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Oscillation Theory of h-fractional Difference Equations*

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Abstract In this paper, we initiate the oscillation theory for h-fractional difference equations of the form

$$\begin{cases} {}_a\Delta_h^\alpha x(t) + r(t)x(t) = e(t) + f(t, x(t)), & t \in \mathbb{T}_h^a, \quad 1 < \alpha < 2, \\ x(a) = c_0, \quad \Delta_h x(a) = c_1, & c_0, c_1 \in \mathbb{R}, \end{cases}$$

where ${}_a\Delta_h^\alpha$ is the Riemann-Liouville h-fractional difference of order α, $\mathbb{T}_h^a := \{a + kh, \ k \in \mathbb{Z}^+ \cup \{0\}\}$, and $a \geqslant 0$, $h > 0$. We study the oscillation of h-fractional difference equations with Riemann-Liouville derivative, and obtain some sufficient conditions for oscillation of every solution. Finally, we give an example to illustrate our main results.

Keywords h-deference equations, Oscillation, Fractional.

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1. Introduction

The fractional calculus (calculus with derivatives of arbitrary order) is an important research field in several different areas such as physics (including classical and quantum mechanic as well as thermodynamics), chemistry, biology, economic, and control theory [7, 10, 11, 13–15]. Fractional difference equations, which is the discrete counterpart of the corresponding fractional differential equations, have recently gained an intensive interest among researchers in the last years. The discrete fractional calculus has been developed much after the appearance of time scale calculus [6]. The researchers started to use delta and nabla analysis by employing the jumping operators σ and ρ.

Computer simulations show that the time scale $(h\mathbb{Z})_a$ is particularly interesting because when h tends to zero one recovers previous fractional continuous-time results [5, 8]. However, the development of the qualitative features of these type of equations is still considered to be at its first stage of progress. Therefore, it is pertinent to develop the oscillation theory of fractional h-difference equations. The last few years, there are new papers which have studied oscillation of fractional difference equations, see the papers [1, 2, 4, 12].

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Marian et al. [12] studied the oscillation criteria for forced nonlinear fractional difference equations of the form

$$\begin{cases}\Delta^{\alpha}x(t)+f_1(t,x(t+a))=v(t)+f_2(t,x(t+a)), & t\in\mathbb{N}_0, \quad 0<\alpha\leqslant 1,\\ \Delta^{\alpha-1}x(t)|_{t=0}=x_0, \end{cases}$$

where Δ^{α} denotes the Riemann-Liouville like discrete fractional difference operator of order α.

Abdalla et al. [1] studied the oscillation of solutions of nonlinear forced fractional difference equations of the form

$$\begin{cases}\nabla^{q}_{a(q)-1}x(t)+f_1(t,x(t))=r(t)+f_2(t,x(t)), & t\in\mathbb{N}_{a(q)},\\ \nabla^{-(m-q)}_{a(q)-1}x(t)|_{t=a(q)}=x(a(q))=c, & c\in\mathbb{R},\end{cases}$$

where $q>0$, $m=[q]+1$, $m\in\mathbb{N}$, $[q]$ is the greatest integer less than or equal to q , $\nabla^{-q}_{a(q)-1}$ and $\nabla^{q}_{a(q)-1}$ are the Riemann-Liouville sum and difference operators, respectively.

Following this trend and there are no results available in the literature regarding the oscillation of solutions of h-fractional difference equations, we are concerned equations of the form

$$\begin{cases}{}_a\Delta^{\alpha}_{h}x(t)+r(t)x(t)=e(t)+f(t,x(t)), & t\in\mathbb{T}^a_h, \quad 1<\alpha<2,\\ x(a)=c_0, \quad \Delta_h x(a)=c_1, & c_0,c_1\in\mathbb{R},\end{cases} \tag{1.1}$$

where ${}_a\Delta^{\alpha}_{h}$ is the Riemann-Liouville h-fractional difference of order α, Δ_h is the forward h-difference operator, $\mathbb{T}^a_h := \{a+kh,\ k\in\mathbb{Z}^+\cup\{0\}\}$, $a\geqslant 0$, $h>0$ and $e(t)$ is a continuous function. The problems will be studied under the following assumptions:

(H1) $r(t)$ is a positive real-valued continuous function on $\mathbb{R}$;
(H2) $x(t)f(t,x(t))>0,\ x(t)\neq 0,\ \ t\in\mathbb{T}^a_h$;
(H3) $|f(t,x(t))|\leqslant p(t)|x(t)|^{\gamma},\ \ x(t)\neq 0,\ t\geqslant a$;
(H4) $|f(t,x(t))|\geqslant p(t)|x(t)|^{\gamma},\ \ x(t)\neq 0,\ t\geqslant a$,

where $p(t)\in C(\mathbb{T}^a_h,\mathbb{R}^+)$ and γ is a the quotient of two positive odd numbers.

We only consider these solutions of (1.1) which exist on $\mathbb{T}^a_h$. If $x(t)$ satisfies (1.1) on $\mathbb{T}^a_h$, then the function $x(t)$ is called a solution of (1.1). A solution $x(t)$ of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is called nonoscillatory. The equation itself is called oscillatory if all of its solutions are oscillatory.

The paper is organized as follows. In Section 2, we present some basic definitions and useful results from the theory of discrete fractional calculus which we rely in the later section. In Section 3, we intend to use the inequalities to obtain some sufficient conditions for oscillation for oscillation of every solution of (1.1). In Section 4, we give an example to illustrate our results.

2. Preliminaries

Before stating and proving our results, we introduce some definitions and notations. Let $\mathbb{T}^a_h := \{a+kh,\ k\in\mathbb{Z}^+\cup\{0\}\}$, where $a\geqslant 0$, $h>0$. Let us denote by $\mathcal{F}_{\mathbb{T}}$ the set of real valued functions defined on $\mathbb{T}^a_h$, $\sigma(t)=t+h$ and $\rho(t)=t-h$.

Definition 2.1. [7] For a function $\psi \in \mathcal{F}_{\mathbb{T}}$ the forward h-difference operator is defined as

$$(\Delta_h \psi)(t) = \frac{\psi(\sigma(t)) - \psi(t)}{h}, \quad t \in \mathbb{T}_h^a,$$

while the h-difference sum is given by

$$({}_a\Delta_h^{-1}\psi)(t) = h \sum_{k=0}^{n-1} \psi(a+kh), \quad t \in \mathbb{T}_h^a.$$

Definition 2.2. [3] For arbitrary $t, \alpha \in \mathbb{R}$ and $h > 0$, the delta h-factorial function is defined by

$$t_h^{(\alpha)} := h^\alpha \frac{\Gamma(\frac{t}{h} + \alpha)}{\Gamma(\frac{t}{h})}.$$

Considering $t \in \mathbb{T}_h^a$, it follows immediately that $\sigma(t + \alpha h) = t + (\alpha+1)h \notin \mathbb{T}_h^a$ for noninteger value α. On this account we introduce the "shifted" time scale

$$\mathbb{T}_h^{a,\alpha} := \{a + (\alpha + k)h, \quad k \in \mathbb{Z}^+ \cup \{0\}\},$$

where $h > 0$ and $\alpha \in \mathbb{R}$. Now we can state the following definition.

Definition 2.3. [5] (Delta h-fractional sum) Let ψ be defined on $\mathbb{T}_h^a$. Then the delta h-fractional sum of order $\alpha > 0$ is defined by

$$({}_a\Delta_h^{-\alpha}\psi)(t) = \sum_{k=\frac{a}{h}}^{\frac{t}{h}-\alpha} (t - \sigma(kh))_h^{(\alpha-1)} \psi(kh)$$

for $t \in \mathbb{T}_h^{a,\alpha}$.

Definition 2.4. [3] (Delta h-RL fractional difference) Let ψ be defined on $\mathbb{T}_h^a$. Then the delta h-fractional difference of order $\alpha > 0$ is defined by

$$({}_a\Delta_h^{\alpha}\psi)(t) = (\Delta_h^n {}_a\Delta_h^{-(n-\alpha)}\psi)(t), \quad t \in \mathbb{T}_h^{a,-\alpha},$$

where $n = [\alpha] + 1$.

