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Eigenvalues and Eigenfunctions of a Schrödinger Operator Associated with a Finite Combination of Dirac-Delta Functions and CH Peakons*

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Abstract In this paper, we first study the Schrödinger operators with the following weighted function $\sum_{i=1}^{n} p_i\delta(x-a_i)$, which is actually a finite linear combination of Dirac-Delta functions, and then discuss the same operator equipped with the same kind of potential function. With the aid of the boundary conditions, all possible eigenvalues and eigenfunctions of the self-adjoint Schrödinger operator are investigated. Furthermore, as a practical application, the spectrum distribution of such a Dirac-Delta type Schrödinger operator either weighted or potential is well applied to the remarkable integrable Camassa-Holm (CH) equation.

Keywords Schrödinger operator, Boundary conditions, Soliton, Peakon solution, Cammassa-Holm equation.

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1. Introduction

With the recent development of quantum mechanics, nonlinear mathematical physics, and nonlinear waves with peaked soliton (peakon) solutions, some new mathematical problems arise to be solved. The classical Schördinger equation with the Dirac-Delta function $\delta(x)$ or its high-order derivatives $\delta^{(n)}$ as a potential function is the new type of spectral problem endowed with a generalized function or a weighted generalized function. Over eighty years since last centenary, much work has been done on the Schrödinger equation with the $\delta(x)$ or δ'-interaction as a potential function. In [10], the author studied a class of solvable one-dimensional Schrödinger equation in which the Hamiltonian can formally be written as

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \sum_{y\in Y} v_y \delta'(\cdot - y), \tag{1.1}$$

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where Y is a discrete subset of R, finite or infinite, and δ' denotes the derivative of Dirac's δ-function. The existence of model (1.1) was shown by Grossmann et al [11].

In [22], the author mentioned that in the study of the quantized Davey-Stewartson system with two particles ($N = 2$), there was the following Schrödinger equation encountered

$$-\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} + c\delta'(x)\varphi = E\varphi, \tag{1.2}$$

where c is the coupling constant, and $\delta'(x)$ is the derivative of the Dirac δ-function. The same equation also appeared in [1] and [10]; however, the author of [22] pointed out that the problem was not dealt with correctly, i.e., the boundary conditions for (1.2) in [1] and [10] at the singular point are irrelevant to the $\delta'(x)$-interaction. The appropriate boundary conditions can be replaced by

$$\begin{cases} \varphi(0^+) = \varphi(0^-) = \varphi(0) = 0, \\ \varphi'(0^+) - \varphi'(0^-) = -c\varphi'(0). \end{cases} \tag{1.3}$$

The author in [22] made remarks that the following boundary conditions in [10] and [11]

$$\begin{cases} \varphi'(0^+) = \varphi'(0^-), \\ \varphi(0^+) - \varphi(0^-) = \beta\varphi'(0), \quad \beta \in R, \end{cases} \tag{1.4}$$

are proposed to substitute for (1.3). Afterwards, in [12] the author points out that in [22] an incorrect and extraneous constraint (1.3) is imposed, as the interesting feature of δ' potential precisely implies that φ is not continuous. Because the potential $\delta'(x)$ has been a source of recurring confusion, using the integration by part from $-\varepsilon(x)$ to $+\varepsilon(x)$, the author provides a derivation of the boundary conditions at the singular point $''0''$, this is

$$\begin{cases} -\dfrac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} + c\delta^{(n)}(x)\varphi = E\varphi, \\ \varphi'(0^+) - \varphi'(0^-) = (-1)^n c\bar{\varphi}^{(n)}(0), \\ \varphi(0^+) - \varphi(0^-) = (-1)^{n-1} nc\bar{\varphi}^{(n-1)}(0). \end{cases} \tag{1.5}$$

When $n = 0$, the potential reads $c\delta(x)$, which usually appears in quantum mechanics. The boundary conditions of (1.5) yield the well-known conditions

$$\begin{cases} \varphi(0^+) = \varphi(0^-), \\ \varphi'(0^+) - \varphi'(0^-) = c\varphi(0). \end{cases} \tag{1.6}$$

When $n = 1$, they become

$$\begin{cases} \varphi(0^+) - \varphi(0^-) = c\bar{\varphi}(0), \\ \varphi'(0^+) - \varphi'(0^-) = c\bar{\varphi}'(0). \end{cases} \tag{1.7}$$

We notice that these conditions are different from the boundary conditions in [22] and the boundary conditions in [1, 10].

