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The Center Conditions and Hopf Cyclicity for a 3D Lotka-Volterra System*

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Abstract The main objective of this paper is not only to find necessary and sufficient conditions for the existence of a center on a local center manifold for a three dimensional Lotka-Volterra system, but also to determine the maximum number of limit cycles that can bifurcate from the positive equilibrium as a fine focus. Firstly, the singular point quantities are computed and simplified to obtain necessary conditions for local integrability, and Darboux method is applied to show the sufficiency. Then, the Hopf bifurcation on the center manifold is investigated, from this, the conclusion of at most five small limit cycles generated in the vicinity of the equilibrium is obtained. To the best of our knowledge, this is the first case with five possible limit cycles for the cyclicity of 3D Lotka-Volterra systems.

Keywords 3D Lotka-Volterra system, Hopf bifurcation, Center problem, Singular point quantities.

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1. Introduction

We consider the three-dimensional (3D) Lotka-Volterra system:

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = x_i(r_i - \sum_{j=1}^{3} a_{ij}x_i), \quad i = 1, 2, 3 \tag{LV}$$

when $r_i > 0, a_{ij} > 0$ $(i, j = 1, 2, 3)$, (LV) is called the competitive 3D LV system, which is very classical one to describe the relations of three species that share and compete for the same resources, habitat or territory (interference competition). Since 1994, Hofbauer and So [9] conjectured the number of limit cycles is at most two for the competitive (LV), the intensive investigations on the limit cycle bifurcation have been triggered, which are generally based on the remarkable result of Hirsch,

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Zeeman [28]: the competitive 3D LV systems have only 33 classes of all possible stable phase portraits and only classes 26-31 of those can have limit cycles.

Though Xiao and Li [25] proved that the number is finite for of the 3D competitive LV system without a heteroclinic polycycle, the maximum number of limit cycles from the interior fixed point in Zeeman's six classes 26-31 remains open up till now (see e.g. [8,15,16,22,27]). At present, four limit cycles is the maximum number given respectively by Yu et al. [27] for class 26 and class 27 by Wang et al. [22] for class 29. As for the non-competitive 3D LV system, the limit cycles bifurcation has been paid little attention to, an example of four possible limit cycles is given by Wang et al. in [21], here we will continue to consider a 3D non-competitive system as follows

$$\dot{x} = \mathrm{diag}(x)A(x-E)', \tag{1.1}$$

where $x = (x_1, x_2, x_3)$, $E = (1,\ 1,\ 1)$ and $()'$ denotes the transpose of a vector, and the interaction matrix

$$A = \begin{pmatrix} 0 & -n & -\nu \\ -h & \lambda & -s \\ -1 & 0 & 0 \end{pmatrix}$$

where n, h, s, ν and λ are real numbers. Obviously, E is the unique positive equilibrium of system (1.1).

In fact, the works on limit cycle bifurcation of systems in $\mathbb{R}^3$ are not unusually seen, especially for the Hopf bifurcation, except on the above 3D LV systems, extensive investigations on chaotic systems have been carried out (see [1,23] and the references therein). Moreover, for the more general higher-dimensional systems, the maximal number of limit cycles which may exist in the vicinity of a Hopf singular point under proper perturbations, i.e. the cyclicity of Hopf bifurcation, is attracting more and more attentions as a very challenging problem. Some good outcomes have also appeared, see for example [26] and references therein.

Furthermore, the problem about the number of limit cycles bifurcated at Hopf point, i.e., the cyclicity, is closely related to center-focus determination. For the center problem on the center manifold, there have been some valid research approaches, such as the averaging theory considered in [2,11], the technique of inverse Jacobi multiplier studied in [3,4], the simplest normal form method given in [20] and the formal first integral method given in [7]. Here, in order to investigate the center problem, we apply the formal series method introduced in [24] to find all the necessary conditions of local integrability, as an upgraded version of the method given in [10] for planar systems, which is also very valid to study the Hopf bifurcation (see e.g. [22,23]).

This paper is organized as follows. In Section 2, after system (1.1) being transformed, the corresponding singular point quantities are computed via the recursion formula derived. In Section 3, the singular point quantities are simplified and the Darboux theory is applied to show the sufficiency of integrability, then the center conditions on the center manifold are determined. In Section 4, the multiple Hopf bifurcations at the equilibrium is investigated for the corresponding system, and it is proved at most five small limit cycles from the positive equilibrium of system (1.1) via Hopf bifurcation. This is an interesting example and also the first case with five possible limit cycles for the cyclicity problem of the 3D LV systems.