Lemma 2.1. *[3] For $\alpha > 0$, $h > 0$ and ψ defined on T_h^a we have for $t \in \mathbb{T}_h^{a+nh} \subset \mathbb{T}_h^a$*

$$({}_a\Delta_h^{-\alpha} {}_a\Delta_h^{\alpha}\psi)(t) = \psi(t) - \sum_{k=0}^{n-1} \frac{(t-a)_h^{(k)}}{k!} \Delta_h^k \psi(a),$$

where $n = [\alpha] + 1$.

Lemma 2.2. *[9] (Young's inequality)*

(i) Let $X, Y \geqslant 0$, $u > 1$ and $\frac{1}{u} + \frac{1}{v} = 1$, then $XY \leqslant \frac{1}{u}X^u + \frac{1}{v}Y^v$,

(ii) Let $X \geqslant 0$, $Y \geqslant 0$, $0 < u < 1$ and $\frac{1}{u} + \frac{1}{v} = 1$, then $XY \geqslant \frac{1}{u}X^u + \frac{1}{v}Y^v$, where equality hold if and only if $Y = X^{u-1}$.

Lemma 2.3. *[16] (Stirling's formula) For $\varepsilon > 0$, we have*

$$\lim_{n\to\infty} \frac{\Gamma(n) n^\varepsilon}{\Gamma(n+\varepsilon)} = 1.$$

3. Main Results

Now, we are in a position to state and prove some new results which guarantee that every solution of (1.1) oscillates.

Theorem 3.1. *Assume that $f(t,x(t))=0$ in (1.1) and (H1)hold. If*

$$\liminf_{t\to\infty}\sum_{s=T+1}^{t-\alpha h} e(s)=-\infty \tag{3.1}$$

for sufficiently large T, then every solution of (1.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1). We suppose that $x(t)$ is an eventually positive solution of (1.1). Then there exists a $T>a$ such that $x(t)>0$ for all $t\geqslant T$. Let $F(t,x(t))=e(t)-f(t,x(t))-r(t)x(t)$. Then by Definition 2.3, Lemma 2.1 and (H1), we have

$$\begin{aligned}
x(t) =& c_0+c_1(t-a)+\frac{1}{\Gamma(\alpha)}\sum_{s=a}^{T}(t-\sigma(s))_h^{(\alpha-1)}F(s,x(s))\\
&+\frac{1}{\Gamma(\alpha)}\sum_{s=T+1}^{t-\alpha h}(t-\sigma(s))_h^{(\alpha-1)}(e(s)-r(s)x(s))\\
<& c_0+c_1(t-a)+\frac{1}{\Gamma(\alpha)}\sum_{s=a}^{T}(t-\sigma(s))_h^{(\alpha-1)}F(s,x(s))\\
&+\frac{1}{\Gamma(\alpha)}\sum_{s=T+1}^{t-\alpha h}(t-\sigma(s))_h^{(\alpha-1)}e(s),
\end{aligned}$$

multiplying both sides of the above inequality by $\Gamma(\alpha)(t-a)^{-1}$, we obtain

$$\begin{aligned}
0<\Gamma(\alpha)(t-a)^{-1}x(t) <& \Gamma(\alpha)c_0(t-a)^{-1}+\Gamma(\alpha)c_1+\sum_{s=a}^{T}(t-a)^{-1}(t-\sigma(s))_h^{(\alpha-1)}F(s,x(s))\\
&+\sum_{s=T+1}^{t-\alpha h}(t-a)^{-1}(t-\sigma(s))_h^{(\alpha-1)}e(s).
\end{aligned} \tag{3.2}$$

By Definition 2.2 and Lemma 2.3, we observe that

$$\begin{aligned}
\lim_{t\to\infty}(t-a)^{-1}(t-\sigma(s))_h^{(\alpha-1)} &= \lim_{t\to\infty}(t-a)^{-1}h^{\alpha-1}\frac{\Gamma\left(\frac{t-\sigma(s)}{h}+\alpha-1\right)}{\Gamma\left(\frac{t-\sigma(s)}{h}\right)\left(\frac{t-\sigma(s)}{h}\right)^{\alpha-1}}\left(\frac{t-\sigma(s)}{h}\right)^{\alpha-1}\\
&=\lim_{t\to\infty}\frac{(t-\sigma(s))^{\alpha-1}}{t-a}\\
&=0.
\end{aligned}$$