In 1993, the authors of [1] stated in [2] that the reasoning in [22] is seriously flawed and subsequently wrong conclusions have been reached. They think that the last condition of (1.3) is ill-defined since the symbol $\varphi'(0)$ is not explained in [22], and especially, the first condition of (1.3) itself is already self-adjoint. Due to this fact any additional boundary condition, such as the last condition of (1.3) together with its first condition necessarily represents a non-self-adjoint operator which is entirely unacceptable as a Hamiltonian in a quantum mechanical context. And they also claimed that the boundary conditions of (1.4) used in [10] and [1], which define a family of self-adjoint extensions, are correct.

From the above discussions, we see that it is very important to understand the meaning of the potential $\delta'(x)$ in a mathematical language. We notice that the conditions of (1.4) are only a kind of self-adjoint domains, and if conditions (1.7) are self-adjoint boundary conditions of the Schrödinger operator with a δ or δ' interaction, it may be helpful to address this question for both mathematicians and physicists. In [14], we gave out the complete description of self-adjoint boundary conditions of the Schrödinger operator associated with a $\delta(x)$ or $\delta'(x)$ interaction for one dimension potential function. All self-adjoint boundary conditions could be chosen with constants α, β, θ, m, n, r, s $(ms-rn=1)$ in many different ways.

$$\begin{cases} \cos\alpha\varphi(0^-)+\sin\alpha\varphi'(0^-)=0, \\ \cos\beta\varphi(0^+)+\sin\beta\varphi'(0^+)=0; \end{cases} \tag{1.8}$$

or

$$\varphi'(0^+)-\varphi'(0^-)=-r\mathrm{e}^{-i\theta}\varphi'(0^+)+s\mathrm{e}^{i\theta}\varphi'(0^-)+r\mathrm{e}^{-i\theta}\varphi(0^+)+r\mathrm{e}^{i\theta}\varphi(0^-), \tag{1.9}$$

or

$$\varphi(0^+)-\varphi(0^-)=n\mathrm{e}^{-i\theta}\varphi'(0^+)+n\mathrm{e}^{i\theta}\varphi'(0^-)-s\mathrm{e}^{-i\theta}\varphi(0^+)+m\mathrm{e}^{i\theta}\varphi(0^-), \tag{1.10}$$

together with conditions

$$\begin{cases} m\mathrm{e}^{i\theta}y(0^-)+n\mathrm{e}^{i\theta}y'(0^-)=y(0^+), \\ r\mathrm{e}^{i\theta}y(0^-)+s\mathrm{e}^{i\theta}y'(0^-)=y'(0^+), \end{cases} \tag{1.11}$$

will yield three families of the self-adjoint boundary conditions, respectively. The different choices of α, β, θ, m, n, r, s may correspond to the different interpretation on $\delta(x)$ and $\delta'(x)$ interaction in physics.

However, a new problem rises – how are about the eigenvalues and eigenfunctions of those self-adjoint Schrödinger operators?

In [4], the authors studied eigenvalue problems with Dirac delta function potentials and accordingly presented the corresponding eigenvalues and eigenfunctions, but it seems that they did not discus if the eigenvalues they obtained are all eigenvalues of the problem. In our paper, we first prove that all squared integrable solutions to the Schrödinger equation with the Dirac function potential satisfy a peakon condition at a discontinuous point of the potential function, but satisfy a smooth condition at other discontinuous points (see §2 for details). Then in §4, we show that all Schrödinger operators associated with discontinuity points, where peakon condition or smooth conditions are cast, are self-adjoint. Therefore, from the viewpoint of operator theory, we verify all eigenvalues of the problem are real and discrete. Also, we present the equations satisfied with all eigenvalues and prove that the equations have unique real root.

The Camassa-Holm (CH) equation

$$u_t + 2k^2 u_x - u_{xxt} + 3uu_x = 2u_x u_{xxx} + uu_{xxx}, \tag{1.12}$$

where the real constant $k > 0$, was derived as a model to describe shallow water waves by Camassa and Holm in 1993 [5]. This equation is integrable with the following Lax pair:

$$\begin{cases} \phi_{xx} = \lambda(u - u_{xx} + k^2)\phi + \dfrac{1}{4}\phi, \\ \phi_t = (\dfrac{1}{2\lambda} - u)\phi_x + \dfrac{1}{2}u_x\phi. \end{cases} \tag{1.13}$$

Considerable interest was paid on the CH equation in recent two decades about its integrable properties and various kinds of exact solutions [7,8,16,18].

Because the parameter k is in the linear term of equation (1.12), without any loss of generality, let us consider a cleaned CH equation with $k = 0$ [19]

$$m_t + m_x u + 2mu_x = 0, \quad m = u - u_{xx}, \quad x \in R. \tag{1.14}$$

Its Lax pair reads now:

$$\begin{cases} L\Psi := -\Psi_{xx} + \dfrac{1}{4}\Psi = \lambda m(x)\Psi, & \text{(space part)} \\ \Psi_t = N\Psi := -(\lambda + u)\Psi_x + \dfrac{1}{2}u_x\Psi, & \text{(time part)} \end{cases} \tag{1.15}$$

and apparently the space part of the Lax pair is the Sturm-Liouville problem with weighted function $m(x)$, where $m(x) = u - u_{xx}$.