2. Calculation of singular point quantities

Firstly, we transform the equilibrium E to the origin, set $\tilde{x} = x - E$, then system (1.1) takes the form

$$\dot{x} = \mathrm{diag}(E + x)Ax' \tag{2.1}$$

we still use x_i instead of $\tilde{x}_i$ for $i = 1, 2, 3$. Thus, for the center problem and Hopf bifurcation of the positive equilibrium E for system (1.1), we only need to investigate the cases of the origin in system (2.1). Furthermore, we choose the interaction matrix A such that the origin of system (2.1) can generate a Hopf bifurcation, that is, suppose A has a pair of purely imaginary eigenvalues $\pm\mathbf{i}\omega(\omega > 0)$ and one negative real eigenvalue. To satisfy the necessary eigenvalue conditions, we need

$$\mathrm{Det}(A) = (A_{11} + A_{22} + A_{33})\mathrm{tr}(A),$$

where $\mathrm{tr}(A) = \sum_{i=1}^{3} a_{ii}, A_{11} = a_{22}a_{33} - a_{23}a_{32}, A_{22} = a_{11}a_{33} - a_{13}a_{31}, A_{33} = a_{22}a_{11} - a_{12}a_{21}$. It yields that

$$\nu = -\omega^2 - hn, \quad \lambda = \tfrac{s}{h} \tag{2.2}$$

where $hs < 0$. Thus, one can construct a matrix P which transforms A to be a diagonal one, namely

$$P^{-1}AP = \begin{pmatrix} \mathbf{i}\omega & 0 & 0 \\ 0 & -\mathbf{i}\omega & 0 \\ 0 & 0 & sh^{-1} \end{pmatrix} \quad \text{where} \quad P = \begin{pmatrix} -\mathbf{i}\omega & \mathbf{i}\omega & -nsh^{-1} \\ h & h & hn + s^2h^{-2} + \omega^2 \\ 1 & 1 & n \end{pmatrix}$$

Moreover, using the transformation: $(x_1, x_2, x_3)' = P\,(z, w, u)'$, and after a time scaling: $T = \mathbf{i}\omega t$, system (2.1) can become the following complex system:

$$\begin{cases} \frac{\mathrm{d}z}{\mathrm{d}T} = z + a_{101}uz + a_{011}uw + a_{200}z^2 + a_{020}w^2 + a_{002}u^2 = Z, \\ \frac{\mathrm{d}w}{\mathrm{d}T} = -(w + b_{011}uz + b_{101}uw + b_{020}z^2 + b_{200}w^2 + b_{002}u^2) = -W, \\ \frac{\mathrm{d}u}{\mathrm{d}T} = \mathbf{i}d_{001}u + d_{101}uz + d_{011}uw + d_{200}z^2 + d_{020}w^2 + d_{002}u^2 = U \end{cases} \tag{2.3}$$

where $u \in \mathbb{R}$, $z, w, T \in \mathbb{C}$, and

$$a_{020} = \tfrac{(h^3n - h^2n - h\omega^2 - s\omega^2) - \mathbf{i}\omega(s - h\omega^2)}{2\omega(\mathbf{i}s + h\omega)},$$
$$a_{002} = \tfrac{h^2s^2n^2(h+s)\omega - \mathbf{i}ns(h^5\,n^2 - h^6\,n^2 - 2\,h^3\,n\,s^2 - h\,n\,s^3 - s^4 + (h^4\,n - 2\,h^5\,n - 2\,h^2\,s^2)\,\omega^2 - h^4\,\omega^4)}{2h^4\omega^2n(\mathbf{i}s + h\omega)},$$
$$a_{011} = \tfrac{hn(s^2 - h\omega^2 - h^2n + h^2\omega^2 + h^3n - s\omega^2) - \mathbf{i}(h+s)ns\,\omega}{2h^2\omega^2},$$
$$a_{101} = \tfrac{n(\mathbf{i}s - h\omega)(h^4\,n - h^3\,n + h\,s^2 - h^2\,\omega^2 + h^3\,\omega^2 + h\,s\,\omega^2 - \mathbf{i}(h-s)s\omega)}{2h^2\omega^2(\mathbf{i}s + h\omega)},$$
$$a_{200} = -a_{020}, b_{kjl} = \bar{a}_{kjl}, (kjl = 200, 020, 002, 011, 101),$$
$$d_{200} = \tfrac{(h-1)h^3}{s^2 + h^2\omega^2}, \quad d_{020} = -d_{200}, \quad d_{011} = \tfrac{h^3\,n - h^2\,n + s^2 + h^2\,\omega^2}{\omega(\mathbf{i}s - h\omega)}, \quad d_{101} = \tfrac{h^3\,n - h^2\,n + s^2 + h^2\,\omega^2}{\omega(\mathbf{i}s + h\omega)},$$
$$d_{002} = \tfrac{\mathbf{i}s(h^5\,n^2 - h^6\,n^2 - 2\,h^3\,n\,s^2 - s^4 - 2\,h^5\,n\,\omega^2 - 2\,h^2\,s^2\,\omega^2 - h^4\,\omega^4)}{h^3\omega(s^2 + h^2\omega^2)}, \quad d_{001} = -\tfrac{\mathbf{i}s}{h\omega}.$$

