Journal of Nonlinear Modeling and Analysis
Volume 3, Number 1, March 2021, 93–104
http://jnma.ca; http://jnma.ijournal.cn
DOI:10.12150/jnma.2021.93

Generalized Coupled Fixed Point Results on Complex Partial Metric Space Using Contractive Condition

Mani Gunaseelan[1], Lakshmi Narayan Mishra[2] and Vishnu Narayan Mishra[3],†

Abstract In this paper, we obtain coupled fixed point results on complex partial metric space under contractive condition. An example to support our result is presented.

Keywords Coupled fixed point, Complex partial metric space.

MSC(2010) 47H10, 54H25.

1. Introduction

In many branches of science, economics, computer science, engineering and the development of nonlinear analysis, the fixed point theory is one of the most important tool. In 1989, Backhtin [4] introduced the concept of b-metric space. In 1993, Czerwik [6] extended the results of b-metric spaces. Altun, Sola and Simsek [1] introduced generalized contractions on partial metric spaces. Rao, Kishore, Tas, Satyanaraya and Prasad [10] introduced common coupled fixed point results in ordered partial metric spaces. Azam, Fisher and Khan [3] introduced new spaces called complex valued metric spaces and established the existence of fixed point theorems under the contraction condition. Dhivya and Marudai [7] introduced new spaces called complex partial metric space and established the existence of common fixed point theorems under the contraction condition of rational expression. Bhaskar and Lakshmikantham [8] introduced the concept of coupled fixed point. Ćirić and Lakshmikantham [5] investigated some more coupled fixed point theorems in partially ordered sets. Aydi [2] introduced coupled fixed point results on partial metric spaces. Gunaseelan and Mishra [9] introduced coupled fixed point theorems on complex partial metric space using different type of contractive conditions. In this paper, we introduced generalized coupled fixed point results on complex partial metric spaces under the contractive condition.

†the corresponding author.
Email address: vishnunarayanmishra@gmail.com(V. N. Mishra)
[1]Department of Mathematics, Sri Sankara Arts and Science College (Autonomous), Affiliated to Madras University, Enathur, Kanchipuram, Tamil Nadu 631 561, India
[2]Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology (VIT) University, Vellore 632 014, Tamil Nadu, India
[3]Department of Mathematics, Indira Gandhi National Tribal University, Laipur, Amarkantak, Anuppur, Madhya Pradesh 484887, India

2. Preliminaries

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\preceq$ on $\mathbb{C}$ as follows:
$z_1 \preceq z_2$ if and only if $Re(z_1) \leq Re(z_2)$, $Im(z_1) \leq Im(z_2)$.
Consequently,one can infer that $z_1 \preceq z_2$ if one of the following conditions is satisfied:
(i) $Re(z_1) = Re(z_2)$,$Im(z_1) < Im(z_2)$,
(ii)$Re(z_1) < Re(z_2)$,$Im(z_1) = Im(z_2)$,
(iii)$Re(z_1) < Re(z_2)$,$Im(z_1) < Im(z_2)$,
(iv)$Re(z_1) = Re(z_2)$,$Im(z_1) = Im(z_2)$.
In particular, we write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of $(i), (ii)$ and (iii) is satisfied and we write $z_1 \prec z_2$ if only (iii) is satisfied. Notice that
(a) If $0 \preceq z_1 \precnsim z_2$, then $|z_1| < |z_2|$,
(b) If $z_1 \preceq z_2$ and $z_2 \prec z_3$ then $z_1 \prec z_3$,
(c) If $a, b \in \mathbb{R}$ and $a \leq b$ then $az \preceq bz$ for all $z \in \mathbb{C}$.

Definition 2.1. [7] A complex partial metric on a non-empty set Y is a function $\sigma_c : Y \times Y \to \mathbb{C}^+$ such that for all $p, r, s \in Y$:
(i) $0 \preceq \sigma_c(p,p) \preceq \sigma_c(p,r)$(small self-distances)
(ii) $\sigma_c(p,r) = \sigma_c(r,p)(symmetry)$
(iii) $\sigma_c(p,p) = \sigma_c(p,r) = \sigma_c(r,r) \Leftrightarrow p = r(equality)$
(iv) $\sigma_c(p,r) \preceq \sigma_c(p,s) + \sigma_c(s,r) - \sigma_c(s,s)(triangularity)$.
A complex partial metric space is a pair (Y, σ_c) such that Y is a non empty set and σ_c is complex partial metric on Y.

For the complex partial metric σ_c on Y, the function $d_{\sigma_c} : Y \times Y \to \mathbb{C}^+$ given by $\sigma_c^t = 2\sigma_c(p,r) - \sigma_c(p,p) - \sigma_c(r,r)$ is a (usual) metric on Y. Each complex partial metric σ_c on Y generates a topology τ_{σ_c} on Y with the base family of open σ_c-balls $\{B_{\sigma_c}(p,\epsilon) : p \in Y, \epsilon > 0\}$, where $B_{\sigma_c}(p,\epsilon) = \{r \in Y : \sigma_c(p,r) < \sigma_c(p,p) + \epsilon\}$ for all $p \in Y$ and $0 < \epsilon \in \mathbb{C}^+$.

