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Stationary Distribution and Extinction of Stochastic HTLV-I Infection Model with CTL Immune Response under Regime Switching*

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Abstract In this paper, the stochastic HTLV-I infection model with CTL immune response is investigated. Firstly, we show that the stochastic system exists unique positive global solution originating from the positive initial value. Secondly, we obtain that the existence of ergodic stationary distribution of the model by stochastic *Lyapunov* functions. Thirdly, we establish sufficient conditions for extinction of the infected cells. Finally, numerical simulations are carried out to illustrate the theoretical results.

Keywords Stochastic HTLV-I infection model, Ergodic stationary distribution, Extinction, Markov switching.

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1. Introduction

Human T-cell leukemia virus includes type-I (HTLV-I) and type-II (HTLV-II), it is pathogen that causes T-cell leukemia and lymphoma in adults. HTLV-I can be transmitted by blood transfusions, injection or sexual contact, or by placenta, birth canal as well as breast-feeding which harms public health, human society and world economy seriously [1–3]. To fight against HTLV-I, which is a kind of infectious disease, we need to pay enough attention to inventing effective drugs and updating treatment methods. What's more, it is known that the dynamic nature of virus spread also has practical significance for disease prevention and control [4, 5].

The Cytotoxic T lymphocyte (CTL) play an important role in antiviral mechanism and they are the main immune factor inhibiting cell replication [6]. In certain infectious diseases, specific CTL kills infected cells, not viruses, such as hepatitis B. Therefore, the dynamics of virus infection model with CTL response has attracted lots of researchers attention [7, 8], which is essential for identifying risk factors of the HAM/TSP development and taking therapeutic measures [9].

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The CTL immune response to a single pool of infected cells has been considered in the HTLV-I infection. This interaction can be described by the following system [10].

$$\begin{cases} \dot{x}(t) = \lambda - \beta x(t)y(t) - d_1 x(t), \\ \dot{y}(t) = \beta x(t)y(t) - ay(t)z(t) - d_2 y(t), \\ \dot{z}(t) = py(t) - d_3 z(t), \end{cases} \tag{1.1}$$

where $x(t), y(t)$ and $z(t)$ are numbers of uninfected of cells, infected cells, and CTL immune cells, respectively. Wang etc obtained the global dynamics of differential system (1.1) which is determined by one important threshold parameter $R_0^* = \lambda\beta/(d_1 d_2)$. If $R_0^* > 1$, then the system (1.1) has two steady states, the infection free steady state and the endemic steady state. It is well-known that if a basic reproductive number $R_0^* < 1$, the infection free steady state is locally asymptotically stable and the endemic steady state does not exist [10].
The meaning of the parameters in the model (1.1) is given in the following list:

λ: the production of healthy $CD4^+T$ cells rate;
β: a constant means the infection rate;
a: the rate of CTL elimination;
$py(t)$: the proliferation rate of CTL cells by contacting the infected cells;
d_1: the natural mortality rate of $x(t)$;
d_2: the mortality rate of infected cells caused by virus;
d_3: the natural mortality rate of $z(t)$;

Those magnificent works provide a great perspective of the epidemic model. But in the real world, the virus dynamics model will inevitably be affected by random fluctuations. Aimed to make the virus dynamic model (1.1) better reflect the actual situation, it is essential to take into account the real random interference in the disease dynamics model.

The pathogenesis of different stages of HTLV-I infection is different, chemotherapy treatment, drug, and cell transplantation have the diverse effects. The external environment will influence people's body and mind. As a result, the system possibly changed from one environmental regime to another.

Note that the epidemic models may be perturbed by telegraph noise which can cause the system to switch from one environmental regime to another [11]. Almost the switching between environmental regimes is usually memoryless and the waiting time for the next switching follows the exponential distribution [12]. Therefore the regime switching can be described by a continuous time Markov chain $r(t)_{t>0}$ with values in a finite state space.

In [13], Jiang and Qi thought that the deterministic model (1.1) disturbed by the telegraph noises and white noises. Furthermore, they consider the standard incidence $\beta xy/(x+y)$ instead of the bilinear incidence βxy. Then model (1.1) under regime switching reduces to

$$\begin{cases} dx = [\lambda(r(t)) - \frac{\beta(r(t))x(t)y(t)}{x(t)+y(t)} - d_1(r(t))x(t)]dt + \sigma_1(r(t))x(t)dB_1, \\ dy = [\frac{\beta(r(t))x(t)y(t)}{x(t)+y(t)} - a(r(t))y(t)z(t) - d_2(r(t))y(t)]dt + \sigma_2(r(t))y(t)dB_2, \\ dz = [p(r(t))y(t) - d_3(r(t))z(t)]dt + \sigma_3(r(t))z(t)dB_3, \end{cases} \tag{1.2}$$

where $B_i (i = 1, 2, 3)$ is independent standard Brownian motions and $\sigma_i(r(t)) (i = 1, 2, 3)$ is the intensity of $B_i (i = 1, 2, 3)$.

For system (1.2), Jiang and Qi provided sufficient conditions for the virus to go extinct exponentially and got the existence of ergodic stationary distribution of system (1.2). In fact, May and Anderson pointed out that the standard incidence $\beta xy/(x+y)$ is usually suitable for humans, social animals that a group with self-protection awareness during the spread of disease [4]. However, on the one hand, HTLV-I infection is achieved in body through intercellular contact between healthy cells and infected cells [7]. On the other hand, the conditions of the result obtained in [13] are so strict that the application range of the result of the actual data [2] is too narrow.

Motivated by the works of Jiang etc [13] and Bangham [7]. In this paper, we think the bilinear incidence βxy instead of the standard incidence $\beta xy/(x+y)$. The stochastic model based on the deterministic system (1.1) takes the following form

$$\begin{cases} dx = [\lambda(r(t)) - \beta(r(t))x(t)y(t) - d_1(r(t))x(t)]dt + \sigma_1(r(t))x(t)dB_1, \\ dy = [\beta(r(t))x(t)y(t) - a(r(t))y(t)z(t) - d_2(r(t))y(t)]dt + \sigma_2(r(t))y(t)dB_2, \\ dz = [p(r(t))y(t) - d_3(r(t))z(t)]dt + \sigma_3(r(t))z(t)dB_3. \end{cases} \tag{1.3}$$

We found that there is no research literature on the model (1.3) so far. In this paper, the existence of the stationary distribution of the model (1.3) and the extinction of the virus is explored. As a result, better performances than [13] that in actual data [2] are obtained.

The thesis is organized as follows. In Section 2, some preliminaries and results which are applied in later content are presented. In Section 3, we devoted to verify that the global existence and positivity of solution of system (1.3). In Section 4, we verifying the system (1.3) have a unique ergodic stationary distributions under some simple conditions. In section 5, we establish sufficient conditions for extinction of the infected cells. In section 6, we provide explanation of the theoretical analysis with some numerical simulations.