In fact, by means of transformation: $z = y_1 + \mathbf{i}y_2$, $w = y_1 - \mathbf{i}y_2$, $T = \mathbf{i}\,t$, we can get its real conjugate system from system (2.3):

$$\begin{cases} \frac{\mathrm{d}y_1}{\mathrm{d}t} = -y_2 + X_2(y_1, y_2, u) = X, \\ \frac{\mathrm{d}y_2}{\mathrm{d}t} = y_1 + Y_2(y_1, y_2, u) = Y, \\ \frac{\mathrm{d}u}{\mathrm{d}t} = \frac{s}{h\omega}\, u + U_2(y_1, y_2, u) = U \end{cases} \tag{2.4}$$

where X_2, Y_2 and U_2 are all quadratic homogeneous polynomials in (y_1, y_2, u) determined by the coefficients of system (2.3).

According to the center manifold theorem [5], the three-dimensional system (2.4) has an approximation to the center manifold taking the form

$$u = u(y_1, y_2) = \frac{4(h-1)h^4\omega}{(s^2 + h^2\omega^2)(s^2 + 4h^2\omega^2)}(h\omega y_1^2 + sy_1y_2 - h\omega y_2^2) + \cdots . \tag{2.5}$$

Substituting (2.5) into the equations of system (2.4), we get the following real planar polynomial differential system

$$\begin{cases} \frac{\mathrm{d}y_1}{\mathrm{d}t} = -y_2 + X_2(y_1, y_2, u(y_1, y_2)) = X(y_1, y_2), \\ \frac{\mathrm{d}y_2}{\mathrm{d}t} = y_1 + Y_2(y_1, y_2, u(y_1, y_2)) = Y(y_1, y_2) \end{cases} \tag{2.6}$$

System (2.6) is also called the equations on the center manifold or reduction system of (2.4). It is well-known, the origin of system (2.6) is center-focus type, we transform system (2.6) into the following form under the polar coordinates: $y_1 = r\cos\theta, y_2 = r\sin\theta$,

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{r\sum\limits_{k=1}^{\infty}\varphi_k(\theta)r^k}{1+\sum\limits_{k=1}^{\infty}\psi_k(\theta)r^k} \tag{2.7}$$

where $\varphi_k(\theta)$ and $\psi_k(\theta)$ are homogeneous polynomials in $\cos\theta$ and $\sin\theta$.

For sufficiently small $\hbar$, let

$$\Delta(\hbar) = r(2\pi, \hbar) - \hbar,\ r = r(\theta, \hbar) = \sum_{m=1}^{\infty} v_m(\theta)\hbar^m \tag{2.8}$$

be the Poincaré succession function and the solution of Eq.(2.7) satisfying the initial-value condition $r|_{\theta=0} = \hbar$. Moreover, for (2.8) we have

$$v_1(\theta) = 1,\ v_m(0) = 0,\ m = 2, 3, \cdots . \tag{2.9}$$

Definition 2.1 ([24]). For the succession function in (2.8), if $v_2(2\pi) = v_3(2\pi) = \cdots = v_{2k}(2\pi) = 0$ and $v_{2k+1}(2\pi) \neq 0$, then the origin is called the fine focus or weak focus of order k, and the quantity of $v_{2k+1}(2\pi)$ is called the k-th focal value at the origin on local center manifold of system (2.4) or (2.3), $k = 1, 2, \cdots$.

In order to investigate Hopf cyclicity of the origin for system (2.4) or system (2.3) restricted to the center manifold, by applying the formal series method, and from Theorem 3.1 in [24], we have the following theorem.