Definition 2.2. [7] Let (Y, σ_c) be a complex partial metric space(CPMS). A sequence (p_n) in a CPMS (Y, σ_c) is convergent to $p \in Y$, if for every $0 \prec \epsilon \in \mathbb{C}^+$ there is $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ we get $p_n \in B_{\sigma_c}(p, \epsilon)$

Definition 2.3. [7] Let (Y, σ_c) be a complex partial metric space. A sequence (p_n) in a CPMS (Y, σ_c) is called Cauchy if there is $a \in \mathbb{C}^+$ such that for every $\epsilon \prec 0$ there is $N \in \mathbb{N}$ such that for all $n, m \geq N$; $|\sigma_c(p_n, p_m) - a| < \epsilon$.

Definition 2.4. [7] Let (Y, σ_c) be a complex partial metric space(CPMS).
(1) A CPMS (Y, σ_c) is said to be complete if a Cauchy sequence (p_n) in Y converges, with respect to τ_{σ_c}, to a point $p \in Y$ such that $\sigma_c(p,p) = \lim_{n,m\to\infty} \sigma_c(p_n, p_m)$.
(2) A mapping $H : Y \to Y$ is said to be continuous at $p_0 \in Y$ if for every $\epsilon \prec 0$, there exists $\delta > 0$ such that $H(B_{\sigma_c}(p_0, \delta)) \subset B_{\sigma_c}(H(p_0, \epsilon))$.

Lemma 2.1. *[7] Let (Y, σ_c) be a complex partial metric space. A sequence $\{y_n\}$ is Cauchy sequence in the CPMS (Y, σ_c) then $\{y_n\}$ is Cauchy in a metric space (Y, σ_c^t).*

Definition 2.5. Let (Y, σ_c) be a complex partial metric space(CPMS).Then an element $(p, r) \in Y \times Y$ is said to be a coupled fixed point of the mapping F :

$Y \times Y \to Y$ if $F(p, r) = p$ and $F(r, p) = r$.

3. Main results

Theorem 3.1. *Let (Y, σ_c) be a complete complex partial metric space. Suppose that the mapping $\psi : Y \times Y \to Y$ satisfies the following contractive condition for all $p, q, r, s \in Y$*

$$\sigma_c(\psi(p, q), \psi(r, s)) \preceq k\sigma_c(p, r) + l\sigma_c(q, s),$$

where k, l are nonnegative constants with $k + l < 1$. Then, ψ has a unique coupled fixed point.

Proof. Choose $p_0, q_0 \in Y$ and set $p_1 = \psi(p_0, q_0)$ and $q_1 = \psi(q_0, p_0)$. Continuing this process, set $p_{n+1} = \psi(p_n, q_n)$ and $q_{n+1} = \psi(q_n, p_n)$.
Then,

$$\begin{aligned} \sigma_c(p_n, p_{n+1}) &= \sigma_c(\psi(p_{n-1}, q_{n-1}), \psi(p_n, q_n)) \\ &\preceq k\sigma_c(p_{n-1}, p_n) + l\sigma_c(q_{n-1}, q_n) \end{aligned}$$

which implies that

$$|\sigma_c(p_n, p_{n+1})| \leq k|\sigma_c(p_{n-1}, p_n)| + l|\sigma_c(q_{n-1}, q_n)| \tag{3.1}$$

Similarly, one can prove that

$$|\sigma_c(q_n, q_{n+1})| \leq k|\sigma_c(q_{n-1}, q_n)| + l|\sigma_c(p_{n-1}, p_n)| \tag{3.2}$$

From (3.1) and (3.2), we get

$$\begin{aligned} |\sigma_c(p_n, p_{n+1})| + |\sigma_c(q_n, q_{n+1})| &\leq (k + l)(|\sigma_c(q_{n-1}, q_n)| + |\sigma_c(p_{n-1}, p_n)|) \\ &= \alpha(|\sigma_c(q_{n-1}, q_n)| + |\sigma_c(p_{n-1}, p_n)|) \end{aligned}$$

where $\alpha = k + l < 1$.
Also,

$$|\sigma_c(p_{n+1}, p_{n+2})| \leq k|\sigma_c(p_n, p_{n+1})| + l|\sigma_c(q_n, q_{n+1})| \tag{3.3}$$

$$|\sigma_c(q_{n+1}, q_{n+2})| \leq k|\sigma_c(q_n, q_{n+1})| + l|\sigma_c(p_n, p_{n+1})| \tag{3.4}$$

From (3.3) and (3.4), we get

$$\begin{aligned} |\sigma_c(p_{n+1}, p_{n+2})| + |\sigma_c(q_{n+1}, q_{n+2})| &\leq (k + l)(|\sigma_c(q_n, q_{n+1})| + |\sigma_c(p_n, p_{n+1})|) \\ &= \alpha(|\sigma_c(q_n, q_{n+1})| + |\sigma_c(p_n, p_{n+1})|) \end{aligned}$$