2. Preliminaries

In this section, we recall some basic knowledge of stochastic differential equations, necessary lemmas and hypotheses in the process of theorem argumentation that we shall use in the rest of the paper.

Throughout this essay, suppose $r(t)_{t\geq 0}$ be a right-continuous Markov chain define on the complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$ with values in a finite space $\mathcal{M} = \{1, 2, \cdots, N\}$. For convenience, we use the notation $\hat{\Psi} = \min_{k\in\mathcal{M}} \Psi(k)$, $\check{\Psi} = \max_{k\in\mathcal{M}} \Psi(k)$, where $\{\Psi(k)\}_{k\in\mathcal{M}}$ is a constant vector. We also denote

$$D(X,k) = [d_{ij}(X,k)], \quad for\ each \quad k \in \mathcal{M}.$$

$$\mathbb{R}^n_+ = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_i > 0, i = 1, \cdots, n\}.$$

$$\sigma(k) = \max\{\sigma_1(k), \sigma_2(k), \sigma_3(k)\}, \quad d_0(k) = \min\{d_1(k), d_2(k)/2, d_3(k)\}.$$

Assume the generator $\Gamma = [\gamma_{ij}]_{N\times N}$ of the Markov chain is given by

$$\mathcal{P}\{r(t+\Delta) = j \mid r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta), & if \quad i \neq j, \\ 1 + \gamma_{ii}\Delta + o(\Delta), & if \quad i = j, \end{cases}$$

where $\Delta > 0, \gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while $\sum_{j=1}^{N} \gamma_{ij} = 0$.

We suppose that Markov chain and Brownian motion are independent. Suppose further that Markov chain $r(t)$ is irreducible, which means that the stochastic system can switch from one regime to the other regime. This implies that Markov chain $r(t)$ has a unique stationary distribution $\pi = \{\pi_1, \pi_2, \cdots, \pi_N\}$ which can be determined by linear equation $\pi\Gamma = 0$ subject to $\sum_{h=1}^{N} \pi_h = 1$, and $\pi_h > 0, \forall h \in \mathcal{M}$. In this paper, we assume $\gamma_{ij} > 0$ for $i \neq j$ and all parameters are positive constants for any $k \in \mathcal{M}$.

We now present some basic theories on the stationary distribution for stochastic differential equations under regime switching. More general, we consider the d-dimensional diffusion process described by the following equation

$$dX(t) = b(X(t), r(t))dt + g(X(t), r(t))dB(t), \quad X(0) = x_0, \quad r(0) = r, \tag{2.1}$$

where $B(t)$ denote an d-dimensional standard Brownian motion defined on the complete probability space$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$, $b(\cdot,\cdot) : \mathbb{R}^n \times \mathcal{M} \to \mathbb{R}^n$, and $\sigma(\cdot,\cdot) : \mathbb{R}^n \times \mathcal{M} \to \mathbb{R}^{n\times d}$.

Denote by $C^{2,1}(\mathbb{R}^d \times [t_0, \infty];\ \mathbb{R}^+)$ the family of all nonnegative functions $V(\cdot, k)$ defined on $\mathbb{R}^d \times [t_0, \infty]$, such that they are continuously differentiable twice on X and once on t.

The differential operator $\mathcal{L}$ can be defined by

$$\mathcal{L}V(X,k) = \sum_{i=1}^{n} b_i(X,k)\frac{\partial V(X,k)}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{n} d_{ij}(X,k)\frac{\partial^2 V(X,k)}{\partial x_i \partial x_j} + \sum_{l=1}^{N} \gamma_{kl} V(X,l),$$

By It$\hat{o}$'s formula, if $X \in \mathbb{R}^d$, we have

$$dV(X,t) = \mathcal{L}V(X,t)dt + V_X(X,t)g(X,t)dB(t).$$

Lemma 2.1 (Lemma 2.1, [14]). *If the following conditions are satisfied:*

(i) $\gamma_{ij} > 0$ *for any* $i \neq j$*;*
(ii) for any $k \in \mathcal{M}, D(X,k)$ *is symmetric and satisfies*

$$\rho|\zeta|^2 \leq \zeta^T D(X,k)\zeta \leq \rho^{-1}|\zeta|^2 \qquad for\ all\ \zeta \in \mathbb{R}^n,$$

with some constant $\rho \in (0,1]$ *for all* $X \in \mathbb{R}^n$.

(iii) there exists a nonempty bounded open subset $\mathcal{U} \in \mathbb{R}^n$ *with a regular boundary satisfying that for each* $k \in \mathcal{M}$, *there is a non-negative twice continuously differential function* $V(X,k) : \mathcal{U}^c \to \mathbb{R}$ *such that for some* $\alpha > 0$,

$$\mathcal{L}V(X,k) \leq -\alpha, \qquad (X,k) \in \mathcal{U}^c \times \mathcal{M},$$

then the diffusion process $(X(t), r(t))$ *for system (2.1) is positive recurrent and ergodic. That is to say, there exists a unique stationary distribution* $\mu(\cdot,\cdot)$ *such that for any Borel measurable function* $f(\cdot,\cdot) : \mathbb{R}^n \times \mathcal{M} \to \mathbb{R}$ *satisfying*

$$\sum_{k\in\mathcal{M}} \int_{\mathbb{R}^n} |f(X,k)|\mu(dX,k) < \infty,$$

we have

$$\mathcal{P}\Big\{\lim_{T\to\infty} \frac{1}{T}\int_0^T f(X(t), r(t))dt = \sum_{k\in\mathcal{M}} \int_{\mathbb{R}^n} f(X,k)\mu(dX,k)\Big\} = 1.$$

.

3. Existence of the positive solution

The global existence of positive solutions is the most important for studying the dynamic behavior of stochastic epidemic models. Next, we will prove by the following theorem that for any given positive initial value, the positive solution of the stochastic system (1.3) exists globally and is unique.

Theorem 3.1. *For any initial value condition* $(x(0), y(0), z(0), r(0)) \in \mathbb{R}^3_+ \times \mathcal{M}$*. The system (1.3) has a unique positive solution* $(x(t), y(t), z(t), r(t)) \in \mathbb{R}^3_+ \times \mathcal{M}$ *on* $t \geq 0$ *with probability one.*

The proof is analogous to that of Theorem 2 in [13]. Hence, we omit it here.

4. Existence of the stationary distribution

A significant aspect of the qualitative analysis of the disease, it has been prevalent in the crowd for a long time and will disappear eventually. In this part, we show that stochastic system (1.3) exists a stationary distribution.