Theorem 2.1. For the system (2.3), setting $c_{110} = 1, c_{101} = c_{011} = c_{200} = c_{020} = 0, c_{kk0} = 0, k = 2, 3, \cdots$, we can derive successively and uniquely the terms of the following formal series:

$$F(z, w, u) = zw + \sum_{\alpha+\beta+\gamma=3}^{\infty} c_{\alpha\beta\gamma} z^{\alpha} w^{\beta} u^{\gamma} \tag{2.10}$$

such that

$$\frac{\mathrm{d}F}{\mathrm{d}T} = \frac{\partial F}{\partial z} Z - \frac{\partial F}{\partial y} W + \frac{\partial F}{\partial u} U = \sum_{m=1}^{\infty} \mu_m (zw)^{m+1} \tag{2.11}$$

and if $\alpha \neq \beta$ or $\alpha = \beta$ and $\gamma \neq 0$, $c_{\alpha\beta\gamma}$ is determined by following recursive formula:

$$\begin{aligned} c_{\alpha\beta\gamma} =& \tfrac{1}{(\alpha-\beta)+\mathbf{i}d_{001}\gamma}[b_{002}(1+\beta)c_{\alpha,\beta+1,\gamma-2} + (d_{002} - a_{101}\alpha + \ b_{101}\beta - d_{002}\gamma)c_{\alpha,\beta,\gamma-1} - (b_{200} - \\ & b_{200}\beta + d_{011}\gamma)c_{\alpha,\beta-1,\gamma} - d_{020}(1+\gamma)c_{\alpha,\beta-2,\gamma+1} + (a_{200} - a_{200}\alpha - d_{101}\gamma)c_{\alpha-1,\beta,\gamma} + \\ & b_{011}(1+\beta)c_{\alpha-1,\beta+1,\gamma-1} + b_{020}(1+\beta)c_{\alpha-2,\beta+1,\gamma} - d_{200}(1+\gamma)c_{\alpha-2,\beta,\gamma+1} - \\ & (1+\alpha)(a_{020}c_{\alpha+1,\beta-2,\gamma} + a_{011}c_{\alpha+1,\beta-1,\gamma-1} + a_{002}c_{\alpha+1,\beta,\gamma-2})] \end{aligned}$$

and for any positive integer m, μ_m is determined by following recursive formula:

$$\begin{aligned} \mu_m =& d_{200}c_{m-2,m,1} - b_{020}(m+1)c_{m-2,m+1,0} + a_{200}(m-1)c_{m-1,m,0} + d_{020}c_{m,m-2,1} - \\ & b_{200}(m-1)c_{m,m-1,0} + a_{020}(m+1)c_{m+1,m-2,0} \end{aligned}$$

and when $\alpha < 0$ or $\beta < 0$ or $\gamma < 0$ or $\gamma = 0$ and $\alpha = \beta$, we have let $c_{\alpha,\beta,\gamma} = 0$.

Definition 2.2. The μ_m in (2.11) is called the m-th singular point quantity at the origin of system (2.3), and more if all the singular point quantities vanish, i.e. $\mu_m = 0$, $m = 1, 2, \cdots$, then the origin of system (2.4) or (2.3) is a center on the local center manifold at the origin.

Lemma 2.2 ([24])**.** For system (2.3), the singular point quantity μ_m is algebraic equivalent to the m-th focal value v_{2m+1} at the origin of system (2.4), i.e., for any positive integer $m = 2, 3, \cdots$, if $v_3 = v_5 = \cdots = v_{2m-1} = 0$ and $\mu_1 = \mu_2 = \cdots = \mu_{m-1} = 0$ hold, then $v_{2m+1} = \mathbf{i}\pi\mu_m$.

Thus, from the Lemma 2.2, by the calculation of singular point quantities of the origin for system (2.3), one can figure out the stability and Hopf bifurcation at the origin of system (2.4) on the center manifold. Now, applying the recursive formulas in Theorem 2.1 in the Mathematica symbolic computation system, we can obtain the first five singular point quantities easily:

$$\begin{aligned} \mu_1 &= \mathbf{i}(1-h)h^3 nsM_1[\omega\, d_1\, d_2]^{-1}, \\ \mu_2 &= \mathbf{i}(1-h)h^3 nsM_2\,[3\omega^3\, d_1^3\, d_2^2\, d_3]^{-1}, \\ \mu_3 &= \mathbf{i}(1-h)h^3 nsM_3\,[144\omega^5\, d_1^5\, d_2^4\, d_3^2\, d_4]^{-1}, \\ \mu_4 &= \mathbf{i}(1-h)h^3 nsM_4\,[4320\omega^7\, d_1^6\, d_2^6\, d_3^3\, d_4^2\, d_5]^{-1}, \\ \mu_5 &= \mathbf{i}(1-h)h^3 nsM_5\,[1555200\omega^9\, d_1^9\, d_2^8\, d_3^4\, d_4^3\, d_5^2\, d_6]^{-1} \end{aligned}$$

