)}{1+\beta_1}}\|\Lambda^{\beta_1}j\|_{L^2}^{\frac{2(1+\alpha)}{1+\beta_1}}\|\omega\|_{L^2}^2 \\ & + C\|b\|_{L^2}^{\frac{2r}{1+r}}\|\Lambda^r j\|_{L^2}^{\frac{2}{1+r}}\|\Lambda^{\alpha}\omega\|_{L^2}^2 \\ \leq & M\epsilon\|\mathcal{L}^{\beta}\Lambda^r j\|_{L^2}^2 + \epsilon\|j\|_{L^2}^2 + C\|b\|_{L^2}^{\frac{2(\beta_1-\alpha)}{1+\beta_1}}\|\Lambda^{\beta_1}j\|_{L^2}^{\frac{2(1+\alpha)}{1+\beta_1}}\|\omega\|_{L^2}^2 \\ & + C\|b\|_{L^2}^{\frac{2r}{1+r}}\|\Lambda^r j\|_{L^2}^{\frac{2}{1+r}}\|\Lambda^{\alpha}\omega\|_{L^2}^2, \\ A_3 \leq & \epsilon\|\Lambda^{\beta_1+r}j\|_{L^2}^2 + C\|j\|_{L^2}^2\|\omega\|_{L^2}^{\frac{2(\beta_1+r)}{2\beta_1-1}} \\ \leq & M\epsilon\|\mathcal{L}^{\beta}\Lambda^r j\|_{L^2}^2 + \epsilon\|j\|_{L^2}^2 + C\|j\|_{L^2}^2\|\omega\|_{L^2}^{\frac{2(\beta_1+r)}{2\beta_1-1}}. \end{aligned}$$

Finally, putting the above results of $A_1 - A_3$ into (3.10), we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\Lambda^r j\|_{L^2}^2 + \|\mathcal{L}^\beta \Lambda^r j\|_{L^2}^2 \le\, & C\|u\|_{L^2}^{1-2\alpha}\|\omega\|_{L^2}^{1+2\alpha}\|j\|_{L^2}^{2-\frac{1}{\beta_1}}\|\Lambda^{\beta_1} j\|_{L^2}^{\frac{1}{\beta_1}} \\
& + C\|u\|_{L^2}\|\omega\|_{L^2}\|j\|_{L^2}^{\frac{2\beta_1-2\alpha-1}{\beta_1}}\|\Lambda^{\beta_1} j\|_{L^2}^{\frac{1+2\alpha}{\beta_1}} \\
& + C\|b\|_{L^2}^{\frac{2(\beta_1-\alpha)}{1+\beta_1}}\|\Lambda^{\beta_1} j\|_{L^2}^{\frac{2(1+\alpha)}{1+\beta_1}}\|\omega\|_{L^2}^2 \\
& + C\|b\|_{L^2}^{\frac{2r}{1+r}}\|\Lambda^r j\|_{L^2}^{\frac{2}{1+r}}\|\Lambda^\alpha \omega\|_{L^2}^2 \\
& + \epsilon\|j\|_{L^2}^2 + M\epsilon\|\mathcal{L}^\beta \Lambda^r j\|_{L^2}^2 \\
& + C\|j\|_{L^2}^2\|\omega\|_{L^2}^{\frac{2(\beta_1+r)}{2\beta_1-1}}.
\end{aligned}$$

taking ϵ small enough so that $\epsilon = \frac{1}{M}$, and by Gronwall's inequality and Lemma 3.2, we obtain

$$\|\Lambda^r j\|_{L^2}^2 + \int_0^t \|\mathcal{L}^\beta \Lambda^r j\|_{L^2}^2 d\tau \le C(u_0, b_0, T).$$

The proof of the lemma is completed. □

4. Proof of Theorem 1.1

In this section, we devoted to prove Theorem 1.1:

Proof. Combining Lemma 3.1 and Lemma 3.2, we can move on to H^2 estimates. Differentiating (3.3)-(3.4), we get

$$(\partial_i\omega)_t + u\cdot\nabla(\partial_i\omega) = -(\partial_i u)\cdot\nabla\omega + (\partial_i b)\cdot\nabla j + b\cdot\nabla(\partial_i j) - \Lambda^{2\alpha}(\partial_i\omega), \quad (4.1)$$

$$\begin{aligned}
(\partial_i j)_t + u\cdot\nabla(\partial_i j) = & -(\partial_i u)\cdot\nabla j + (\partial_i b)\cdot\nabla\omega + b\cdot\nabla(\partial_i\omega) \\
& + \partial_i(T(\nabla u,\nabla b)) - \mathcal{L}^{2\beta}(\partial_i j).
\end{aligned} \quad (4.2)$$