We denote

$$R_1^* = \frac{\hat{\beta} \sum_{h\in\mathcal{M}} \pi_h \lambda(h)}{\check{d}_1 \sum_{h\in\mathcal{M}} \pi_h (d_2(h) + \frac{1}{2}\sigma_2^2(h))}.$$

Theorem 4.1. *Suppose the stochastic system (1.3) takes initial value in* $\mathbb{R}^3_+ \times \mathcal{M}$*,* $\hat{d}_0 > \check{\sigma}^2/2$ *and* $R_1^* > 1$*, then there exists a stationary distribution* $\mu(\cdot,\cdot)$ *for system (1.3).*

Proof. To prove Theorem 4.1, it is only necessary to verify conditions (i), (ii) and (iii) in Lemma 2.1. Presumption $\gamma_{ij} > 0$ if $i \neq j$ in Section 2 means that Lemma 2.1 condition (i) holds. Besides this, the diffusion matrix $D(x, y, z, k) =$ diag $\{\sigma_1^2(k)x^2, \sigma_2^2(k)y^2, \sigma_3^2(k)z^2\}$ is positive definitely for stochastic systems (1.3), which means Lemma 2.1 condition (ii) is satisfied. Next, we prove the condition (iii) in Lemma 2.1 is satisfied.

Define a C^2 function

$$\begin{aligned} g(x, y, z, k) = {} & M\Big(-x - y - \frac{\check{d}_1}{\hat{\beta}} \ln y + \frac{\check{d}_1 \check{a}}{\hat{\beta}\hat{d}_3} z + \omega_k\Big) + \frac{1}{m+2}\Big(x + y + \frac{\hat{d}_2}{2\check{p}} z\Big)^{m+2} \\ & - \ln x - \ln z, \end{aligned}$$

where $m > 0$ is a small enough constant and M is a reasonable large number.

Note that

$$\lim_{(x,y,z)\to 0^+} g(x, y, z, k) = +\infty, \qquad \lim_{(x,y,z)\to +\infty} g(x, y, z, k) = +\infty, \quad k \in \mathcal{M}.$$

According to the continuity of $g(x, y, z, k)$, it must exists a minimum point (x_0, y_0, z_0, k). $g(x_0, y_0, z_0, k)$ is the minimum value of $g(x, y, z, k)$. We define the non-negative C^2 function

$$\begin{aligned} V(x, y, z, k) = {} & M\Big(-x - y - \frac{\check{d}_1}{\hat{\beta}} \ln y + \frac{\check{d}_1 \check{a}}{\hat{\beta}\hat{d}_3} z + \omega_k\Big) + \frac{1}{m+2}\Big(x + y + \frac{\hat{d}_2}{2\check{p}} z\Big)^{m+2} \\ & - \ln x - \ln z - g(x_0, y_0, z_0, k). \end{aligned}$$

As a matter of convenience, denote

$$V_1 = -x - y - \frac{\check{d}_1}{\hat{\beta}} \ln y + \frac{\check{d}_1 \check{a}}{\hat{\beta}\hat{d}_3} z + \omega_k, \qquad V_2 = -\ln x,$$

$$V_3 = -\ln z, \qquad V_4 = \frac{1}{m+2}(x + y + \frac{\hat{d}_2}{2\check{p}} z)^{m+2}.$$

Using Itô′s formula, we can get that

$$\begin{aligned}\mathcal{L}V_1 &= -\lambda(k) + d_1(k)x + d_2(k)y + a(k)yz - \frac{\check{d}_1}{\hat{\beta}}\beta(k)x + \frac{\check{d}_1}{\hat{\beta}}a(k)z + \frac{\check{d}_1}{\hat{\beta}}d_2(k) \\ &+ \frac{1}{2}\frac{\check{d}_1}{\hat{\beta}}\sigma_2^2(k) + \frac{\check{d}_1}{\hat{\beta}}\frac{\check{a}}{\hat{d}_3}p(k)y - \frac{\check{d}_1}{\hat{\beta}}\frac{\check{a}}{\hat{d}_3}d_3(k)z + \sum_{l\in\mathcal{M}}\gamma_{kl}\omega_l \\ &\le -\lambda(k) + a(k)yz + d_2(k)y + \frac{\check{d}_1}{\hat{\beta}}\frac{\check{a}}{\hat{d}_3}p(k)y + \frac{\check{d}_1}{\hat{\beta}}d_2(k) + \frac{1}{2}\frac{\check{d}_1}{\hat{\beta}}\sigma_2^2(k) \\ &+ \sum_{l\in\mathcal{M}}\gamma_{kl}\omega_l. \end{aligned} \tag{4.1}$$

Let

$$R_0(k) = -\lambda(k) + \frac{\check{d}_1}{\hat{\beta}}d_2(k) + \frac{\check{d}_1}{2\hat{\beta}}\sigma_2^2(k).$$

Because of the generator matrix Γ is irreducible, hence, for $R_0 = (R_0(1), ..., R_0(N))^T$, let $\omega = (\omega(1), ..., \omega(N))$ be the solution of the following Poisson system

$$\Gamma\omega + R_0 = \sum_{h\in\mathcal{M}}\pi_h R_0(h)\overrightarrow{1},$$

where $\overrightarrow{1}$ is the N-dimensional column vector composed of 1, which means that

$$R_0(k) + \sum_{l\in\mathcal{M}}\gamma_{kl}\omega_l = \sum_{h\in\mathcal{M}}\pi_h R_0(h).$$

Substituting this equality into (4.1), we obtain

$$\begin{aligned}\mathcal{L}V_1 &\le \sum_{h\in\mathcal{M}}\pi_h R_0(h) + d_2(k)y + \frac{\check{d}_1}{\hat{\beta}}\frac{\check{a}}{\hat{d}_3}p(k)y + a(k)yz \\ &= -(R_1^* - 1)\sum_{h\in\mathcal{M}}\pi_h[\frac{\check{d}_1}{\hat{\beta}}d_2(h) + \frac{\check{d}_1}{2\hat{\beta}}\sigma_2^2(h)] + d_2(k)y + \frac{\check{d}_1}{\hat{\beta}}\frac{\check{a}}{\hat{d}_3}p(k)y + a(k)yz \\ &= -A + d_2(k)y + \frac{\check{d}_1}{\hat{\beta}}\frac{\check{a}}{\hat{d}_3}p(k)y + a(k)yz, \end{aligned} \tag{4.2}$$

where

$$A = (R_1^* - 1)\sum_{h\in\mathcal{M}}\pi_h[\frac{\check{d}_1}{\hat{\beta}}d_2(h) + \frac{\check{d}_1}{2\hat{\beta}}\sigma_2^2(h)].$$