Parker [16] studied the bilinear form for the CH equation and gave its multi-soliton solutions. Dullin, Gottwald and Holm [8] dealt with the traveling-wave solutions of a generalized version of the CH equation and also studied some exact solutions.

The present paper provides an approach to study all possible eigenvalue distributions related to the peakon solutions to the CH equation (1.14). Our strategy is to use the discontinuity of the first-order derivative and the Dirac distribution skills for the CH equation (see Eq. (1.12) and Theorem 1.1).

Let us consider the traveling wave solution of the CH equation (1.14) through the setting $u(x,t) = v(x - ct)$, where c is the wave speed. Let $\xi = x - ct$, then substituting $u(x,t) = v(\xi)$ into the CH equation (1.14) yields

$$(v - c)(v - v'')' + 2v'(v - v'') = 0, \tag{1.16}$$

where $v' = v_\xi, v'' = v_{\xi\xi}, v''' = v_{\xi\xi\xi}$.

The CH equation has the peakon solution [5] $u(x,t) = v(\xi) = ce^{-|x-ct-\xi_0|}$ ($\xi_0 = x_0 - ct_0$) with the following properties:

$$v(\xi_0) = c, \quad v(\pm\infty) = 0, \quad v'(\xi_0-) = c, \quad v'(\xi_0+) = -c, \tag{1.17}$$

where $v'(\xi_0-)$ and $v'(\xi_0+)$ represent the left-derivative and the right-derivative at ξ_0, respectively.

Theorem 1.1 (Proposition 1, [19]). *The CH equation* (1.14) *has the following weak traveling wave solution :*

$$u(x,t) = -a\sinh(|x-ct-\xi_0|) + ce^{-|x-ct-\xi_0|}, \tag{1.18}$$

where $a \in R$ is an arbitrary constant, c is the wave speed, and $\xi_0 = x_0 - ct_0$ is an arbitrarily real constant.

In particular, if we take $a = 0$ in this theorem, then (1.18) exactly gives the regular peakon solution $u(x,t) = ce^{-|x-ct-\xi_0|}$ which was described by Camassa and Holm [5]. The most attractive fact is that CH equation admits peaked soliton(peakon) solutions: $u(x,t) = ce^{-|x-ct|}$ in the case $k = 0$ (Camassa and Holm [5] 1993).

Substituting $u(x,t) = ce^{-|x-ct-\xi_0|}$ to the eigenvalue problem (i.e. spacial part) of the Lax pair for the CH equation reads as

$$-\Psi_{xx} + \frac{1}{4}\Psi(x) = 2\lambda\delta(x-a)\Psi(x), \tag{1.19}$$

which exactly coincides with the one-peakon of CH equation: $u = ce^{-|x-ct|}$ if one takes $a = ct$.

Let us now consider the two-peakon of the CH equation in the form of

$$u(x,t) = p_1(t)e^{-|x-q_1(t)|} + p_2(t)e^{-|x-q_2(t)|}, \tag{1.20}$$

where $p_1(t), p_2(t)$ and $q_1(t), q_2(t)$ are four functions of t and stand for the amplitude and the peak positions of the two-peakon. The two-peakon dynamical system can be figured out [5] through substituting it to the CH equation. But, the first equation (i.e. spacial part) of the Lax pair for the CH equation (1.14) would be

$$-\Psi_{xx} + \frac{1}{4}\Psi(x) = 2\lambda\big(p_1\delta(x-q_1) + p_2\delta(x-q_2)\big)\Psi(x), \tag{1.21}$$

which corresponds to the special potential with two-peakon solution.

In general, the N-peakon admits the following formulation:

$$u(x,t) = \sum_{j=1}^{N} p_j(t)e^{-|x-q_j(t)|}, \tag{1.22}$$

where $p_j(t)$ and $q_j(t)$ $(j = 1...N)$ are $2N$ functions of t standing for the amplitudes and the peak positions of the N-peakon, respectively. The N-peakon dynamical system has already been presented [5] through substituting it to the CH equation (1.14). For the N-peakon set as the special potential, the first equation (i.e. the spacial part) in the Lax pair of CH equation (1.14) becomes

$$-\Psi_{xx} + \frac{1}{4}\Psi(x) = 2\lambda\sum_{j=1}^{n} p_j\delta(x-q_j)\Psi(x). \tag{1.23}$$

Then, a natural question rises – how are about the eigenvalues and eigenfunctions of those Schrödinger problems with the Dirac weighted function?