Repeating this way, we get

$$\begin{aligned} |\sigma_c(p_n, p_{n+1})| + |\sigma_c(q_n, q_{n+1})| &\leq \alpha(|\sigma_c(q_{n-1}, q_n)| + |\sigma_c(p_{n-1}, p_n)|) \\ &\leq \alpha^2(|\sigma_c(q_{n-2}, q_{n-1})| + |\sigma_c(p_{n-2}, p_{n-1})|) \\ &\leq \cdots \leq \alpha^n(|\sigma_c(q_0, q_1)| + |\sigma_c(p_0, p_1)|) \end{aligned}$$

Now, if $|\sigma_c(p_n, p_{n+1})| + |\sigma_c(q_n, q_{n+1})| = s_n$, then

$$s_n \leq \alpha s_{n-1} \leq \cdots \leq \alpha^n s_0 \tag{3.5}$$

If $s_0 = 0$ then $|\sigma_c(p_0, p_1)| + |\sigma_c(q_0, q_1)| = 0$. Hence $p_0 = p_1 = \psi(p_0, q_0)$ and $q_0 = q_1 = \psi(q_0, p_0)$, which implies that (p_0, q_0) is a coupled fixed point of ψ.
Let $s_0 > 0$. For each $n \geq m$, we have

$$\begin{aligned}\sigma_c(p_n, p_m) &\preceq \sigma_c(p_n, p_{n-1}) + \sigma_c(p_{n-1}, p_{n-2}) - \sigma_c(p_{n-1}, p_{n-1}) \\ &\quad + \sigma_c(p_{n-2}, p_{n-3}) + \sigma_c(p_{n-3}, p_{n-4}) - \sigma_c(p_{n-3}, p_{n-3}) \\ &\quad + \cdots + \sigma_c(p_{m+2}, p_{m+1}) + \sigma_c(p_{m+1}, p_m) - \sigma_c(p_{m+1}, p_{m+1}) \\ &\preceq \sigma_c(p_n, p_{n-1}) + \sigma_c(p_{n-1}, p_{n-2}) + \cdots + \sigma_c(p_{m+1}, p_m)\end{aligned}$$

which implies that

$$|\sigma_c(p_n, p_m)| \leq |\sigma_c(p_n, p_{n-1})| + |\sigma_c(p_{n-1}, p_{n-2})| + \cdots + |\sigma_c(p_{m+1}, p_m)|.$$

Similarly, one can prove that

$$|\sigma_c(q_n, q_m)| \leq |\sigma_c(q_n, q_{n-1})| + |\sigma_c(q_{n-1}, q_{n-2})| + \cdots + |\sigma_c(q_{m+1}, q_m)|.$$

Thus,

$$\begin{aligned}|\sigma_c(p_n, p_m)| + |\sigma_c(q_n, q_m)| &\leq s_{n-1} + s_{n-2} + s_{n-3} + \cdots + s_m \\ &\leq (\alpha^{n-1} + \alpha^{n-2} + \cdots + \alpha^m)s_0 \\ &\leq \frac{\alpha^m}{1-\alpha} s_0 \to 0 \quad n \to \infty.\end{aligned}$$

which implies that $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in (Y, σ_c). Since the partial metric space (Y, σ_c) is complete, there exist $p, q \in Y$ such that $\{p_n\} \to p$ and $q_n \to q$ as $n \to \infty$ and $\sigma_c(p, p) = \lim_{n\to\infty} \sigma_c(p, p_n) = \lim_{n,m\to\infty} \sigma_c(p_n, p_m) = 0$, $\sigma_c(q, q) = \lim_{n\to\infty} \sigma_c(q, q_n) = \lim_{n,m\to\infty} \sigma_c(q_n, q_m) = 0$. We now show that $p = \psi(p, q)$. We suppose on the contrary that $p \neq \psi(p, q)$ and $q \neq \psi(q, p)$ so that $0 \prec \sigma_c(p, \psi(p, q)) = l_1$ and $0 \prec \sigma_c(q, \psi(q, p)) = l_2$
then

$$\begin{aligned}l_1 = \sigma_c(p, \psi(p, q)) &\preceq \sigma_c(p, p_{n+1}) + \sigma_c(p_{n+1}, \psi(p, q)) \\ &= \sigma_c(p, p_{n+1}) + \sigma_c(\psi(p_n, q_n), \psi(p, q)) \\ &\preceq \sigma_c(p, p_{n+1}) + k\sigma_c(p_n, p) + l\sigma_c(q_n, q)\end{aligned}$$

which implies that

$$|l_1| \leq |\sigma_c(p, p_{n+1})| + k|\sigma_c(p_n, p)| + l|\sigma_c(q_n, q)|$$

As $n \to \infty$, $|l_1| \leq 0$. Which is a contradiction, therefore $|\sigma_c(p, \psi(p, q))| = 0$ which implies that $p = \psi(p, q)$. Similarly we can prove that $q = \psi(q, p)$. Thus (p, q) is a coupled fixed point of ψ. Now, if (g, h) is another coupled fixed point of ψ, then

$$\sigma_c(p, g) = \sigma_c(\psi(p, q), \psi(g, h)) \preceq k\sigma_c(p, g) + l\sigma(q, h),$$