where

$$\begin{aligned}
&d_1 = s^2 + h^2\omega^2,\ d_2 = s^2 + 4h^2\omega^2,\ d_3 = s^2 + 9h^2\omega^2,\\
&d_4 = (s^2 + 16h^2\omega^2)(4s^2 + h^2\omega^2),\\
&d_5 = (9s^2 + 4h^2\omega^2)(9s^2 + h^2\omega^2)(4s^2 + 9h^2\omega^2)(s^2 + 25h^2\omega^2),\\
&d_6 = (16s^2 + h^2\omega^2)(4s^2 + 25h^2\omega^2)(s^2 + 36h^2\omega^2),\\
&M_1 = (h + h^2 + s)\omega^2 + s^2 + n(h^3 - h^2)
\end{aligned}$$

and we have

$$M_2 = \sum_{i=0}^{3} M_{2i}n^i, \quad M_3 = \sum_{i=0}^{5} M_{3i}n^i, \quad M_4 = \sum_{i=0}^{7} M_{4i}n^i, \quad M_5 = \sum_{i=0}^{9} M_{5i}n^i$$

where the $M_{3i}, M_{3i}, M_{4i}, M_{5i}$ are all polynomials only in h, s and ω^2 (which are available in Email address of the corresponding author).

3. Center conditions of the system

The maximum number of limit cycles generated from the weak focus via Hopf bifurcation is related closely to its highest order, from this, we must analyze the above singular point quantities obtained in the last section. Obviously, being $d_i \neq 0$, $i = 1, 2, \cdots, 6$, in order to obtain more higher order of the weak focus, we set $(1-h)h\,n\,s \neq 0$, and from $\mu_1 = 0$, we let $M_1 = 0$, then get

$$n = \frac{s^2 + h\,\omega^2 + h^2\,\omega^2 + s\,\omega^2}{(1-h)h^2}, \quad (h \neq 0, 1). \tag{3.1}$$

And substituting n of (3.1) into $M_i, i = 2, 3, 4, 5$ yields four expressions:

$$\begin{aligned}
M_2 &= 4h(h+s)\omega^4 d_1^2 d_2 G_2,\\
M_3 &= 8h(h+s)\omega^4 d_1^3 d_2^2 d_7 G_3,\\
M_4 &= 8h(h+s)\omega^4 d_1^2 d_2^3 d_7 d_8 G_4,\\
M_5 &= 40h(h+s)\omega^4 d_1^3 d_2^4 d_7 d_8 d_9 G_5
\end{aligned}$$

where

$$\begin{aligned}
&d_7 = 4s^2 + h^2\omega^2,\\
&d_8 = (9\,s^2 + h^2\,\omega^2)\,(9\,s^2 + 4\,h^2\,\omega^2)\,(4\,s^2 + 9\,h^2\,\omega^2),\\
&d_9 = (16\,s^2 + h^2\,\omega^2)(4\,s^2 + 25\,h^2\,\omega^2),\\
&G_2 = h\,s^2 - 3\,h^2\,\omega^2 + s\,\omega^2 + 4\,h\,s\,\omega^2 + 3\,h\,\omega^4
\end{aligned}$$

and G_3, G_4, G_5 are all polynomials only in h, s and ω^2, we would not present them here due to their lengthy expressions, but one can easily calculate them.

One can note easily that d_7, d_8, d_9 also never become zero, and $h(h+s) \neq 0$ from the limited coefficients conditions (2.2) in advance for system (2.1). Similarly, in order to detect more higher order of the weak focus, we compute respectively the polynomial remainders of G_3, G_4, G_5 for G_2 as follows

$$\begin{aligned}
\text{PolynomialRemainder}[G_3, G_2, s] &= \tfrac{12\omega^{12}(h^2+\omega^2)}{h^{10}}\tilde{G}_3,\\
\text{PolynomialRemainder}[G_4, G_2, s] &= \tfrac{12\omega^{28}(h^2+\omega^2)}{h^{26}}\tilde{G}_4,\\
\text{PolynomialRemainder}[G_5, G_2, s] &= \tfrac{12\omega^{52}(h^2+\omega^2)}{h^{50}}\tilde{G}_5,
\end{aligned} \tag{3.2}$$

where $\tilde{G}_3, \tilde{G}_4, \tilde{G}_5$ are also all polynomials only in h, s and ω^2, and the highest degree is 1 for s in each $\tilde{G}_i$, $i = 3, 4, 5$, which is not difficult to understand due to the highest degree 2 of s in G_2.