In this paper, we shall discuss the eigenvalue λ's distribution under the potential function $m = \sum_{j=1}^{n} p_j\delta(x-q_j)$. The whole paper is organized as follows.

In Section 2, we provide the solutions of the Schrödinger equation with the Dirac function potential In Section 3, we present the eigenvalues and corresponding eigenfunctions of the self-adjoint Schrödinger operator with the Dirac function weight in one dimensional space. In Section 4, we obtain the eigenvalues and corresponding eigenfunctions of the self-adjoint Schrödinger operator with Dirac functions as a potential in one dimension. In Section 5, we study the relationship of the Schrödinger operator between the one with the Dirac function potential and the one with the Dirac function weight.

2. Schrödinger equation with the Dirac function potential

Let $-\infty =: a_0 < a_1 < \cdots < a_n < a_{n+1} := +\infty$ be the singular points for some Schrödinger equation. From integrable system theory, we only consider the solution $z(x)$ of thus equations at every singular point a_k $(k = 1, \cdots, n)$ satisfy either smooth conditions:

$$z^{(j)}(a_k - 0) = z^{(j)}(a_k + 0), \quad j = 0, 1, \tag{2.1}$$

or peakon conditions:

$$z(a_k - 0) = z(a_k + 0), \quad z'(a_k - 0) = -z'(a_k + 0), \tag{2.2}$$

$$z'(a_k - 0) \cdot z''(a_k - 0) > 0. \tag{2.3}$$

Remark 2.1. The peakon condition is the character of peakon solution $y = c_1 \exp(-c_2|x - c_3|)$, where $c_1 \neq 0, c_2 > 0, c_3 \in R$ are constants.

The conditions (2.1) and (2.2) can be written as the following matrix form

$$\begin{pmatrix} z(a_k - 0) \\ z'(a_k - 0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z(a_k + 0) \\ z'(a_k + 0) \end{pmatrix} \tag{2.4}$$

and

$$\begin{pmatrix} z(a_k - 0) \\ z'(a_k - 0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} z(a_k + 0) \\ z'(a_k + 0) \end{pmatrix}, \tag{2.5}$$

respectively.

Lemma 2.1. *Let $-\infty =: a_0 < a_1 < \cdots < a_n < a_{n+1} := +\infty$, and $s \in C$ be constants, then any solution of the degenerated Schrödinger equation*

$$z''(x) = s^2 z(x) \quad on \quad \bigcup_{i=0}^{n} (a_i, a_{i+1}) \tag{2.6}$$

can be written as

$$z(x) = c_i \mathrm{e}^{|s|x} + d_i \mathrm{e}^{-|s|x}, \quad x \in (a_i, a_{i+1}), \quad i = 0, 1, \cdots, n, \tag{2.7}$$

where c_i's and d_i's are complex constants.

Remark 2.2. It is clear that when $s \in \mathrm{i}R$ (i.e. $s^2 \leq 0$), $z(x) \notin L^2(-\infty, +\infty)$ for all c_1 and c_2 (at lest one of them does not equals 0). When $s \in R^* := R \setminus \{0\}$ (i.e. $s^2 > 0$), $z(x) \in L^2(-\infty, +\infty)$ must have $d_0 = c_n = 0$.

We only consider the non-trivial real solution $z(x)$ of (2.6) when $s^2 > 0$, it is unique up to a non-zero constant multiple. By above Remark, $c_0 \neq 0$.

Lemma 2.2. *If $z(x)$ at a_k satisfies the smooth conditions, then $c_k = c_{k-1}$, $d_k = d_{k-1}$, that is smooth conditions do not change the form of solution of* (2.6).

Proof. From (2.7) and smooth conditions,

$$c_{k-1}\mathrm{e}^{|s|a_k} + d_{k-1}\mathrm{e}^{-|s|a_k} = c_k\mathrm{e}^{|s|a_k} + d_k\mathrm{e}^{-|s|a_k}, \tag{2.8}$$

$$c_{k-1}|s|\mathrm{e}^{|s|a_k} - d_{k-1}|s|\mathrm{e}^{-|s|a_k} = c_k|s|\mathrm{e}^{|s|a_k} - d_k|s|\mathrm{e}^{-|s|a_k}, \tag{2.9}$$

the result will be obtained by solving above equations, since $s \neq 0$ by the Remark 2.2. □