Thus,

$$\sigma_c(p, g) \preceq \frac{l}{1-k} \sigma_c(q, h) \tag{3.6}$$

which implies that

$$|\sigma_c(p,g)| \leq \frac{l}{1-k}|\sigma_c(q,h)| \tag{3.7}$$

Similarly,

$$|\sigma_c(q,h)| \leq \frac{l}{1-k}|\sigma_c(p,g)| \tag{3.8}$$

From (3.7) and (3.8), we get

$$|\sigma_c(p,g)| + |\sigma_c(q,h)| \leq \frac{l}{1-k}[|\sigma_c(p,g)| + |\sigma_c(q,h)|]$$

$$(1 - \frac{l}{1-k})(|\sigma_c(p,g)| + |\sigma_c(q,h)|) \leq 0$$

Since $k + l < 1$, this implies that $|\sigma_c(p,g)| + |\sigma_c(q,h)| \leq 0$. Therefore $p = g$ and $q = h \implies (p,q) = (g,h)$.
Thus, ψ has a unique coupled fixed point. □

Corollary 3.1. *Let (Y, σ_c) be a complete complex partial metric space. Suppose that the mapping $\psi : Y \times Y \to Y$ satisfies the following contractive condition for all $p, q, r, s \in Y$*

$$\sigma_c(\psi(p,q), \psi(r,s)) \preceq \frac{k}{2}(\sigma_c(p,r) + \sigma_c(q,s)), \tag{3.9}$$

where $0 \leq k < 1$. Then, ψ has a unique coupled fixed point.

Example 3.1. Let $Y = [0,\infty)$ endowed with the usual complex partial metric $\sigma_c \colon Y \times Y \to [0,\infty)$ defined by $\sigma_c(p,q) = \max\{p,q\}(1+i)$.The complex partial metric space (Y,σ_c) is complete because (Y,σ_c^t) is complete. Indeed, for any $p,q \in Y$

$$\begin{aligned}\sigma_c^t &= 2\sigma_c(p,r) - \sigma_c(p,p) - \sigma_c(r,r)\\ &= 2\max\{p,q\}(1+i) - (p+ip) - (q+iq)\\ &= |p-q| + i|p-q|.\end{aligned}$$

Thus, (Y,σ_c) is the Euclidean complex metric space which is complete. Consider the mapping $\psi \colon Y \times Y \to Y$ defined by $\psi(p,q) = \frac{p+q}{2}$. For any $p,q,g,h \in Y$, we have

$$\begin{aligned}\sigma_c(\psi(p,q),\psi(g,h)) &= \frac{1}{12}\max\{p+q, g+h\}(1+i)\\ &\leq \frac{1}{12}[\max\{p,g\} + \max\{q,h\}](1+i)\\ &= \frac{1}{12}[\sigma_c(p,u) + \sigma_c(q,h)].\end{aligned}$$

which is the contractive condition (3.9) for $k = \frac{1}{6}$. Therefore, by Corollary 3.1, ψ has a unique coupled fixed point, which is $(0,0)$. Note that if the mapping $\psi \colon Y \times Y \to Y$ is given by $\psi(p,q) = \frac{p+q}{2}$, then ψ satisfies the contractive condition (3.9) for $k = 1$, that is,

$$\sigma_c(\psi(p,q),\psi(g,h)) = \frac{1}{2}\max\{p+q, g+h\}(1+i)$$

$$\leq \frac{1}{2}[\max\{p,g\}+\max\{q,h\}](1+i)$$
$$= \frac{1}{2}[\sigma_c(p,g)+\sigma_c(q,h)].$$

In this case, $(0,0)$ and $(1,1)$ are both coupled fixed points of ψ, and hence, the coupled fixed point of ψ is not unique. This shows that the condition $k<1$ in Corollary 3.1, and hence $k+l<1$ in Theorem 3.1 cannot be omitted in the statement of the aforesaid results.

Theorem 3.2. *Let (Y,σ_c) be a complete complex partial metric space. Suppose that the mapping $\psi : Y\times Y \to Y$ satisfies the following contractive condition for all $p,q,r,s\in Y$*

$$\sigma_c(\psi(p,q),\psi(r,s)) \preceq k\sigma_c(\psi(p,q),r)+l\sigma_c(\psi(r,s),p),$$

where k,l are nonnegative constants with $k+2l<1$. Then, ψ has a unique coupled fixed point.