Multiplying by $\partial_i\omega$ and $\partial_i j$ both sides of (4.1)-(4.2) respectively, integrating over $\mathbb{R}^2$ and taking the divergence free property into account, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla\omega\|_{L^2}^2 + \|\nabla j\|_{L^2}^2) + \|\Lambda^\alpha\nabla\omega\|_{L^2}^2 + \|\mathcal{L}^\beta\nabla j\|_{L^2}^2 \\
&= -\int_{\mathbb{R}^2}\sum_{i=1}^2[(\partial_i u)\cdot\nabla\omega]\partial_i\omega dx + \int_{\mathbb{R}^2}\sum_{i=1}^2[(\partial_i b)\cdot\nabla j]\partial_i\omega dx \\
&- \int_{\mathbb{R}^2}\sum_{i=1}^2[(\partial_i u)\cdot\nabla j]\partial_i j dx + \int_{\mathbb{R}^2}\sum_{i=1}^2[(\partial_i b)\cdot\nabla\omega]\partial_i j dx \\
&+ \int_{\mathbb{R}^2}\sum_{i=1}^2[\partial_i(T(\nabla u,\nabla b))]\partial_i j dx \\
&\le C(I_1+I_2+I_3+I_4+I_5).
\end{aligned} \quad (4.3)$$

Now, we are ready to give the estimate for the right hand of (4.3).

I_1 can be estimated like [16, p483]. For completeness of the article, it will provided in the appendix of this paper,

$$I_1 \leq C\varepsilon\|\omega\|_{L^p}^{\frac{p\alpha}{p\alpha-1}}\|\nabla\omega\|_{L^2}^2 + C(\varepsilon)\|\Lambda^{1+\alpha}\omega\|_{L^2}^2. \tag{4.4}$$

where we know the fact $\alpha > 0$ and $p > \frac{1}{\alpha}$.

We can estimate $\|\omega\|_{L^p}$ like [16, p483], the detailed process about (4.5) in appendix, we have

$$\frac{d}{dt}\|\omega\|_{L^p} \leq \|b\|_{L^\infty}\|\nabla j\|_{L^p}. \tag{4.5}$$

combining Lemma 3.3 and Sobolev embedding, we get

$$j \in L^2(0,T;H^{\beta_1+r}) \ \Rightarrow \ b \in L^2(0,T;L^\infty), \nabla j \in L^2(0,T;L^p).$$

In order to get $\|\nabla j\|_{L^p}$ bounded by using the Gagliardo-Nirenberg inequality,

$$\|\nabla j\|_{L^p} \leq \|j\|_{L^2}^{1-\theta}\|\Lambda^{r+\beta_1}j\|_{L^2}^{\theta}.$$

where

$$\theta = (1-\frac{1}{p})\frac{2}{r+\beta_1}\ ,\ 0<\theta<1 \quad \Rightarrow \quad p < \frac{2}{2-(r+\beta_1)}.$$

because of $r = \alpha + \beta_1 - 1$ and $\beta > \beta_1$,

$$\frac{1}{\alpha} < p < \frac{2}{3-(\alpha+2\beta)}.$$

so if $\alpha + 2\beta < 3$, we get

$$\frac{1}{\alpha} < \frac{2}{3-(2\beta+\alpha)} \ \Rightarrow \ 3\alpha + 2\beta > 3;$$

on the other hand, $\alpha + 2\beta \geq 3$, thus we can choose any number, such that

$$\frac{1}{\alpha} < p < \infty.$$

therefore, when $3\alpha + 2\beta > 3$, we have

$$\|\omega\|_{L^p} \leq \|\omega_0\|_{L^p} + \|b\|_{L^2(0,T;L^\infty)}\|\nabla j\|_{L^2(0,T;L^p)} \leq C(\omega_0, T). \tag{4.6}$$

For I_2 and I_4, we can estimate in a straight way (see [14, p4201]),

$$I_2 = I_4 \leq C(\varepsilon)\|j\|_{L^2}^2 + \|\nabla j\|_{L^2}^2\|\nabla\omega\|_{L^2}^2 + \varepsilon\|\Lambda\nabla j\|_{L^2}^2. \tag{4.7}$$