Corollary 2.1. *If $z(x)$ at $a_1, \cdots, a_k$ satisfies smooth conditions, then*

$$z(x)\Big|_{(a_0, a_{k+1})} = c_0\mathrm{e}^{|s|x}. \tag{2.10}$$

Lemma 2.3. *If $z(x)$ at a_k satisfies peakon conditions, then*

$$c_k = d_{k-1}\mathrm{e}^{-2|s|a_k}, \quad d_k = c_{k-1}\mathrm{e}^{2|s|a_k}, \tag{2.11}$$

and

$$d_{k-1}^2 < c_{k-1}^2\mathrm{e}^{2|s|a_k}. \tag{2.12}$$

Proof. From (2.7) and peakon conditions,

$$c_{k-1}\mathrm{e}^{|s|a_k} + d_{k-1}\mathrm{e}^{-|s|a_k} = c_k\mathrm{e}^{|s|a_k} + d_k\mathrm{e}^{-|s|a_k}, \tag{2.13}$$

$$c_{k-1}|s|\mathrm{e}^{|s|a_k} - d_{k-1}|s|\mathrm{e}^{-|s|a_k} = -c_k|s|\mathrm{e}^{|s|a_k} + d_k|s|\mathrm{e}^{-|s|a_k}, \tag{2.14}$$

$$\left(c_{k-1}|s|\mathrm{e}^{|s|a_k} - d_{k-1}|s|\mathrm{e}^{-|s|a_k}\right)\left(c_{k-1}s^2\mathrm{e}^{|s|a_k} + d_{k-1}s^2\mathrm{e}^{-|s|a_k}\right) > 0, \tag{2.15}$$

we have (2.11) from (2.13) and (2.14), also, the (2.12) will be obtained from (2.15). □

Corollary 2.2. *If $z(x)$ at $a_1, \cdots, a_{k-1}$ satisfies smooth conditions, but at a_k satisfies peakon conditions, then*

$$z(x)\Big|_{(a_k, a_{k+1})} = c_0\mathrm{e}^{|s|(-x+2a_k)}. \tag{2.16}$$

Corollary 2.3. *If $z(x)\Big|_{(a_{k-1}, a_k)} = C\mathrm{e}^{-|s|x}$, where $C \in R$, is a constant, then $z(x)$ does not satisfy the peakon conditions at a_k .*

Proof. Here $c_{k-1} = 0, d_{k-1} = C \neq 0$, so

$$d_{k-1}^2 - c_{k-1}^2\mathrm{e}^{2|s|a_k} = C^2 \geq 0, \tag{2.17}$$

does not satisfy the condition (2.12) that is equivalent to (2.3) in peakon conditions. □

Theorem 2.1. *For any $s \in R^*$, the equation* (2.6) *has exactly n linearly independent solutions*

$$z_k(x) = \begin{cases} c_0 \mathrm{e}^{|s|x}, & x \in (-\infty, a_k], \\ c_0 \mathrm{e}^{|s|(-x+2a_k)}, & x \in [a_k, +\infty), \end{cases} \tag{2.18}$$

$k = 1, 2, \cdots, n$, in $L^2(-\infty, +\infty)$, where $c_0 \neq 0$ is a constant.

Proof. From Corollary 2.1, if $z(x)$ satisfy smooth condition at all singular points, then $z(x) = c_0 \mathrm{e}^{|s|x}$ on hole real axis, this is a contradiction with $z(x) \in L^2(-\infty, +\infty)$. Therefore $z(x)$ satisfy the peacon conditions at some singular points.

Let a_k be first singular point such that $z(x)$ satisfies the peakon conditions, from Corollary 2.1 again and Corollary 2.2, we have

$$y(x, \lambda)\Big|_{(a_k, a_{k+1})} = c_0 \mathrm{e}^{|s|(-x+2a_k)}. \tag{2.19}$$

By Lemma 2.2 and Corollary 2.3, $z(x)$ does not satisfy the peakon conditions at $a_{k+1}, \cdots, a_n$, but satisfy the smooth condition, from Lemma 2.2 again, we also obtain

$$z(x)\Big|_{(a_k, +\infty)} = c_0 \mathrm{e}^{|s|(-x+2a_k)}. \tag{2.20}$$

Hence, $z_1, \cdots, z_n$ are all linearly independent solutions of (2.6).