Proof. Choose $p_0,q_0\in Y$ and set $p_1=\psi(p_0,q_0)$ and $q_1=\psi(q_0,p_0)$. Continuing this process, set $p_{n+1}=\psi(p_n,q_n)$ and $q_{n+1}=\psi(q_n,p_n)$.
Then,

$$\begin{aligned}
\sigma_c(p_n,p_{n+1}) &= \sigma_c(\psi(p_{n-1},q_{n-1}),\psi(p_n,q_n))\\
&\preceq k\sigma_c(\psi(p_{n-1},q_{n-1}),p_n)+l\sigma_c(\psi(p_n,q_n),p_{n-1})\\
&= k\sigma_c(p_n,p_n)+l\sigma_c(p_{n+1},p_{n-1})\\
&\preceq k\sigma_c(p_n,p_{n+1})+l\sigma_c(p_{n+1},p_{n-1})\\
&\preceq k\sigma_c(p_n,p_{n+1})+l(\sigma_c(p_{n+1},p_n)+\sigma_c(p_n,p_{n-1})-\sigma_c(p_n,p_n))\\
&\preceq k\sigma_c(p_n,p_{n+1})+l(\sigma_c(p_{n+1},p_n)+\sigma_c(p_n,p_{n-1}))\\
&\preceq \frac{l}{1-(k+l)}\sigma_c(p_n,p_{n-1})
\end{aligned}$$

which implies that

$$|\sigma_c(p_n,p_{n+1})| \leq \frac{l}{1-(k+l)}|\sigma_c(p_n,p_{n-1})| \tag{3.10}$$

Similarly, one can prove that

$$|\sigma_c(q_n,q_{n+1})| \leq \frac{l}{1-(k+l)}|\sigma_c(q_n,q_{n-1})| \tag{3.11}$$

From (3.10) and (3.11), we get

$$\begin{aligned}
|\sigma_c(p_n,p_{n+1})|+|\sigma_c(q_n,q_{n+1})| &\leq \frac{l}{1-(k+l)}(|\sigma_c(p_n,p_{n-1})|+|\sigma_c(q_n,q_{n-1})|)\\
&= \alpha(|\sigma_c(p_n,p_{n-1})|+|\sigma_c(q_n,q_{n-1})|)
\end{aligned}$$

where $\alpha=\frac{l}{1-(k+l)}<1$.
Also,

$$|\sigma_c(p_{n+1},p_{n+2})| \leq \frac{l}{1-(k+l)}|\sigma_c(p_n,p_{n-1})| \tag{3.12}$$

$$|\sigma_c(q_{n+1}, q_{n+2})| \leq \frac{l}{1-(k+l)}|\sigma_c(q_n, q_{n-1})| \tag{3.13}$$

From (3.12) and (3.13), we get

$$\begin{aligned}|\sigma_c(p_{n+1}, p_{n+2})| + |\sigma_c(q_{n+1}, q_{n+2})| &\leq \frac{l}{1-(k+l)}(|\sigma_c(p_n, p_{n-1})| + |\sigma_c(q_n, q_{n-1})|)\\ &= \alpha(|\sigma_c(p_n, p_{n-1})| + |\sigma_c(q_n, q_{n-1})|)\end{aligned}$$

Repeating this way, we get

$$\begin{aligned}|\sigma_c(p_n, p_{n+1})| + |\sigma_c(q_n, q_{n+1})| &\leq \alpha(|\sigma_c(p_n, p_{n-1})| + |\sigma_c(q_n, q_{n-1})|)\\ &\leq \alpha^2(|\sigma_c(q_{n-2}, q_{n-1})| + |\sigma_c(p_{n-2}, p_{n-1})|)\\ &\leq \cdots \leq \alpha^n(|\sigma_c(q_0, q_1)| + |\sigma_c(p_0, p_1)|)\end{aligned}$$

Now, if $|\sigma_c(p_n, p_{n+1})| + |\sigma_c(q_n, q_{n+1})| = s_n$, then

$$s_n \leq \alpha s_{n-1} \leq \cdots \leq \alpha^n s_0 \tag{3.14}$$

If $s_0 = 0$ then $|\sigma_c(p_0, q_1)| + |\sigma_c(q_0, q_1)| = 0$. Hence $p_0 = p_1 = \psi(p_0, q_0)$ and $q_0 = q_1 = \psi(q_0, p_0)$, which implies that (p_0, q_0) is a coupled fixed point of ψ.
Let $s_0 > 0$. For each $n \geq m$,we have

$$\begin{aligned}\sigma_c(p_n, p_m) \preceq\ &\sigma_c(p_n, p_{n-1}) + \sigma_c(p_{n-1}, p_{n-2}) - \sigma_c(p_{n-1}, p_{n-1})\\ &+ \sigma_c(p_{n-2}, p_{n-3}) + \sigma_c(p_{n-3}, p_{n-4}) - \sigma_c(p_{n-3}, p_{n-3})\\ &+ \cdots + \sigma_c(p_{m+2}, p_{m+1}) + \sigma_c(p_{m+1}, p_m) - \sigma_c(p_{m+1}, p_{m+1})\\ &\preceq \sigma_c(p_n, p_{n-1}) + \sigma_c(p_{n-1}, p_{n-2}) + \cdots + \sigma_c(p_{m+1}, p_m)\end{aligned}$$

which implies that

$$|\sigma_c(p_n, p_m)| \leq |\sigma_c(p_n, p_{n-1})| + |\sigma_c(p_{n-1}, p_{n-2})| + \cdots + |\sigma_c(p_{m+1}, p_m)|.$$