About this term $\|\Lambda\nabla j\|_{L^2}^2$, we have

$$\begin{aligned}
\|\Lambda\nabla j\|_{L^2}^2 &= \int |\xi|^4\widehat{j}\overline{\widehat{j}}d\xi \\
&= \int_{|\xi|\leq 1}|\xi|^4\widehat{j}\overline{\widehat{j}}d\xi + \int_{|\xi|>1}|\xi|^4\widehat{j}\overline{\widehat{j}}d\xi \\
&\leq \int_{|\xi|\leq 1}\widehat{j}\overline{\widehat{j}}d\xi + \int_{|\xi|>1}|\xi|^4\widehat{j}\overline{\widehat{j}}d\xi,
\end{aligned}$$

where

$$\begin{aligned}\int_{|\xi|>1}|\xi|^4\widehat{j}\overline{\widehat{j}}d\xi &= \int_{|\xi|>1}|\xi|^2\frac{|\xi|^{2\beta}}{g^2(|\xi|)}\widehat{j}\overline{\widehat{j}}\cdot|\xi|^2\frac{g^2(|\xi|)}{|\xi|^{2\beta}}d\xi\\ &\leq \sup_{|\xi|>1}\frac{g^2(|\xi|)}{|\xi|^{2(\beta-1)}}\int_{|\xi|>1}|\xi|^2\frac{|\xi|^{2\beta}}{g^2(|\xi|)}\widehat{j}\overline{\widehat{j}}d\xi\\ &\leq \sup_{|\xi|>1}\frac{\ln(e+|\xi|)}{|\xi|^{2(\beta-1)}}\int_{|\xi|>1}|\xi|^2\frac{|\xi|^{2\beta}}{g^2(|\xi|)}\widehat{j}\overline{\widehat{j}}d\xi\\ &\leq M\|\mathcal{L}^\beta\nabla j\|_{L^2}^2.\end{aligned}$$

when $\beta > 1$, the function is bounded. $g(|\xi|)$ such that there exists an absolute constant $c \geq 0$ satisfying

$$g^2(|\xi|) \leq c\ln(e+|\xi|).$$

then, we can get

$$\|\Lambda\nabla j\|_{L^2}^2 \leq \|j\|_{L^2}^2 + M\|\mathcal{L}^\beta\nabla j\|_{L^2}^2. \tag{4.8}$$

putting (4.8) into (4.7), we get

$$I_2 = I_4 \leq C(\varepsilon)\|j\|_{L^2}^2 + \|\nabla j\|_{L^2}^2\|\nabla\omega\|_{L^2}^2 + M\varepsilon\|\mathcal{L}^\beta\nabla j\|_{L^2}^2,$$

thus, I_3 and I_5 also can be estimated like [14, p4202], and using (4.8), we obtain

$$\begin{aligned}I_3 &\leq C(\varepsilon)\|\omega\|_{L^2}^2\|\nabla j\|_{L^2}^2 + \varepsilon\|\Lambda\nabla j\|_{L^2}^2\\ &\leq C(\varepsilon)\|\omega\|_{L^2}^2\|\nabla j\|_{L^2}^2 + C(\varepsilon)\|j\|_{L^2}^2 + M\varepsilon\|\mathcal{L}^\beta\nabla j\|_{L^2}^2,\\ I_5 &\leq C(\varepsilon)\|j\|_{L^2}^2 + \|\nabla j\|_{L^2}^2\|\nabla\omega\|_{L^2}^2 + C(\varepsilon)\|\omega\|_{L^2}^2\|\nabla j\|_{L^2}^2 + \varepsilon\|\Lambda\nabla j\|_{L^2}^2\\ &\leq C(\varepsilon)\|j\|_{L^2}^2 + \|\nabla j\|_{L^2}^2\|\nabla\omega\|_{L^2}^2 + C(\varepsilon)\|\omega\|_{L^2}^2\|\nabla j\|_{L^2}^2 + M\varepsilon\|\mathcal{L}^\beta\nabla j\|_{L^2}^2.\end{aligned}$$

Finally, putting the above results of $I_1 - I_5$ into (4.3), we deduce

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\nabla\omega\|_{L^2}^2 + \|\nabla j\|_{L^2}^2) + \|\Lambda^\alpha\nabla\omega\|_{L^2}^2 + \|\mathcal{L}^\beta\nabla j\|_{L^2}^2\\ &\leq C(\varepsilon)(\|\omega\|_{L^2}^2 + \|\nabla j\|_{L^2}^2 + \|\omega\|_{L^p}^{\frac{p\alpha}{p\alpha-1}})(\|\nabla\omega\|_{L^2}^2 + \|\nabla j\|_{L^2}^2)\\ &+ C(\varepsilon)\|j\|_{L^2}^2 + C\varepsilon\|\Lambda^{1+\alpha}\omega\|_{L^2}^2 + M\varepsilon\|\mathcal{L}^\beta\nabla j\|_{L^2}^2.\end{aligned} \tag{4.9}$$