We can also get

$$\mathcal{L}V_2 \le -\frac{\hat{\lambda}}{x} + \check{\beta}y + \check{d}_1 + \frac{1}{2}\check{\sigma}_1^2, \tag{4.3}$$

and

$$\mathcal{L}V_3 \le -\hat{p}\frac{y}{z} + \check{d}_3 + \frac{1}{2}\check{\sigma}_3^2. \tag{4.4}$$

$$\begin{aligned}
\mathcal{L}V_4 &= (x+y+\frac{\hat{d}_2}{2\check{p}}z)^{m+1}[\lambda(k) - d_1(k)x - a(k)yz + \frac{\hat{d}_2}{2\check{p}}p(k)y - d_2(k)y - \frac{\hat{d}_2}{2\check{p}}d_3(k)z] \\
&\quad + \frac{1}{2}(m+1)(x+y+\frac{\hat{d}_2}{2\check{p}}z)^m[\sigma_1^2(k)x^2 + \sigma_2^2(k)y^2 + \frac{\hat{d}_2^2}{4\check{p}^2}\sigma_3^2(k)z^2] \\
&\le \check{\lambda}(x+y+\frac{\hat{d}_2}{2\check{p}}z)^{m+1} + \frac{1}{2}(m+1)\check{\sigma}^2(x+y+\frac{\hat{d}_2}{2\check{p}}z)^{m+2} - \hat{d}_0(x+y+\frac{\hat{d}_2}{2\check{p}}z)^{m+2} \\
&\le B - \frac{1}{2}[\hat{d}_0 - \frac{1}{2}(m+1)\check{\sigma}^2][x^{m+2} + y^{m+2} + (\frac{\hat{d}_2}{2\check{p}}z)^{m+2}],
\end{aligned} \tag{4.5}$$

where

$$B = \sup_{(x,y,z)\in\mathbb{R}_+^3}\left\{\check{\lambda}(x+y+\frac{\hat{d}_2}{2\check{p}}z)^{m+1} - \frac{1}{2}[\hat{d}_0 - \frac{1}{2}(m+1)\check{\sigma}^2](x+y+\frac{\hat{d}_2}{2\check{p}}z)^{m+2}\right\}.$$

In view of (4.2)-(4.5), we can obtain that

$$\begin{aligned}
\mathcal{L}V &\le -MA + M\check{d}_2 y + M\frac{\check{d}_1}{\hat{\beta}}\frac{\check{a}}{\hat{d}_3}\check{p}y + M\check{a}yz - \frac{\hat{\lambda}}{x} + \check{\beta}y + \check{d}_1 + \frac{1}{2}\check{\sigma}_1^2 \\
&\quad - \hat{p}\frac{y}{z} + \check{d}_3 + \frac{1}{2}\check{\sigma}_3^2 + B - \frac{1}{2}[\hat{d}_0 - \frac{1}{2}(m+1)\check{\sigma}^2][x^{m+2} + y^{m+2} + (\frac{\hat{d}_2}{2\check{p}}z)^{m+2}].
\end{aligned}$$

Let us construct a bounded closed set as follows

$$D_\varepsilon = \{(x,y,z)\in\mathbb{R}_+^3 \mid \varepsilon_1 \le x \le \frac{1}{\varepsilon_1}, \varepsilon_2 \le y \le \frac{1}{\varepsilon_2}, \varepsilon_3 \le z \le \frac{1}{\varepsilon_3}\},$$

where $\varepsilon_2 = \varepsilon_1^2, \varepsilon_3 = \varepsilon_2^2$ are small enough satisfying the following conditions

$$E - \frac{\lambda(k)}{\varepsilon_1} \le -1, \quad -MA + F \le -1, \quad -\hat{p}\frac{\varepsilon_2}{\varepsilon_3} + G \le -1, \tag{4.6}$$

$$-\frac{1}{8}[\hat{d}_0 - \frac{1}{2}(m+1)\check{\sigma}^2](\frac{1}{\varepsilon_1})^{m+2} + H \le -1, \tag{4.7}$$

$$-\frac{1}{8}[\hat{d}_0 - \frac{1}{2}(m+1)\check{\sigma}^2](\frac{1}{\varepsilon_2})^{m+2} + I \le -1, \tag{4.8}$$

$$-\frac{1}{8}[\hat{d}_0 - \frac{1}{2}(m+1)\check{\sigma}^2](\frac{1}{\varepsilon_3})^{m+2} + J \le -1, \tag{4.9}$$

where E, F, G, H, I and J are constants, which are respectively defined in (4.11), (4.13), (4.15), (4.17), (4.19) and (4.21).

As a matter of convenience, we can divide the complementary set of D_ε into the following six domains

$$D_\varepsilon^1 = \{(x,y,z)\in\mathbb{R}_+^3 \mid 0 < x < \varepsilon_1\}, \quad D_\varepsilon^2 = \{(x,y,z)\in\mathbb{R}_+^3 \mid 0 < y < \varepsilon_2, x \ge \varepsilon_1\},$$

$$D_\varepsilon^3 = \{(x,y,z) \in \mathbb{R}_+^3 \mid 0 < z < \varepsilon_3, y \geq \varepsilon_2\}, \quad D_\varepsilon^4 = \{(x,y,z) \in \mathbb{R}_+^3 \mid x > \frac{1}{\varepsilon_1}\},$$

$$D_\varepsilon^5 = \{(x,y,z) \in \mathbb{R}_+^3 \mid y > \frac{1}{\varepsilon_2}\}, \quad D_\varepsilon^6 = \{(x,y,z) \in \mathbb{R}_+^3 \mid z > \frac{1}{\varepsilon_3}\}.$$

Then, $\mathbb{R}_+^3 \setminus D_\varepsilon = \bigcup_{i=1}^6 D_\varepsilon^i$. Next, we will show that

$$\mathcal{L}V(x,y,z,k) \leq -1, \qquad (x,y,z,k) \in (\mathbb{R}_+^3 \setminus D_\varepsilon) \times \mathcal{M},$$

which is equivalent to verifying on the above six domains.
Case 1. If $(x,y,z,k) \in D_\varepsilon^1 \times \mathcal{M}$, we have