$\square$

Let $c_0 = \mathrm{e}^{-|s|a_k}$, and $z_k(x)$ can be normalized as

$$z_k(x) = \mathrm{e}^{-|s(x-a_k)|} = \begin{cases} \mathrm{e}^{|s|(x-a_k)}, & x \in (-\infty, a_k], \\ \mathrm{e}^{|s|(-x+a_k)}, & x \in [a_k, +\infty), \end{cases}. \tag{2.21}$$

Lemma 2.4. *For any $k : 1 \leq k \leq n$,*

$$\int_{-\infty}^{+\infty} z_k(x) \mathtt{d}x = \frac{2}{|s|}. \tag{2.22}$$

Proof. Because

$$\int_{-\infty}^{a_k} z_k(x) \mathtt{d}x = \int_{-\infty}^{a_k} \mathrm{e}^{|s|(x-a_k)} \mathtt{d}x = \frac{1}{|s|} \mathrm{e}^{|s|(x-a_k)} \Big|_{-\infty}^{a_k} = \frac{1}{|s|}, \tag{2.23}$$

$$\int_{a_k}^{+\infty} z_k(x) \mathtt{d}x = \int_{a_k}^{+\infty} \mathrm{e}^{|s|(-x+a_k)} \mathtt{d}x = -\frac{1}{|s|} \mathrm{e}^{|s|(-x+a_k)} \Big|_{a_k}^{+\infty} = \frac{1}{|s|}, \tag{2.24}$$

then

$$\int_{-\infty}^{+\infty} z_k(x) \mathtt{d}x = \int_{-\infty}^{a_k} z_k(x) \mathtt{d}x + \int_{a_k}^{+\infty} z_k(x) \mathtt{d}x = \frac{2}{|s|}. \tag{2.25}$$

$\square$

3. Self-adjoint Schrödinger operator with the Dirac weighted function

In order to investigate the peaked soliton (peakon solution) of CH equation (1.13), we must have the solutions of differential equation (1.23). So, we need to study the singular Sturm-Liouville problem

$$-y'' + \frac{1}{4}y = \lambda D(x)y, \quad x \in (-\infty, +\infty), \tag{3.1}$$

where

$$D(x) = 2\sum_{i=1}^{n} p_i \delta(x - a_i), \tag{3.2}$$

$p_i \in R,\ p_i > 0\,(i = 1, \cdots, n)$, $a_1 < a_2 < \cdots < a_n$, and the $\delta(x)$ is Dirac function

$$\delta(x) = \begin{cases} 0, & x \neq 0, \\ \infty, & x = 0, \end{cases} \tag{3.3}$$

Also,

$$\int_{-\infty}^{+\infty} \delta(x)\mathtt{d}x = 1, \tag{3.4}$$

and need to give out its eigenvalues and eigenfunctions.

This function have following property also.

Lemma 3.1. *For any function $f(x)$,*

$$\int_{-\infty}^{+\infty} f(x)\delta(x)\mathtt{d}x = f(0). \tag{3.5}$$

The points $-\infty =: a_0,\ a_1,\ a_2,\ \cdots,\ a_n,\ a_{n+1} := +\infty$ are singular points of (3.1).

By the deficiency theory of singular differential operators, we define the Sturm-Liouville operator L as following:

$$Ly = -y'' + \frac{1}{4}y, \tag{3.6}$$

$$\begin{aligned} y \in \mathfrak{D}(L) := \Big\{ & y \in L^2(-\infty, +\infty) \mid y, y' \in AC_{\mathtt{Loc}}\left(\cup_{i=0}^{n}(a_i, a_{i+1})\right), \\ & -y'' + \frac{1}{4}y \in L^2(-\infty, +\infty), \sum_{i=1}^{n} Y(a_i - 0) + A_i Y(a_i + 0) = 0 \Big\}, \end{aligned} \tag{3.7}$$

where $Y = (y,\ y')^{\mathrm{T}}$, and

$$A_i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.8}$$

then the L is is a self-adjoint operator since the restrictions of L to (a_0, a_1) and (a_n, a_{n+1}) are all limit-point symmetric operator. We notice that the wight function $D(x)$ is a real function and $D(x) \geq 0$ on $(-\infty, +\infty)$, so, the spectrum of self-adjoint operator L consists of discrete spectrum and continuous spectrum, and are on the real axis. [6,9].

For any complex $\lambda \in C$, the solution $y(x, \lambda)$ of Sturm-Liouville problem (3.1) which is restricted on every subinterval (a_k, a_{k+1}) $(k = 0, 1, \cdots, n)$ satisfy the differential equation:

$$-y'' + \frac{1}{4}y = 0. \tag{3.9}$$

By Theorem 2.1, generalized Sturm-Liouville problem (3.1) have n linear independent solutions

$$\varphi_k(x, \lambda) = \exp\left(\frac{|x - a_k|}{2}\right), \quad k = 1, 2, \cdots, n. \tag{3.10}$$

Theorem 3.1. *Generalized Sturm-Liouville problem* (3.1) *have only n eigenvalues (multiple eigenvalue be denoted by its multiplicity)*

$$\lambda_k = \left(2\sum_{i=1}^{n}\left(p_i \exp\left(\frac{|a_i - a_k|}{2}\right)\right)\right)^{-1}, \; k = 1, 2, \cdots, n \tag{3.11}$$

and their corresponding eignfunctions are $\varphi_k(x, \lambda)$, $k = 1, 2, \cdots, n$.