Hence, there exists a $M>0$ such that

$$-M<(t-a)^{-1}(t-\sigma(s))_h^{(\alpha-1)}<M,\quad t\in\mathbb{T}_h^a.$$

It follows from (3.2) that

$$0 < C_0(t-a)^{-1} + C_1 + C_2(T) + M \sum_{s=T+1}^{t-\alpha h} e(s), \quad t \in \mathbb{T}_h^a,$$

where $C_0 = \Gamma(\alpha)c_0$, $C_1 = \Gamma(\alpha)c_1$ and $C_2(T) = M \sum_{s=a}^{T} F(s, x(s))$, i.e.

$$\sum_{s=T+1}^{t-\alpha h} e(s) > -\frac{1}{M}(C_0(t-a)^{-1} + C_1 + C_2(T)), \quad t \in \mathbb{T}_h^a. \tag{3.3}$$

Taking limit inferior in (3.3) as $t \to \infty$, we have

$$\begin{aligned}\liminf_{t\to\infty} \sum_{s=T+1}^{t-\alpha h} e(s) &\geqslant \liminf_{t\to\infty}\left(-\frac{1}{M}(C_0(t-a)^{-1} + C_1 + C_2(T))\right)\\ &= -\limsup_{t\to\infty}\frac{1}{M}(C_0(t-a)^{-1} + C_1 + C_2(T))\\ &> -\infty,\end{aligned}$$

i.e.

$$\liminf_{t\to\infty} \sum_{s=T+1}^{t-\alpha h} e(s) > -\infty,$$

which contradicts condition (3.1). In case $x(t)$ eventually negative, one can proceed in the same way and reach to a contradiction with (3.1). The proof is complete. □

Theorem 3.2. *Assume that (H1)-(H3) hold for $0 < \gamma < 1$. If*

$$\liminf_{t\to\infty} \sum_{s=T+1}^{t-\alpha h} (e(s) + H(s)) = -\infty \tag{3.4}$$

and

$$\limsup_{t\to\infty} \sum_{s=T+1}^{t-\alpha h} (H(s) - e(s)) = \infty, \tag{3.5}$$

for sufficiently large T, where

$$H(s) = \frac{1-\gamma}{\gamma} r^{\frac{\gamma}{\gamma-1}}(s) \left(\gamma p(s)\right)^{\frac{1}{1-\gamma}}, \tag{3.6}$$

then every solution of (1.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1).

Case 1: Suppose that $x(t)$ is an eventually positive solution of (1.1). Then there exists a $T > a$ such that $x(t) > 0$ for all $t \geqslant T$. For each fixed s, let $X = x^\gamma(s)$, $Y = \frac{\gamma p(s)}{r(s)}$, $u = \frac{1}{\gamma}$ and $v = \frac{1}{1-\gamma}$, then from part (i) of Lemma 2.2 we get

$$\begin{aligned}p(s)x^\gamma(s) - r(s)x(s) &= \frac{r(s)}{\gamma}\left(x^\gamma(s)\frac{\gamma p(s)}{r(s)} - \gamma(x^\gamma(s))^{\frac{1}{\gamma}}\right)\\ &= \frac{r(s)}{\gamma}(XY - \frac{1}{u}X^u) \leqslant \frac{r(s)}{\gamma}\frac{1}{v}Y^v = H(s),\end{aligned} \tag{3.7}$$

where $H(s)$ is defined as in (3.6). In view of (H3), by following similar steps to the proof of Theorem 3.1, we get