$$\begin{aligned}
\mathcal{L}V &\leq M\check{d}_2 y + M\frac{\check{d}_1}{\hat{\beta}}\frac{\check{a}}{\hat{d}_3}\check{p}y + M\check{a}yz - \frac{\hat{\lambda}}{x} + \check{\beta}y + \check{d}_1 + \frac{1}{2}\check{\sigma}_1^2 + \check{d}_3 + \frac{1}{2}\check{\sigma}_3^2 \\
&\quad + B - \frac{1}{2}[\hat{d}_0 - \frac{1}{2}(m+1)\check{\sigma}^2][x^{m+2} + y^{m+2} + (\frac{\hat{d}_2}{2\check{p}}z)^{m+2}] \\
&\leq M\check{d}_2 y + M\frac{\check{d}_1}{\hat{\beta}}\frac{\check{a}}{\hat{d}_3}\check{p}y + \check{\beta}y - \frac{1}{4}[\hat{d}_0 - \frac{1}{2}(m+1)\check{\sigma}^2]y^{m+2} \\
&\quad + M\check{a}yz - \frac{1}{4}[\hat{d}_0 - \frac{1}{2}(m+1)\check{\sigma}^2][y^{m+2} + (\frac{\hat{d}_2}{2\check{p}}z)^{m+2}] \\
&\quad + \check{d}_1 + \frac{1}{2}\check{\sigma}_1^2 + \check{d}_3 + \frac{1}{2}\check{\sigma}_3^2 + B - \frac{\hat{\lambda}}{x} \\
&\leq -\frac{\hat{\lambda}}{\varepsilon_1} + E,
\end{aligned} \tag{4.10}$$

where

$$\begin{aligned}
E = \sup_{(x,y,z)\in\mathbb{R}_+^3} &\{M\check{d}_2 y + M\frac{\check{d}_1}{\hat{\beta}}\frac{\check{a}}{\hat{d}_3}\check{p}y + M\check{a}yz + \check{\beta}y + \check{d}_1 + \frac{1}{2}\check{\sigma}_1^2 + \check{d}_3 + \frac{1}{2}\check{\sigma}_3^2 \\
&+ B - \frac{1}{2}[\hat{d}_0 - \frac{1}{2}(m+1)\check{\sigma}^2][x^{m+2} + y^{m+2} + (\frac{\hat{d}_2}{2\check{p}}z)^{m+2}]\}.
\end{aligned} \tag{4.11}$$

From (4.6), we can easy show that

$$\mathcal{L}V \leq -1, \qquad (x,y,z) \in D_\varepsilon^1.$$

Case 2. If $(x,y,z,k) \in D_\varepsilon^2 \times \mathcal{M}$, we get that

$$\begin{aligned}
\mathcal{L}V \leq &-MA + M\check{d}_2\varepsilon_2 + M\frac{\check{d}_1}{\hat{\beta}}\frac{\check{a}}{\hat{d}_3}\check{p}\varepsilon_2 + M\check{a}\varepsilon_2 z + \check{\beta}\varepsilon_2 + \check{d}_1 + \frac{1}{2}\check{\sigma}_1^2 + \check{d}_3 + \frac{1}{2}\check{\sigma}_3^2 \\
&+ B - \frac{1}{2}[\hat{d}_0 - \frac{1}{2}(m+1)\check{\sigma}^2][x^{m+2} + y^{m+2} + (\frac{\hat{d}_2}{2\check{p}}z)^{m+2}].
\end{aligned}$$

In addition

$$\begin{aligned}
\mathcal{L}V \leq &-MA + M\check{d}_2\varepsilon_2 + M\frac{\check{d}_1}{\hat{\beta}}\frac{\check{a}}{\hat{d}_3}\check{p}\varepsilon_2 + M\check{a}\varepsilon_2(z^{m+2}+1) + \check{\beta}\varepsilon_2 + \check{d}_1 + \frac{1}{2}\check{\sigma}_1^2 \\
&+ \check{d}_3 + \frac{1}{2}\check{\sigma}_3^2 + B - \frac{1}{2}[\hat{d}_0 - \frac{1}{2}(m+1)\check{\sigma}^2][x^{m+2} + y^{m+2} + (\frac{\hat{d}_2}{2\check{p}}z)^{m+2}].
\end{aligned}$$

Then

$$\begin{aligned}\mathcal{L}V &\leq -MA + M\check{d}_2\varepsilon_2 + M\frac{\check{d}_1}{\hat{\beta}}\frac{\check{a}}{\hat{d}_3}\check{p}\varepsilon_2 + M\check{a}\varepsilon_2 + \check{\beta}\varepsilon_2 + \check{d}_1 + \frac{1}{2}\check{\sigma}_1^2 + \check{d}_3 \\ &\quad + \frac{1}{2}\check{\sigma}_3^2 + B - \frac{1}{2}(\frac{\hat{d}_2}{2\check{p}})^{m+2}[\hat{d}_0 - \frac{1}{2}(m+1)\check{\sigma}^2]z^{m+2} + M\check{a}\varepsilon_2 z^{m+2} \\ &\leq -MA + F, \end{aligned} \tag{4.12}$$

where

$$\begin{aligned}F = \sup_{(x,y,z)\in\mathbb{R}_+^3} &\{M\check{d}_2\varepsilon_2 + M\frac{\check{d}_1}{\hat{\beta}}\frac{\check{a}}{\hat{d}_3}\check{p}\varepsilon_2 + M\check{a}\varepsilon_2 + \check{\beta}\varepsilon_2 + \check{d}_1 + \frac{1}{2}\check{\sigma}_1^2 + \check{d}_3 \\ &+ \frac{1}{2}\check{\sigma}_3^2 + B - \frac{1}{2}(\frac{\hat{d}_2}{2\check{p}})^{m+2}[\hat{d}_0 - \frac{1}{2}(m+1)\check{\sigma}^2]z^{m+2} + M\check{a}\varepsilon_2 z^{m+2}\}. \end{aligned} \tag{4.13}$$

In view of (4.6), we can obtain that

$$\mathcal{L}V \leq -1, \qquad (x,y,z) \in D_\varepsilon^2.$$

Case 3. If $(x,y,z,k) \in D_\varepsilon^3 \times \mathcal{M}$, we know that

$$\begin{aligned}\mathcal{L}V &\leq -\hat{p}\frac{y}{z} + \check{d}_1 + \frac{1}{2}\check{\sigma}_1^2 + \check{d}_3 + \frac{1}{2}\check{\sigma}_3^2 + B + M\check{d}_2 y + M\frac{\check{d}_1}{\hat{\beta}}\frac{\check{a}}{\hat{d}_3}\check{p}y \\ &\quad + M\check{a}\varepsilon_3 y + \check{\beta}y - \frac{1}{2}[\hat{d}_0 - \frac{1}{2}(m+1)\check{\sigma}^2]y^{m+2} \leq -\hat{p}\frac{\varepsilon_2}{\varepsilon_3} + G, \end{aligned} \tag{4.14}$$

where

$$\begin{aligned}G = \sup_{(x,y,z)\in\mathbb{R}_+^3} &\{\check{d}_1 + \frac{1}{2}\check{\sigma}_1^2 + \check{d}_3 + \frac{1}{2}\check{\sigma}_3^2 + B + M\check{d}_2 y + M\frac{\check{d}_1}{\hat{\beta}}\frac{\check{a}}{\hat{d}_3}\check{p}y \\ &+ M\check{a}\varepsilon_3 y + \check{\beta}y - \frac{1}{2}[\hat{d}_0 - \frac{1}{2}(m+1)\check{\sigma}^2]y^{m+2}\}. \end{aligned} \tag{4.15}$$