taking ε small enough so that $C\varepsilon = M\varepsilon = \frac{1}{2}$, and using the Gronwall's inequality and (4.6), we get

$$(\|\nabla\omega\|_{L^2}^2 + \|\nabla j\|_{L^2}^2) + \int_0^t(\|\Lambda^\alpha\nabla\omega\|_{L^2}^2 + \|\mathcal{L}^\beta\nabla j\|_{L^2}^2)d\tau \leq C(T).$$

therefore, we have $\omega, j \in L^2(0,T;L^\infty)$.

When $0 < \alpha < \frac{1}{2}, \beta > 1, 3\alpha + 2\beta > 3$, this completes the proof of Theorem1.1. □

5. Appendix

In this appendix, we will provides the detailed proof in the previous sections.

Proof of (3.6) **of Lemma 3.2 :**
As the previous reason, we have (see [15, p129])

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\omega\|_{L^2}^2+\|j\|_{L^2}^2)+\|\Lambda^\alpha\omega\|_{L^2}^2+\|\mathcal{L}^\beta j\|_{L^2}^2\\
&=\int_{\mathbb{R}^2}T(\nabla u,\nabla b)jdx\\
&\le C\|\omega\|_{L^2}\|j\|_{L^4}^2\\
&\le C\|\omega\|_{L^2}\|j\|_{L^2}\|\nabla j\|_{L^2}\\
&\le C\|\omega\|_{L^2}^2\|j\|_{L^2}^2+\frac{1}{2\varepsilon}\|\nabla j\|_{L^2}^2.
\end{aligned}$$

where we have used the Gagliardo-Nirenberg inequality:

$$\|j\|_{L^4}\le\|j\|_{L^2}^{\frac{1}{2}}\|\nabla j\|_{L^2}^{\frac{1}{2}}.$$

Proof of (4.4) **of the proof of Theorem 1.1 :**
We are ready to give the estimate I_1(see [16, p483]),

$$\begin{aligned}
I_1=&\int_{\mathbb{R}^2}|\nabla u||\nabla\omega|^2dx\\
\le&C\|\nabla u\|_{L^p}\|\nabla\omega\|_{L^{\frac{2p}{p-1}}}^2\\
\le&C\|\omega\|_{L^p}\|\nabla\omega\|_{L^{\frac{2p}{p-1}}}^2\\
\le&C\|\omega\|_{L^p}\|\nabla\omega\|_{L^2}^{2-\frac{2}{p\alpha}}\|\Lambda^{1+\alpha}\omega\|_{L^2}^{\frac{2}{p\alpha}}\\
\le&C\varepsilon\|\omega\|_{L^p}^{\frac{p\alpha}{p\alpha-1}}\|\nabla\omega\|_{L^2}^2+C(\varepsilon)\|\Lambda^{1+\alpha}\omega\|_{L^2}^2.
\end{aligned}$$

where we used the Gagliardo-Nirenberg inequality

$$\|\nabla\omega\|_{L^{\frac{2p}{p-1}}}\le C\|\nabla\omega\|_{L^2}^{1-\frac{1}{p\alpha}}\|\Lambda^{1+\alpha}\omega\|_{L^2}^{\frac{1}{p\alpha}}\ ,\ p>\frac{1}{\alpha}.$$

Thus we have (4.4).

Proof of (4.5) **of the proof of Theorem 1.1 :**
We multiply both side of (3.3) by $|\omega|^{p-2}\omega(p>2)$ and integrate with respect to x in $\mathbb{R}^2$ to obtain(see [16, p483])

$$\frac{1}{p}\frac{d}{dt}\|\omega\|_{L^p}^p+\int_{\mathbb{R}^2}(\Lambda^\alpha\omega)|\omega|^{p-2}\omega dx=\int_{\mathbb{R}^2}(b\cdot\nabla)j|\omega|^{p-2}\omega dx.$$

where we have used $\nabla\cdot u=0$ and the following property [28]:

$$\int_{\mathbb{R}^2}(\Lambda^\alpha\omega)|\omega|^{p-2}\omega dx\ge0.$$

we have

$$\frac{d}{dt}\|\omega\|_{L^p}\le\|b\|_{L^\infty}\|\nabla j\|_{L^p}.$$

Thus we have proved (4.5).

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