Proof. $\varphi_k(x, \lambda)$ is a solution of (3.1), i.e.

$$-\varphi_k''(x, \lambda) + \frac{1}{4}\varphi_k(x, \lambda) = \lambda\sum_{i=1}^{n} 2p_i\delta(x - a_i) \cdot \varphi_k(x, \lambda), \quad x \in (-\infty, +\infty), \tag{3.12}$$

then

$$-\varphi_k'(x, \lambda)\Big|_{-\infty}^{+\infty} + \frac{1}{4}\int_{-\infty}^{+\infty} \varphi_k(x, \lambda)\mathtt{d}x = 2\lambda\sum_{i=1}^{n} p_i \int_{-\infty}^{+\infty} \delta(x - a_i) \cdot \varphi_k(x, \lambda)\mathtt{d}x, \tag{3.13}$$

by (3.10), Lemma 2.4 , and the property of Dirac function $\delta(x - a)$,

$$1 = 2\lambda\sum_{i=1}^{n} p_i\varphi_k(a_i, \lambda) = 2\lambda\left(\sum_{i=1}^{k} p_i \exp\left(\frac{|a_i - a_k|}{2}\right)\right), \tag{3.14}$$

Hence,

$$\lambda = \left(2\sum_{i=1}^{n}\left(p_i \exp\left(\frac{|a_i - a_k|}{2}\right)\right)\right)^{-1}. \tag{3.15}$$

□

Corollary 3.1. *The generalized Sturm-Liouville problem*

$$-y'' + \frac{1}{4}y = 2\lambda p\delta(x - a)y, \quad x \in (-\infty, +\infty), \quad p > 0 \tag{3.16}$$

have only one eigenvalue $\lambda = 1/(2p)$ and its corresponding eignfunction is $\varphi(x, \lambda) = \exp(-|x - a|/2)$.

In the same way, we can have:

Theorem 3.2. *The generalized Sturm-Liouville problem*

$$-y'' + py = 2\lambda\delta(x-a)y, \quad x \in (-\infty, +\infty), \tag{3.17}$$

have an eigenvalue $\lambda = \sqrt{p}$ *and its corresponding eignfunction is*

$$\varphi(x, \lambda) = \exp\left(-\sqrt{p}|x-a|\right). \tag{3.18}$$

Corollary 3.2. *The generalized Sturm-Liouville problem*

$$-y'' + \frac{1}{4}y = 2\lambda\delta(x)y, \quad x \in (-\infty, +\infty), \tag{3.19}$$

have an eigenvalue $\lambda = 1/2$ *and its corresponding eignfunction is*

$$\varphi(x, \lambda) = \exp\left(-|x|/2\right).$$

4. Self-adjoint Schrödinger operator with the Dirac function as the potential

In this section, we study the eigenvalue and eigenfunction of Sturm-Liouville problem with a Dirac functions interaction (or a potential function $D(x)$)

$$-y'' + D(x)y = \mu y, \quad x \in (-\infty, +\infty), \tag{4.1}$$

where

$$D(x) = \sum_{i=1}^{n} q_i \delta(x - a_i), \tag{4.2}$$

$q_i \in R,\ q_i < 0\,(i = 1, \cdots, n)$, $a_1 < a_2 < \cdots < a_n$.

We define the self-adjoint Sturm-Liouville operator L similarly as following:

$$Ly = -y'', \tag{4.3}$$

$$y \in \mathfrak{D}(L) := \Big\{ y \in L^2(-\infty, +\infty) \mid y, y' \in AC_{\mathtt{Loc}}\left(\cup_{i=0}^{n}(a_i, a_{i+1})\right), \\ -y'' \in L^2(-\infty, +\infty), \sum_{i=1}^{n} Y(a_i - 0) + A_i Y(a_i + 0) = 0 \Big\}, \tag{4.4}$$

where A_i defined as (3.8). So, the spectrum of self-adjoint operator L consists of discrete spectrum and continuous spectrum, and are on the real axis.

By Remark 2.2 and Theorem 2.1, for some $\mu < 0$, the generalized Sturm-Liouville problem (4.1) have n linear independent square-integrable solutions

$$\psi_k(x, \lambda) = \exp\left(-\sqrt{-\mu}\ |x - a_k|\right), \qquad k = 1, 2, \cdots, n. \tag{4.5}$$

Theorem 4.1. *The generalized Sturm-Liouville problem* (4.1) *have only* n *eigenvalues (multiple eigenvalue be denoted by its multiplicity)* μ_k, $(k = 1, 2, \cdots, n)$, *and their corresponding eignfunctions are* $\psi_k(x, \mu_k)$, $k = 1, 2, \cdots, n$.