It follows from (4.6), we can get that

$$\mathcal{L}V \leq -1, \qquad (x,y,z) \in D_\varepsilon^3.$$

Case 4. If $(x,y,z,k) \in D_\varepsilon^4 \times \mathcal{M}$, we obtain that

$$\begin{aligned}\mathcal{L}V &\leq M\check{d}_2 y + M\frac{\check{d}_1}{\hat{\beta}}\frac{\check{a}}{\hat{d}_3}\check{p}y + M\check{a}yz + \check{\beta}y + \check{d}_1 + \frac{1}{2}\check{\sigma}_1^2 + \check{d}_3 + \frac{1}{2}\check{\sigma}_3^2 + B \\ &\quad - \frac{1}{2}[\hat{d}_0 - \frac{1}{2}(m+1)\check{\sigma}^2][x^{m+2} + y^{m+2} + (\frac{\hat{d}_2}{2\check{p}}z)^{m+2}] \\ &\leq M\check{d}_2 y + M\frac{\check{d}_1}{\hat{\beta}}\frac{\check{a}}{\hat{d}_3}\check{p}y + \check{\beta}y - \frac{1}{8}[\hat{d}_0 - \frac{1}{2}(m+1)\check{\sigma}^2]y^{m+2} \\ &\quad + M\check{a}yz - \frac{1}{4}[\hat{d}_0 - \frac{1}{2}(m+1)\check{\sigma}^2][y^{m+2} + (\frac{\hat{d}_2}{2\check{p}}z)^{m+2}] \\ &\quad + \check{d}_1 + \frac{1}{2}\check{\sigma}_1^2 + \check{d}_3 + \frac{1}{2}\check{\sigma}_3^2 + B - \frac{1}{8}[\hat{d}_0 - \frac{1}{2}(m+1)\check{\sigma}^2](\frac{1}{\varepsilon_1})^{m+2}. \end{aligned} \tag{4.16}$$

Hence, we have

$$\mathcal{L}V \le -\frac{1}{8}[\hat{d}_0 - \frac{1}{2}(m+1)\check{\sigma}^2](\frac{1}{\varepsilon_1})^{m+2} + H,$$

where

$$H = \sup_{(x,y,z)\in\mathbb{R}^3_+}\{M\check{d}_2 y + M\frac{\check{d}_1}{\hat{\beta}}\frac{\check{a}}{\hat{d}_3}\check{p}y + \check{\beta}y - \frac{1}{8}[\hat{d}_0 - \frac{1}{2}(m+1)\check{\sigma}^2]y^{m+2} + M\check{a}yz$$
$$- \frac{1}{4}[\hat{d}_0 - \frac{1}{2}(m+1)\check{\sigma}^2][y^{m+2} + (\frac{\hat{d}_2}{2\check{p}}z)^{m+2}] + \check{d}_1 + \frac{1}{2}\check{\sigma}_1^2 + \check{d}_3 + \frac{1}{2}\check{\sigma}_3^2 + B\} \quad (4.17)$$

From inequality (4.7), we can obtain that

$$\mathcal{L}V \le -1, \qquad (x,y,z)\in D^4_\varepsilon.$$

Case 5. If $(x,y,z,k)\in D^5_\varepsilon\times\mathcal{M}$, we have

$$\begin{aligned}\mathcal{L}V &\le M\check{d}_2 y + M\frac{\check{d}_1}{\hat{\beta}}\frac{\check{a}}{\hat{d}_3}\check{p}y + M\check{a}yz + \check{\beta}y + \check{d}_1 + \frac{1}{2}\check{\sigma}_1^2 + \check{d}_3 + \frac{1}{2}\check{\sigma}_3^2 + B\\ &\quad - \frac{1}{2}[\hat{d}_0 - \frac{1}{2}(m+1)\check{\sigma}^2][x^{m+2} + y^{m+2} + (\frac{\hat{d}_2}{2\check{p}}z)^{m+2}]\\ &\le -\frac{1}{8}[\hat{d}_0 - \frac{1}{2}(m+1)\check{\sigma}^2](\frac{1}{\varepsilon_2})^{m+2} + I, \end{aligned} \quad (4.18)$$

where

$$I = \sup_{(x,y,z)\in\mathbb{R}^3_+}\{M\check{d}_2 y + M\frac{\check{d}_1}{\hat{\beta}}\frac{\check{a}}{\hat{d}_3}\check{p}y + \check{\beta}y - \frac{1}{8}[\hat{d}_0 - \frac{1}{2}(m+1)\check{\sigma}^2]y^{m+2} + M\check{a}yz$$
$$- \frac{1}{4}[\hat{d}_0 - \frac{1}{2}(m+1)\check{\sigma}^2][y^{m+2} + (\frac{\hat{d}_2}{2\check{p}}z)^{m+2}] + \check{d}_1 + \frac{1}{2}\check{\sigma}_1^2 + \check{d}_3 + \frac{1}{2}\check{\sigma}_3^2 + B\}. \quad (4.19)$$

Together with inequality (4.8), we can obtain that

$$\mathcal{L}V \le -1, \qquad (x,y,z)\in D^5_\varepsilon.$$

Case 6. If $(x,y,z,k)\in D^6_\varepsilon\times\mathcal{M}$, we get

$$\begin{aligned}\mathcal{L}V &\le M\check{d}_2 y + M\frac{\check{d}_1}{\hat{\beta}}\frac{\check{a}}{\hat{d}_3}\check{p}y + M\check{a}yz + \check{\beta}y + \check{d}_1 + \frac{1}{2}\check{\sigma}_1^2 + \check{d}_3 + \frac{1}{2}\check{\sigma}_3^2 + B\\ &\quad - \frac{1}{2}[\hat{d}_0 - \frac{1}{2}(m+1)\check{\sigma}^2][x^{m+2} + y^{m+2} + (\frac{\hat{d}_2}{2\check{p}}z)^{m+2}]\\ &\le -\frac{1}{8}[\hat{d}_0 - \frac{1}{2}(m+1)\check{\sigma}^2](\frac{1}{\varepsilon_3})^{m+2} + J, \end{aligned} \quad (4.20)$$

where

$$J = \sup_{(x,y,z)\in\mathbb{R}^3_+}\{M\check{d}_2 y + M\frac{\check{d}_1}{\hat{\beta}}\frac{\check{a}}{\hat{d}_3}\check{p}y + \check{\beta}y - \frac{1}{8}[\hat{d}_0 - \frac{1}{2}(m+1)\check{\sigma}^2]y^{m+2} + M\check{a}yz$$
$$- \frac{1}{4}[\hat{d}_0 - \frac{1}{2}(m+1)\check{\sigma}^2][y^{m+2} + (\frac{\hat{d}_2}{2\check{p}}z)^{m+2}] + \check{d}_1 + \frac{1}{2}\check{\sigma}_1^2 + \check{d}_3 + \frac{1}{2}\check{\sigma}_3^2 + B\}. \quad (4.21)$$