$$\begin{aligned} 0 &< C_0(t-a)^{-1} + C_1 + C_2(T) + \sum_{s=T+1}^{t-\alpha h} (t-a)^{-1}(t-\sigma(s))_h^{(\alpha-1)}(e(s) + f(s, x(s)) - r(s)x(s)) \\ &\leqslant C_0(t-a)^{-1} + C_1 + C_2(T) + \sum_{s=T+1}^{t-\alpha h} (t-a)^{-1}(t-\sigma(s))_h^{(\alpha-1)}(e(s) + p(s)x^\gamma(s) - r(s)x(s)) \\ &\leqslant C_0(t-a)^{-1} + C_1 + C_2(T) + \sum_{s=T+1}^{t-\alpha h} (t-a)^{-1}(t-\sigma(s))_h^{(\alpha-1)}(e(s) + H(s)) \\ &< C_0(t-a)^{-1} + C_1 + C_2(T) + M \sum_{s=T+1}^{t-\alpha h} (e(s) + H(s)), \quad t \in \mathbb{T}_h^a, \end{aligned} \tag{3.8}$$

where C_0, C_1 , C_2 and M are defined as in Theorem 3.1. Hence, we get

$$\liminf_{t\to\infty} \sum_{s=T+1}^{t-\alpha h} (e(s) + H(s)) > -\infty,$$

which contradicts condition (3.4).

Case 2: Suppose that $\overline{x}(t)$ is an eventually negative solution of (1.1). Then there exists a $T > a$ such that $\overline{x}(t) < 0$ for all $t \geqslant T$. For each fixed s, let $\overline{X} = \overline{x}(s)^\gamma$, $Y = \frac{\gamma p(s)}{r(s)}$, $u = \frac{1}{\gamma}$ and $v = \frac{1}{1-\gamma}$, then from part (i) of Lemma 2.2 we get

$$\begin{aligned} p(s)\overline{x}^\gamma(s) - r(s)\overline{x}(s) &= \frac{r(s)}{\gamma}\left(\overline{x}^\gamma(s)\frac{\gamma p(s)}{r(s)} - \gamma(\overline{x}^\gamma(s))^{\frac{1}{\gamma}}\right) \\ &= \frac{r(s)}{\gamma}(\overline{X}Y - \frac{1}{u}\overline{X}^u) \geqslant -\frac{r(s)}{\gamma}\frac{1}{v}Y^v = -H(s), \end{aligned}$$

where $H(s)$ is defined as in (3.6). In view of (H3), by following similar steps to the proof of Theorem 3.1, we get

$$\begin{aligned} 0 &> C_0(t-a)^{-1} + C_1 + C_2(T) + \sum_{s=T+1}^{t-\alpha h} (t-a)^{-1}(t-\sigma(s))_h^{(\alpha-1)}(e(s) + f(s, \overline{x}(s)) - r(s)\overline{x}(s)) \\ &\geqslant C_0(t-a)^{-1} + C_1 + C_2(T) + \sum_{s=T+1}^{t-\alpha h} (t-a)^{-1}(t-\sigma(s))_h^{(\alpha-1)}(e(s) + p(s)\overline{x}^\gamma(s) - r(s)\overline{x}(s)) \\ &\geqslant C_0(t-a)^{-1} + C_1 + C_2(T) + \sum_{s=T+1}^{t-\alpha h} (t-a)^{-1}(t-\sigma(s))_h^{(\alpha-1)}(e(s) - H(s)) \\ &> C_0(t-a)^{-1} + C_1 + C_2(T) - M \sum_{s=T+1}^{t-\alpha h} (e(s) - H(s)) \\ &= C_0(t-a)^{-1} + C_1 + C_2(T) + M \sum_{s=T+1}^{t-\alpha h} (H(s) - e(s)), \quad t \in \mathbb{T}_h^a, \end{aligned}$$

where C_0, C_1, C_2 and M are defined as in Theorem 3.1. Hence, we get

$$\limsup_{t\to\infty} \sum_{s=T+1}^{t-\alpha h} (H(s) - e(s)) < \infty,$$

which contradicts condition (3.5). This proof is complete. □

Theorem 3.3. *Assume that (H1), (H2) and (H4) hold for $\gamma > 1$. If*

$$\liminf_{t\to\infty} \sum_{s=T+1}^{t-\alpha h} (e(s) + H(s)) = -\infty \tag{3.9}$$

and

$$\limsup_{t\to\infty} \sum_{s=T+1}^{t-\alpha h} (H(s) - e(s)) = \infty, \tag{3.10}$$