Proof. For any $\mu < 0$ and and $k: \ 1 \leq k \leq n$, The $\psi_k(x,\mu)$ satisfy equation (4.1) when $x \in \cup_{i=0}^{n}(a_i, a_{i+1})$. Putting this solution into the equation (4.1), and taking integral for two side of equation (4.1) on $(-\infty, +\infty)$, and have

$$-\psi_k'(x,\mu)\Big|_{-\infty}^{+\infty} + \sum_{i=1}^{n} q_i \int_{-\infty}^{\infty} \delta(x-a_i)\psi_k(x,\mu)\mathrm{d}x = \mu \int_{-\infty}^{+\infty} \psi_k(x,\mu)\mathrm{d}x. \tag{4.6}$$

By (4.5), Lemma 2.4 , and the property of Dirac function $\delta(x-a)$,

$$\sum_{i=1}^{n} q_i \exp\left(-\sqrt{-\mu}\ |a_i - a_k|\right) = \mu \cdot \frac{2}{\sqrt{-\mu}}, \tag{4.7}$$

i.e.,

$$\sum_{i=1,\ i\neq k}^{n} q_i \exp\left(-\sqrt{-\mu}\ |a_i - a_k|\right) + 2\sqrt{-\mu} + q_k = 0. \tag{4.8}$$

Notice that all $q_i < 0$, the function

$$f_k(t) := \sum_{i=1,\ i\neq k}^{n} q_i \exp\left(-|a_i - a_k|t\right) + 2t + q_k \tag{4.9}$$

is continuous and strictly increasing on $[0, +\infty)$, and

$$\lim_{t\to 0+} f_k(t) = \sum_{i=1}^{n} q_i < 0, \quad \lim_{t\to+\infty} f_k(t) = +\infty. \tag{4.10}$$

So $f_k(t) = 0$ has unique solution, denote it by α_k, on $(0, +\infty)$. And the equation (4.8) has unique solution $\mu_k = -t_k^2$, the corresponding eigenfuction is $\psi_k(x, \mu_k)$. □

Corollary 4.1. *The generalized Sturm-Liouville problem*

$$-y'' + q\delta(x-a)y = \mu y, \quad x \in (-\infty, +\infty), \tag{4.11}$$

where $q, a \in R$ are constants, has only one eigenvalue $\mu = -\dfrac{q^2}{4}$ when $q < 0$, and its corresponding eignfunction is $\varphi(x) = \exp\left(\dfrac{q}{2}|x-a|\right) = \mathrm{e}^{\frac{q}{2}|x-a|}$. It does not have any eigenvalue when $q \geq 0$.

Proof. The equation (4.8) becomes

$$2\sqrt{-\mu} = -q. \tag{4.12}$$

It does not have real solution when $q > 0$.

If $q = 0$, then $\mu = 0$. In this case, the non-trivial real solution of (4.11) is $y(x) \equiv 1$, it does not belong to $L^2(-\infty, +\infty)$.

If $q < 0$, the equation (4.12) solves $\mu = -q^2/4$, and the corresponding eignfunction is $\varphi(x) = \exp\left(\dfrac{q}{2}|x-a|\right)$ by (4.5).

□

Corollary 4.2. *The generalized Sturm-Liouville problem*

$$-y'' - \delta(x)y = \mu y, \quad x \in (-\infty, +\infty), \tag{4.13}$$

have an eigenvalue $\mu = -\dfrac{1}{4}$ *and its corresponding eignfunction is* $\varphi(x, \lambda) = \exp(-|x|/2)$.

From Theorem 3.2 and Corollary 4.1, we find that the problem (4.11) is inverse spectral problem of (3.17) when $\lambda = -q/2$ and $\mu = -p$, i.e. we can obtain p by $\lambda = -q/2$. p is satisfied equation $-q = 2\lambda = 2\sqrt{p}$, then we have $p = \dfrac{q^2}{4}$, so, $\mu = -p = -\dfrac{q^2}{4}$.

In other way, the problem (3.17) is inverse spectral problem of (4.11) when $\mu = -p$. From Corollary 4.1, q is satisfied equation $-p = \mu = -\dfrac{q^2}{4}$, then we have $\lambda = -q = \pm\sqrt{p}$, there is an addition root. By the proof of Corollary 4.1, the generalized Sturm-Liouville problem (4.11) have not any solution in $L^2(-\infty, +\infty)$ when $q \geq 0$. So, this inverse spectral problem have only one root $q = -2\sqrt{p}$, and $\lambda = -q/2 = \sqrt{p}$.

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