By inequality (4.9), we can obtain that

$$\mathcal{L}V \le -1, \qquad (x, y, z) \in D_\varepsilon^6.$$

Clearly, in view of Eqs. (4.10), (4.12), (4.14), (4.14), (4.18) and (4.20), for any small enough $\varepsilon_1, \varepsilon_2$ and ε_3, we can get that

$$\mathcal{L}V \le -1, \qquad (x, y, z) \in \mathbb{R}_+^3 \setminus D_\varepsilon.$$

Hence, the condition (iii) of Lemma 2.1 is satisfied. This completes the proof. □

5. Extinction of the virus

For the dynamic behavior of epidemic models, finding the condition for virus extinction is a major concern. This has very important practical significance for the prevention of infectious diseases [15]. In this section, we will give clear result of the virus extinction in the system (1.3).

We define

$$R_2^* = \frac{\check{\beta} \sum_{h \in \mathcal{M}} \pi_h \lambda(h)}{\hat{d}_1 \sum_{h \in \mathcal{M}} \pi_h (d_2(h) + \frac{1}{2}\sigma_2^2(h))}.$$

Theorem 5.1. *Suppose the stochastic system (1.3) with initial value in $\mathbb{R}_+^3 \times \mathcal{M}$, and $R_2^* < 1$, then $y(t)$ will go to zero exponentially with probability one for system (1.3).*

Proof. Define $V(t) = \ln y(t)$, applying the generalized Itô$'$s formula, we can get that

$$\begin{aligned} dV(t) &= [\beta(r(t))x - a(r(t))z - d_2(r(t)) - \frac{1}{2}\sigma_2^2(r(t))]dt + \sigma_2(r(t))dB_2(t) \\ &\le [\beta(r(t))x - d_2(r(t)) - \frac{1}{2}\sigma_2^2(r(t))]dt + \sigma_2(r(t))dB_2(t). \end{aligned} \tag{5.1}$$

Integrating both sides of (5.1) from 0 to t, yields

$$\ln y(t) - \ln y(0) \le \check{\beta} \int_0^t x(s)ds - \int_0^t [d_2(r(s)) + \frac{1}{2}\sigma_2^2(r(s))]ds + \int_0^t \sigma_2(r(s))dB_2(s).$$

Note that the strong law of large numbers of martingales [16] and $r(t)$ is irreducible, we can obtain

$$\limsup_{t\to\infty} \frac{1}{t} \ln y(t) < \check{\beta} \limsup_{t\to\infty} \frac{1}{t} \int_0^t x(s)ds - \sum_{h\in\mathcal{M}} \pi_h (d_2(h) + \frac{1}{2}\sigma_2^2(h)). \tag{5.2}$$

Consider the following stochastic system

$$\begin{cases} d\psi(t, r(t)) = [\lambda(r(t)) - d_1(r(t))\psi(t)]dt + \sigma_1(r(t))\psi(t)dB_1(t), \\ \psi(0, r(0)) = x(0, r(0)). \end{cases}$$

By the stochastic comparison principle, we can obtain

$$x(t) \le \psi(t), \quad a.s.$$

Clearly, we can get that

$$\int_0^t d_1(r(s))\psi(s)ds = \int_0^t \lambda(r(s))ds - \psi(t) + \psi(0) + \int_0^t \sigma_1(r(s))\psi(s)dB_1(s).$$

Hence, we can obtain that

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t \psi(s)ds \leq \frac{1}{\hat{d}_1}\sum_{k\in\mathcal{M}} \pi_k\lambda(k).$$

Further, we have

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t x(s)ds \leq \limsup_{t\to\infty} \frac{1}{t}\int_0^t \psi(s)ds \leq \frac{1}{\hat{d}_1}\sum_{k\in\mathcal{M}} \pi_k\lambda(k). \tag{5.3}$$

Substituting (5.3) into (5.2), we can obtain that

$$\begin{aligned}\limsup_{t\to\infty} \frac{1}{t}\ln y(t) &< \frac{\check{\beta}}{\hat{d}_1}\sum_{h\in\mathcal{M}} \pi_h\lambda(h) - \sum_{h\in\mathcal{M}} \pi_h(d_2(h) + \frac{1}{2}\sigma_2^2(h)) \\ &= (R_2^* - 1)\sum_{h\in\mathcal{M}} \pi_h(d_2(h) + \frac{1}{2}\sigma_2^2(h)) \\ &< 0.\end{aligned}$$

This completes the proof. □

6. Discussions and numerical simulations

In order to better explain the results of our theorem to the readers, we will provide numerical simulations to the results of the theorem. The numerical method used here is Milstein′s Higher Order method [17]. Set the finite state space of continuous time Markov chain $r(t)_{t>0}$ as $\mathcal{M} = \{1, 2, 3\}$, and its generator Γ as follows

$$\Gamma = \begin{bmatrix} -1/3 & 1/6 & 1/6 \\ 1/4 & -1/2 & 1/4 \\ 1/4 & 1/4 & -1/2 \end{bmatrix}.$$

We can get that the unique stationary distribution $\pi = (\pi_1, \pi_2, \pi_3) = \left(\frac{3}{7}, \frac{2}{7}, \frac{2}{7}\right)$.

In the work of Wang and Liu et al, the author gives the range of values for each parameters. According to the work, the parameters value in the stochastic system (1.3) are selected as follows [2].

$$\lambda(1) = 10, \lambda(2) = 15, \lambda(3) = 20; d_1(1) = 0.1, d_1(2) = 0.12, d_1(3) = 0.15;$$

$$\beta(1) = 0.03, \beta(2) = 0.04, \beta(3) = 0.05; d_2(1) = 0.6, d_2(2) = 1, d_2(3) = 1.8;$$

$$a(1) = 0.1, a(2) = 0.3, a(3) = 0.5; d_3(1) = 0.2, d_3(2) = 0.3, d_3(3) = 0.4;$$

$$p(1) = 0.4, p(2) = 0.5, p(3) = 0.6; \sigma_1(1) = 0.03, \sigma_1(2) = 0.036, \sigma_1(3) = 0.045;$$

$$\sigma_2(1) = 0.12, \sigma_2(2) = 0.2, \sigma_2(3) = 0.36; \sigma_3(1) = 0.06, \sigma_3(2) = 0.09, \sigma_3(3) = 0.12;$$