Similarly,one can prove that

$$|\sigma_c(q_n, q_m)| \leq |\sigma_c(q_n, q_{n-1})| + |\sigma_c(q_{n-1}, q_{n-2})| + \cdots + |\sigma_c(q_{m+1}, q_m)|.$$

Thus,

$$\begin{aligned}|\sigma_c(p_n, p_m)| + |\sigma_c(q_n, q_m)| &\leq s_{n-1} + s_{n-2} + s_{n-3} + \cdots + s_m\\ &\leq (\alpha^{n-1} + \alpha^{n-2} + \cdots + \alpha^m)s_0\\ &\leq \frac{\alpha^m}{1-\alpha}s_0 \to 0 \quad n \to \infty.\end{aligned}$$

which implies that $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in (Y, σ_c). Since the partial metric space (Y, σ_c) is complete, there exist $p, q \in Y$ such that $\{p_n\} \to p$ and $q_n \to q$ as $n \to \infty$ and $\sigma_c(p,p) = \lim_{n\to\infty}\sigma_c(p, p_n) = \lim_{n,m\to\infty}\sigma_c(p_n, p_m) = 0$,$\sigma_c(q,q) = \lim_{n\to\infty}\sigma_c(q, q_n) = \lim_{n,m\to\infty}\sigma_c(q_n, q_m) = 0$. We now show that $p = \psi(p,q)$. We suppose on the contrary that $p \neq \psi(p,q)$ and $q \neq \psi(q,p)$ so that

$0 \prec \sigma_c(p, \psi(p,q)) = l_1$ and $0 \prec \sigma_c(q, \psi(q,p)) = l_2$
then

$$\begin{aligned} l_1 = \sigma_c(p, \psi(p,q)) &\preceq \sigma_c(p, p_{n+1}) + \sigma_c(p_{n+1}, \psi(p,q)) \\ &= \sigma_c(p, p_{n+1}) + \sigma_c(\psi(p_n, q_n), \psi(p,q)) \\ &\preceq \sigma_c(p, p_{n+1}) + k\sigma_c(\psi(p_n, q_n), p) + l\sigma_c(\psi(p,q), p_n) \\ &= \sigma_c(p, p_{n+1}) + k\sigma_c(p_{n+1}, p) + l\sigma_c(\psi(p,q), p_n) \end{aligned}$$

which implies that

$$|l_1| \leq |\sigma_c(p, p_{n+1})| + k|\sigma_c(p_n, p)| + l|\sigma_c(\psi(p,q), p_n)|$$

As $n \to \infty$, $|l_1| \leq 0$. Which is a contradiction, therefore $|\sigma_c(p, \psi(p,q))| = 0$ which implies that $p = \psi(p,q)$. Similarly we can prove that $q = \psi(q,p)$. Thus (p,q) is a coupled fixed point of ψ. Now, if (g,h) is another coupled fixed point of ψ, then

$$\sigma_c(p,g) = \sigma_c(\psi(p,q), \psi(g,h)) \preceq k\sigma_c(\psi(p,q), g) + l\sigma(\psi(g,h), p),$$

Thus,

$$(1 - (k+l))\sigma_c(p,g) \preceq 0 \tag{3.15}$$

which implies that

$$(1 - (k+l))|\sigma_c(p,g)| \leq 0 \tag{3.16}$$

Similarly,

$$(1 - (k+l))|\sigma_c(q,h)| \leq 0 \tag{3.17}$$

From (3.16) and (3.17),since $k + l < 1$. Therefore $p = g$ and $q = h$
which implies that $(p,q) = (g,h)$.
Thus, ψ has a unique coupled fixed point. □

from theorems (3.2) with $k = l$, we get the following corollary.

Corollary 3.2. *Let (Y, σ_c) be a complete complex partial metric space. Suppose that the mapping $\psi : Y \times Y \to Y$ satisfies the following contractive condition for all $p, q, r, s \in Y$*

$$\sigma_c(\psi(p,q), \psi(r,s)) \preceq k(\sigma_c(\psi(p,q), r) + \sigma_c(\psi(r,s), p)),$$

where k is nonnegative constant with $k < \frac{1}{3}$. Then, ψ has a unique coupled fixed point.

Theorem 3.3. *Let (Y, σ_c) be a complete complex partial metric space. Suppose that the mapping $\psi : Y \times Y \to Y$ satisfies*

$$\sigma_c(\psi(p,q), \psi(r,s)) \preceq r \max\{\sigma_c(p,r), \sigma_c(q,s), \sigma_c(\psi(p,q), p), \sigma_c(\psi(r,s), r)\},$$

for all $p, q, r, s \in Y$. If $r \in [0,1)$, then ψ has a unique coupled fixed point.