Furthermore, we consider $\mu_3 = 0$, and only let $\tilde{G}_3 = 0$, then we get

$$s = \tfrac{3h\,s_n}{s_d} \tag{3.3}$$

and substituting it into $G_2, \tilde{G}_4, \tilde{G}_5$, we obtain

$$\begin{aligned}
G_2 &= 9h^{19}(1 + 6h + 9h^2 + \omega^2)f_n f_2 s_d^{-2},\\
\tilde{G}_4 &= 9h^{19}(1 + 6h + 9h^2 + \omega^2)f_n f_4 s_d^{-1},\\
\tilde{G}_5 &= 9h^{19}(1 + 6h + 9h^2 + \omega^2)f_n f_5 s_d^{-1}
\end{aligned} \tag{3.4}$$

where

$$f_n = 9\omega^4 + (1 - 10h + 10h^2)\omega^2 + h^2(h + 3)^2$$

and

$$s_n = \sum_{i=0}^{19} s_{ni}h^i, \quad s_d = \sum_{i=0}^{19} s_{di}h^i, \quad f_2 = \sum_{i=0}^{16} f_{2i}h^i, \quad f_4 = \sum_{i=0}^{46} f_{4i}h^i, \quad f_5 = \sum_{i=0}^{94} f_{5i}h^i$$

where the $s_{ni}, s_{di}, f_{2i}, f_{4i}, f_{5i}$ are all polynomials only in ω^2.

From the above simplifying process, we can obtain the reduced singular point quantities:

$$\begin{aligned}
\mu_1 &= \tfrac{\mathbf{i}}{\omega\, d_1 d_2}\ ns(1 - h)h^3 M_1,\\
\mu_2 &= \tfrac{4\mathbf{i}}{3d_1 d_2 d_3}\ ns(1 - h)(h + s)h^2 G_2,\\
\mu_3 &= \tfrac{2\mathbf{i}\, d_7}{3d_1^2 d_2^2 d_3^2 d_4 h^6}\ ns(1 - h)(h + s)(h^2 + \omega^2)\omega^{11}\tilde{G}_3,\\
\mu_4 &= \tfrac{\mathbf{i}\, d_7 d_8}{5d_1^3 d_2 d_3^2 d_4 s_d h^3}\ ns(1 - h)(h + s)(h^2 + \omega^2)(1 + 6h + 9h^2 + \omega^2)\omega^{25} f_n\, f_4,\\
\mu_5 &= \tfrac{\mathbf{i}\, d_7 d_8 d_9}{360 d_1^6 d_2^4 d_3^4 d_4^3 d_5^2 d_6 s_d h^{27}}\ ns(1 - h)(h + s)(h^2 + \omega^2)(1 + 6h + 9h^2 + \omega^2)\omega^{47} f_n\, f_5.
\end{aligned} \tag{3.5}$$

In the above expressions, we have already let $\mu_1 = 0$ for μ_2 and μ_3, while for μ_4 and μ_5 we have already let $\mu_1 = \mu_2 = \mu_3 = 0$.

One can know that $f_n = 0$ in (3.4) holds if and only if

$$\omega^2 = \frac{10h - 10h^2 - 1 \pm \sqrt{(1 + 4h)^2(1 - 28h + 4h^2)}}{18}. \tag{3.6}$$

However, on the right-hand side of (3.6), the expression can all be proved negative, which contradicts with $\omega^2 > 0$, thus $f_n \neq 0$ in (3.4), at the same time, $1 + 6h +$

$9h^2 + \omega^2 \neq 0$ holds. Therefore, we only need to consider whether f_2, f_4 and f_5 can disappear simultaneously.

Next, by computing the polynomial resultants of f_4, f_5 for f_2 with respect to h via Mathematica, we obtain

$$\begin{aligned} &\text{Resultant}[f_4, f_2, h] = F_4 F_c F_{44}, \\ &\text{Resultant}[f_5, f_2, h] = F_5 F_c F_{84} \end{aligned} \tag{3.7}$$

where F_4, F_5 are two polynomials only in ω^2 and $F_4 F_5 \neq 0$, and more F_c, F_{44}, F_{84} as follows

$$F_c = 8 - 381\,\omega^2 - 459\,\omega^4 + 486\,\omega^6, \quad F_{44} = \sum_{i=0}^{44} a_i\,\omega^{2i}, \quad F_{84} = \sum_{i=0}^{84} b_i\,\omega^{2i} \tag{3.8}$$

where all $a_i,\ b_i \in \mathbb{R}$. On the one hand, we have

$$\text{Resultant}[F_{84}, F_{44}, \omega^2] \neq 0 \tag{3.9}$$

which shows that F_{44} and F_{84} have no common zero point. On the other hand, from