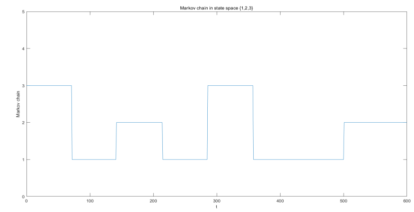

Figure 1. The Markovian chain of the ergodic system, state space $\mathcal{M} = \{1, 2, 3\}$

Obviously

$$R_1^* = \frac{\sum_{h\in\mathcal{M}} \pi_h \lambda(h)}{\frac{\check{d}_1}{\hat{\beta}} \sum_{h\in\mathcal{M}} \pi_h (d_2(h) + \frac{1}{2}\sigma_2^2(h))} = 2.6346 > 1, \quad \hat{d}_0 > \check{\sigma}^2/2.$$

Therefore, according to Theorem 4.1, the solution $(x(t), y(t), z(t))$ of stochastic system (1.3) is positive recurrence and admits a unique ergodic stationary distribution. These assertion are support by Figs 1-2, respectively.

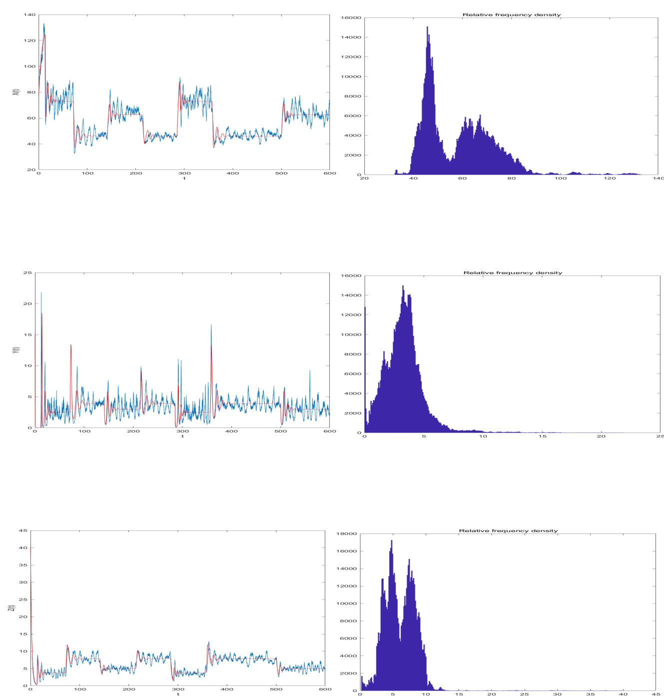

Figure 2. The solution of stochastic system (1.3) and its histogram with initial value condition $(x(0) = 80, y(0) = 15, z(0) = 40, k = 3)$

In the Figure 2, the solution trajectories for stochastic system (1.3) is represented by the green line and the corresponding deterministic system (1.1) is represented by red line. It is vivid show that the stochastic system (1.3) switches from one state to other state according to the law of Markov chain $r(t)$. The right three subgraphs describe the density function of the stationary distribution of $x(t), y(t), z(t)$ respectively. It is displayed directly that the distribution diagrams have local wave crests are corresponding to the three states $\{1, 2, 3\}$ of the Markov chain $r(t)$.

To demonstrate Theorem 5.1, the part parameters value in the stochastic system (1.3) are selected as follows

$$d_1(1) = 0.18, d_1(2) = 0.19, d_1(3) = 0.2; \beta(1) = 0.01, \beta(2) = 0.012, \beta(3) = 0.013,$$

other parameters take the same values as Figure 2.

We can easy show that

$$R_2^* = \frac{\sum_{h\in\mathcal{M}} \pi_h \lambda(h)}{\frac{\hat{d}_1}{\check{\beta}} \sum_{h\in\mathcal{M}} \pi_h (d_2(h) + \frac{1}{2}\sigma_2^2(h))} = 0.9514 < 1.$$

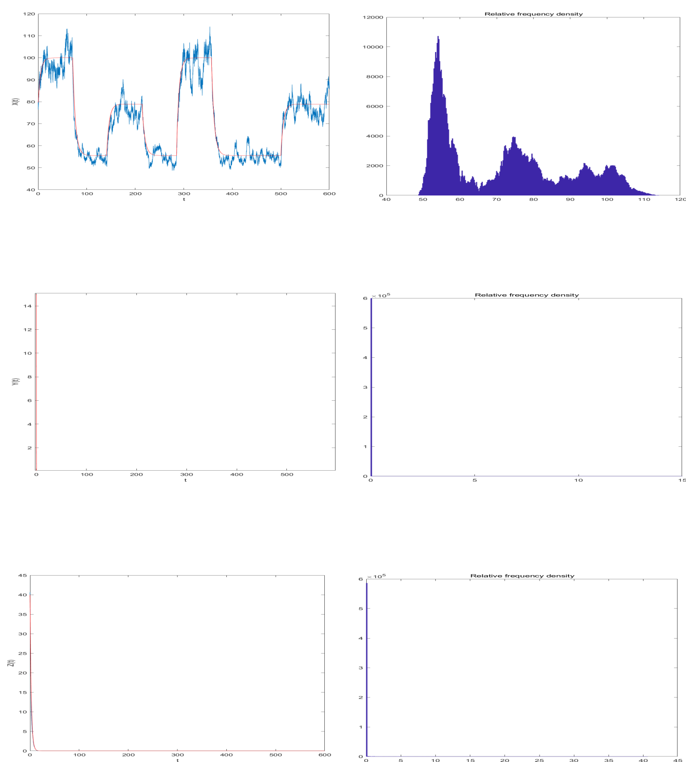

Figure 3. The solution of stochastic system (1.3) and its histogram with initial value condition ($x(0) = 80, y(0) = 15, z(0) = 40, k = 3$)

Obviously, the conditions in Theorem 5.1 are satisfied. In Figure 3, it is displayed directly that the solution $y(t), z(t)$ of the stochastic system (1.3) goes to extinction.

In particular, it can be seen from R_1^* and R_2^* that due to the large value of λ, our results have a smaller limit on the disturbance intensity $\sigma_i(k)(i = 1, 2, 3, k \in \mathcal{M})$ and have a broad application space.

From [10], the threshold R_0^* of the basic reproductive number in the deterministic model (1.1) determines persistence or extinction of the virus. It is easy to show the value of R_0^* consists of λ, β, d_1 and d_2. As can be seen from the results of this paper on the stochastic model (1.3), the persistence and extinction of the virus is also affected by the environment. In addition, we can obtain the results of [10] for the deterministic model (1.1) if $\sigma_i(k) = 0, k \in \mathcal{M} = \{1\}, i = 1, 2, 3$ holds.

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