Proof. Choose $p_0, q_0 \in Y$ and set $p_1 = \psi(p_0, q_0)$ and $q_1 = \psi(q_0, p_0)$. Continuing this process, set $p_{n+1} = \psi(p_n, q_n)$ and $q_{n+1} = \psi(q_n, p_n)$.
Then,

$$\begin{aligned}\sigma_c(p_{n+1}, p_{n+2}) &= \sigma_c(\psi(p_n, q_n), \psi(p_{n+1}, q_{n+1}))\\ &\preceq r\max\{\sigma_c(p_n, p_{n+1}), \sigma_c(q_n, q_{n+1}), \sigma_c(\psi(p_n, q_n), p_n),\\ &\sigma_c(\psi(p_{n+1}, q_{n+1}), p_{n+1})\}\\ &= r\max\{\sigma_c(p_n, p_{n+1}), \sigma_c(q_n, q_{n+1}),\\ &\sigma_c(p_{n+1}, p_n), \sigma_c(p_{n+2}, p_{n+1})\}\\ &\preceq r\max\{\sigma_c(p_n, p_{n+1}), \sigma_c(q_n, q_{n+1})\},\end{aligned}$$

which implies that

$$|\sigma_c(p_{n+1}, p_{n+2})| \le r\max\{|\sigma_c(p_n, p_{n+1})|, |\sigma_c(q_n, q_{n+1})|\}. \tag{3.18}$$

Similarly, one can prove that

$$|\sigma_c(q_{n+1}, q_{n+2})| \le r\max\{|\sigma_c(q_n, q_{n+1})|, |\sigma_c(p_n, p_{n+1})|\}. \tag{3.19}$$

From (3.18) and (3.19), we get

$$\max\{|\sigma_c(p_{n+1}, p_{n+2})|, |\sigma_c(q_{n+1}, q_{n+2})|\} \le r\max\{|\sigma_c(q_n, q_{n+1})|, |\sigma_c(p_n, p_{n+1})|\}. \tag{3.20}$$

Continuing this process, we get

$$\begin{aligned}\max\{|\sigma_c(p_n, p_{n+1})|, |\sigma_c(q_n, q_{n+1})|\} &\le r\max\{|\sigma_c(q_{n-1}, q_n)|, |\sigma_c(p_{n-1}, p_n)|\}\\ &\le r^2\max\{|\sigma_c(q_{n-2}, q_{n-1})|, |\sigma_c(p_{n-2}, p_{n-1})|\}\\ &\vdots\\ &\le r^n\max\{|\sigma_c(q_0, q_1)|, |\sigma_c(p_0, p_1)|\}.\end{aligned}$$

As $n \to \infty$,

$$\lim_{n\to\infty}\max\{|\sigma_c(p_n, p_{n+1})|, |\sigma_c(q_n, q_{n+1})|\} = 0.$$

Therefore,

$$\lim_{n\to\infty}|\sigma_c(p_n, p_{n+1})| = 0, \tag{3.21}$$

$$\lim_{n\to\infty}|\sigma_c(q_n, q_{n+1})| = 0 \tag{3.22}$$

For each $n > m$, we have

$$\begin{aligned}\sigma_c(p_n, p_m) &\preceq \sigma_c(p_n, p_{n-1}) + \sigma_c(p_{n-1}, p_{n-2}) - \sigma_c(p_{n-1}, p_{n-1})\\ &+ \sigma_c(p_{n-2}, p_{n-3}) + \sigma_c(p_{n-3}, p_{n-4}) - \sigma_c(p_{n-3}, p_{n-3})\\ &+ \cdots + \sigma_c(p_{m+2}, p_{m+1}) + \sigma_c(p_{m+1}, p_m) - \sigma_c(p_{m+1}, p_{m+1})\\ &\preceq \sigma_c(p_n, p_{n-1}) + \sigma_c(p_{n-1}, p_{n-2}) + \cdots + \sigma_c(p_{m+1}, p_m)\end{aligned}$$

which implies that

$$|\sigma_c(p_n, p_m)| \le |\sigma_c(p_n, p_{n-1})| + |\sigma_c(p_{n-1}, p_{n-2})| + \cdots + |\sigma_c(p_{m+1}, p_m)|.$$

Therefore,

$$|\sigma_c(p_n, p_m)| \le r^n \max\{|\sigma_c(q_0, q_1)|, |\sigma_c(p_0, p_1)|\}.$$

As $n, m \to \infty$,

$$\lim_{n\to\infty} |\sigma_c(p_n, p_m)| = 0.$$

Similarly, one can prove that

$$|\sigma_c(q_n, q_m)| \le |\sigma_c(q_n, q_{n-1})| + |\sigma_c(q_{n-1}, q_{n-2})| + \cdots + |\sigma_c(q_{m+1}, q_m)|,$$

$$|\sigma_c(q_n, q_m)| \le r^n \max\{|\sigma_c(q_0, q_1)|, |\sigma_c(p_0, p_1)|\},$$

$$\lim_{n\to\infty} |\sigma_c(q_n, q_m)| = 0.$$

which implies that $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in (Y, σ_c).