$$\begin{aligned} &\text{Resultant}[f_2, F_c, \omega^2] = 1594323 h^3 H_{45}, \\ &\text{Resultant}[f_4, F_c, \omega^2] = -3188646 h^3 H_{135}, \\ &\text{Resultant}[H_{45}, H_{135}, h] \neq 0 \end{aligned} \tag{3.10}$$

where H_{45} and H_{135} are all polynomials only in h, with degree 45, 135 respectively, we know that the roots of $F_c = 0$ are not in the solution set of the group of $f_2 = 0$ and $f_4 = 0$, similarly for the group of $f_2 = 0$ and $f_5 = 0$.

Therefore, the above symbolic computations show that $f_2 = 0, f_4 = 0, f_5 = 0$ have no solutions, namely f_2, f_4, f_5 have no common zero, but some values of h and ω^2 should be found such that $f_2 = f_4 = 0$, which will be discussed in the next section.

According to the above analysis, we have the following lemma.

Lemma 3.1. For system (2.3), the first five singular point quantities of the origin: $\mu_i, i = 1, 2, 3, 4, 5$ in (3.5) are all zero if and only if $n(1-h) = 0$, i.e. $n = 0$ or $h = 1$ holds.

Furthermore, from Definition 2.2 and Lemma 2.2, we have

Theorem 3.2. For system (2.4), the origin is a center on the local center manifold if and only if the following condition is satisfied:

$$n = 0 \ \text{ or } \ h = 1. \tag{3.11}$$

Proof. From the lemma 3.1, the proof of the necessity is obvious. Now we prove the sufficient condition, this technique derives from the Darboux theory of integrability (one can see some notions and facts in [6, 12–14, 17–19]).

Case (I): if $h = 1$ in the conditions (3.11) holds, then system (2.4) has the corresponding form as follows

$$\begin{aligned}\tfrac{\mathrm{d}y_1}{\mathrm{d}t} &= -y_2 - 2y_1y_2 + \tfrac{1}{2\omega}ns(n+s^2+\omega^2)u^2,\\ \tfrac{\mathrm{d}y_2}{\mathrm{d}t} &= y_1 + \tfrac{1}{2\omega^2}[4\omega^3y_1y_2 + 2ns(s-\omega^2)uy_1 - 2ns(1+s)\omega uy_2\\ &\quad + ns^2(2n+ns+s^2+\omega^2)u^2],\\ \tfrac{\mathrm{d}u}{\mathrm{d}t} &= \tfrac{s}{\omega}u - \tfrac{1}{\omega}[2suy_1 - 2n\omega uy_2 + s(2n+s^2+\omega^2)u^2].\end{aligned} \tag{3.12}$$

And we can figure out easily one algebraic invariant surface for system (3.15): $F(y_1, y_2, u) = u$, in fact, there exists a polynomial $K(y_1, y_2, u) = \frac{1}{\omega}[s - 2sy_1 + 2n\omega y_2 - s(2n + s^2 - \omega^2)u]$, as the cofactor of $F(x, y, u)$, such that $\frac{\mathrm{d}F}{\mathrm{d}t}\big|_{(3.12)} = KF$. One can observe that $F(y_1, y_2, u) = u = 0$ is just the center eigenspace, i.e., (y_1, y_2)-plane. Thus it forms a local center manifold in a neighborhood of the origin. We substitute $u = 0$ into the first and second equations of the system defined by system (3.15), we have the differential equations

$$\frac{\mathrm{d}y_1}{\mathrm{d}t} = -y_2(1+2y_1), \quad \frac{\mathrm{d}y_2}{\mathrm{d}t} = y_1(1+2\omega y_2) \tag{3.13}$$

which has a first integral:

$$H(y_1, y_2) = \frac{1}{2\omega}(y_2 + \omega y_1) - \frac{1}{4\omega^2}[\ln|1+2\omega y_2| + \omega^2 \ln|1+2y_1|], \tag{3.14}$$

then the origin is a center for systems (3.13). Therefore the origin is a center for the flow of system (2.4) restricted to a center manifold.