for sufficiently large T, where $H(s)$ is defined as in (3.6), then every bounded solution of (1.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory bounded from above solution of (1.1). Then there exists a constant N such that

$$|x(t)| \leqslant N, \quad \textit{for } t \in \mathbb{T}_h^a. \tag{3.11}$$

Suppose that $x(t)$ is an eventually bounded positive solution of (1.1). Then there exists a $T > 0$ such that $x(t) > 0$ for $t \geqslant T$. Using (ii) of Lemma 2.2 and similar to the proof of (3.7), and by (H4) we get

$$f(s, x(s)) - r(s)x(s) \geqslant p(s)x^{\gamma}(s) - r(s)x(s) \geqslant H(s), \quad s \geqslant T.$$

By Lemma 2.1 and similar to (3.8), we obtain

$$\Gamma(\alpha)x(t)(t-a)^{-1} \geqslant C_0(t-a)^{-1} + C_1 + C_2(T) + \sum_{s=T+1}^{t-\alpha h} (t-a)^{-1}(t-\sigma(s))_h^{(\alpha-1)}(e(s)+H(s)), \quad t \in \mathbb{T}_h^a,$$

where C_0, C_1 and C_2 are defined as in Theorem 3.1. In view of (3.11), we have

$$\begin{aligned} N\Gamma(\alpha)(t-a)^{-1} &\geqslant C_0(t-a)^{-1} + C_1 + C_2(T) + \sum_{s=T+1}^{t-\alpha h} (t-a)^{-1}(t-\sigma(s))_h^{(\alpha-1)}(e(s) + H(s)) \\ &> C_0(T)C_0(t-a)^{-1} + C_1 + C_2(T) - M \sum_{s=T+1}^{t-\alpha h} (e(s) + H(s)), \quad t \in \mathbb{T}_h^a, \end{aligned}$$

where M is defined as in Theorem 3.1, i.e.

$$\sum_{s=T+1}^{t-\alpha h} (e(s) + H(s)) > -\frac{1}{M}(N\Gamma(\alpha)(t-a)^{-1} - C_0(t-a)^{-1} - C_1 - C_2(T)), \quad t \in \mathbb{T}_h^a.$$

For $t \geqslant T$, and this implied that

$$\liminf_{t\to\infty} \sum_{s=T+1}^{t-\alpha h} (e(s) + H(s)) > -\infty,$$

which contradicts (3.9). In case $x(t)$ eventually negative, the verifications are similar to Case 2 of Theorem 3.2, so we omit it here. The proof is complete. □

4. Examples

. Consider the equation

$$\begin{cases} \Delta^{\frac{3}{2}}_{\frac{2}{3}} x(t) + x(t)\ln(t+e) = -t\ln(t+e) + x^{\frac{1}{2}}(t)\ln(t+e), \quad t \in \mathbb{T}^a_h, \\ x(a) = 0, \quad \Delta_{\frac{2}{3}} x(a) = \frac{1}{\Gamma(\alpha)}, \end{cases} \tag{4.1}$$

where $a = 0$, $\alpha = \frac{3}{2}$, $h = \frac{2}{3}$, $r(t) = p(t) = \ln(t+e)$, $\gamma = \frac{1}{2}$, $e(t) = -t\ln(t+e)$, $c_0 = 0$ and $c_1 = \frac{1}{\Gamma(\alpha)}$. It is clear that conditions (H1)-(H3) are satisfied. Thus, the function $H(s) = \frac{1-\gamma}{\gamma} r^{\frac{\gamma}{\gamma-1}}(s)\,(\gamma p(s))^{\frac{1}{1-\gamma}} = \frac{1}{4}\ln(t+e)$. Furthermore, we have

$$\liminf_{t\to\infty} \sum_{s=T+1}^{t-\alpha h} (-s\ln(s+e) + \frac{1}{4}\ln(s+e)) = -\infty,$$

and

$$\limsup_{t\to\infty} \sum_{s=T+1}^{t-\alpha h} (\frac{1}{4}\ln(s+e) + s\ln(s+e)) = \infty.$$

By Theorem 3.2 we see that (4.1) is oscillatory.

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