Case (II): if $n = 0$ in the conditions (3.11) holds, then system (2.4) has the corresponding form as follows

$$\begin{aligned}\tfrac{\mathrm{d}y_1}{\mathrm{d}t} &= -y_2(1+2y_1),\\ \tfrac{\mathrm{d}y_2}{\mathrm{d}t} &= y_1(1+2\omega y_2),\\ \tfrac{\mathrm{d}u}{\mathrm{d}t} &= \tfrac{s}{h\omega}u + \tfrac{2s}{\omega}uy_1 - \tfrac{4(h-1)h^3}{s^2+h^2\omega^2}y_1y_2 - \tfrac{2h^3\omega^2}{s^2+h^2\omega^2}uy_1\\ &\quad - \tfrac{2hs^2}{s^2+h^2\omega^2}uy_2 + \tfrac{s(s^2+h^2\omega^2)}{h^3\omega}u^2.\end{aligned} \tag{3.15}$$

Clearly, the first two equations in system (3.15) are independent on the variable u, and just the same as the two in system (3.13), thus (3.14) is also a first integral of system (3.15). Hence, the origin is a center for the flow of system (2.4) restricted to a center manifold. $\square$

4. Hopf cyclicity of the system

In this section, we turn to discussion the maximal number of limit cycles from the origin for system (2.1). We need to figure out the highest order of the origin as a weak focus. From the above analyzing in the last section, we have known that f_2, f_4 and f_5 have no common zero, but f_2 and f_4 should have ones.

In fact, the equations $f_2 = 0$ and $f_4 = 0$ are coupled and there exist possible solutions, only 2 groups (can up to 98 digit points in executing the procedure

"NSolve" of the computer algebraic system Mathematica to solve the two equations) as follows:

$$h^{(1)} \doteq -0.03297395745363060470136947,\ \omega^{2(1)} \doteq 0.01773444864628213446790575;$$
$$h^{(2)} \doteq 0.01941997619263711506246973,\ \omega^{2(2)} \doteq 0.05025449995363412651878426. \tag{4.1}$$

By numeric calculating, we also know that the above 2 groups of solves never make f_5 disappear. Then, the critical values are denoted by

$$\begin{aligned} &h^* = h^{(k)},\ \omega^{2*} = \omega^{2(k)},\ k = 1, 2, \\ &n^* = n(h^*, \omega^{2*}),\ \ s^* = s(h^*, \omega^{2*}) \end{aligned} \tag{4.2}$$

where the expressions of n and s are given by (3.1) and (3.3) respectively. At the same time, similar to the process of (3.7) or (3.10), one can apply the symbolic computation to prove rigidly that the above the critical values (4.2) never make $(h-1)(h+s)hs = 0$ or $s_n = 0$ or $s_d = 0$ hold. Definitely, one can also use the numeric method to verify them quickly. Thus, we obtain the following result.

Lemma 4.1. For system (2.3), the origin is a weak focus of order 5, namely $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 0, \mu_5 \neq 0$ if and only if the following conditions hold:

$$s = s^*,\ h = h^*,\ n = n^*,\ \omega = \omega^*. \tag{4.3}$$

Furthermore, according to Lemma 2.2 and Theorem 3.2, then we have

Theorem 4.2. For the origin of system (2.4) as a weak focus on center manifold, the highest is the fifth order, and its first 5 focal values are as follow

$$v_3 = \mathbf{i}\pi\mu_1, \quad v_5 = \mathbf{i}\pi\mu_2, \quad v_7 = \mathbf{i}\pi\mu_3, \quad v_9 = \mathbf{i}\pi\mu_4, \quad v_{11} = \mathbf{i}\pi\mu_5$$

in the above expression of v_5, we have let $v_3 = 0$, similarly we have let $v_3 = v_5 = 0$ for $v_7 = 0$, let $v_3 = v_5 = v_7 = 0$ for $v_9 = 0$ and let $v_3 = v_5 = v_7 = v_9 = 0$ for $v_{11} = 0$.

Thus, according to Lemma 4.1, we can apply the Hopf bifurcation theory to determine the bifurcation of limit cycles, then via appropriate perturbations of the critical values given in (4.2), five possible small amplitude limit cycles can generate in the neighborhood of the origin of system (2.4).

Furthermore, from the above discussion and Theorem 4.2, and according to the topology equivalence of the affine transformation from system (2.1) to (2.3), it follows that

Theorem 4.3. At most five small amplitude limit cycles can be generated from the origin of system (2.1) or the positive equilibrium of system (1.1) as a fine focus on the center manifold via Hopf bifurcation.

5. Conclusion

In summary, based on precise symbolic computation of singular point quantities at the Hopf singularity for a three-dimensional Lotka-Volterra system, we have found the center conditions and determined that the highest order as a fine focus is the fifth. Furthermore, we obtained that the system has at most 5 small limit cycles from the equilibrium via Hopf bifurcation. We expect that some new outcomes on the cyclicity for 3D LV system will be obtained via our method used in this paper.

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