Since the partial metric space (Y, σ_c) is complete, there exist $p, q \in Y$ such that $\{p_n\} \to p$ and $q_n \to q$ as $n \to \infty$ and $\sigma_c(p,p) = \lim_{n\to\infty} \sigma_c(p, p_n) = \lim_{n,m\to\infty} \sigma_c(p_n, p_m) = 0$, $\sigma_c(q,q) = \lim_{n\to\infty} \sigma_c(q, q_n) = \lim_{n,m\to\infty} \sigma_c(q_n, q_m) = 0$. We now show that $p = \psi(p,q)$. Now

$$\begin{aligned}
\sigma_c(p, \psi(p,q)) &\preceq \sigma_c(p, p_{n+1}) + \sigma_c(p_{n+1}, \psi(p,q)) \\
&= \sigma_c(p, p_{n+1}) + \sigma_c(\psi(p_n, q_n), \psi(p,q)) \\
&\preceq \sigma_c(p, p_{n+1}) + r \max\{\sigma_c(p_n, p), \sigma_c(q_n, q), \sigma_c(\psi(p_n, q_n), p_n), \\
&\sigma_c(\psi(p,q), p)\} \\
&= \sigma_c(p, p_{n+1}) + r \max\{\sigma_c(p_n, p), \sigma_c(q_n, q), \sigma_c(p_{n+1}, p_n, \sigma_c(\psi(p,q), p)\},
\end{aligned}$$

which implies that

$$\begin{aligned}
|\sigma_c(p, \psi(p,q))| \le &|\sigma_c(p, p_{n+1})| + r \max\{|\sigma_c(p_n, p)|, |\sigma_c(q_n, q)|, |\sigma_c(p_{n+1}, p_n)|, \\
&|\sigma_c(\psi(p,q), p)|\}
\end{aligned}$$

As $n \to \infty$, $|\sigma_c(p, \psi(p,q))| \le r|\sigma_c(\psi(p,q), p)|$.
Since $r \in [0, 1)$, therefore $|\sigma_c(p, \psi(p,q))| = 0 \implies p = \psi(p,q)$. Similarly we can prove that $q = \psi(q,p)$. Thus (p,q) is a coupled fixed point of ψ. Now, if (g,h) is another coupled fixed point of ψ, then

$$\begin{aligned}
\sigma_c(p,g) = \sigma_c(\psi(p,q), \psi(g,h)) &\preceq r \max\{\sigma_c(p,g), \sigma_c(q,h), \sigma_c(\psi(p,q), p), \sigma_c(\psi(g,h), g)\} \\
&\preceq r \max\{\sigma_c(p,g), \sigma_c(q,h), \sigma_c(p,p), \sigma_c(g,g)\},
\end{aligned}$$

Since $\sigma_c(p,p) \preceq \sigma_c(p,g)$ and $\sigma_c(g,g) \preceq \sigma_c(p,g)$, we have

$$\sigma_c(p,g) \preceq r \max\{\sigma_c(p,g), \sigma_c(q,h)\}$$

$$|\sigma_c(p,g)| \leq r \max\{|\sigma_c(p,g)|, |\sigma_c(q,h)|\}. \tag{3.23}$$

Similarly, we can prove

$$|\sigma_c(q,h)| \leq r \max\{|\sigma_c(p,g)|, |\sigma_c(q,h)|\}. \tag{3.24}$$

From (3.23) and (3.24), we have

$$\max\{|\sigma_c(p,g)|, |\sigma_c(q,h)|\} \leq r \max\{|\sigma_c(p,g)|, |\sigma_c(q,h)|\} \tag{3.25}$$

Since $r < 1$, we have $\max\{|\sigma_c(p,g)|, |\sigma_c(q,h)|\} = 0$ which implies that $\sigma_c(p,g) = 0$ and $\sigma_c(q,h) = 0$. Therefore $p = g$ and $q = h$ which implies that $(p,q) = (g,h)$. Thus, ψ has a unique coupled fixed point. □

Corollary 3.3. *Let (Y, σ_c) be a complete complex partial metric space. Suppose that the mapping $\psi : Y \times Y \to Y$ satisfies*

$$\sigma_c(\psi(p,q), \psi(r,s)) \preceq a\sigma_c(p,r) + b\sigma_c(q,s) + c\sigma_c(\psi(p,q),p) + d\sigma_c(\psi(r,s),r),$$

for all $p,q,r,s \in Y$ with $a,b,c,d \in [0,1)$, then ψ has a unique coupled fixed point.

Proof. The proof follows from Theorem 3.3.
Note that

$$a\sigma_c(p,r) + b\sigma_c(q,s) + c\sigma_c(\psi(p,q),p) + d\sigma_c(\psi(r,s),r) \leq (a+b+c+d) \\ \max\{\sigma_c(p,r), \sigma_c(q,s), \sigma_c(\psi(p,q),p), \sigma_c(\psi(r,s),r)\}$$

□

Example 3.2. Let $Y = [0,\infty)$ endowed with the usual complex partial metric $\sigma_c \colon Y \times Y \to [0,\infty)$ defined by $\sigma_c(p,q) = \max\{p,q\}(1+i)$. The complex partial metric space (Y, σ_c) is complete because (Y, σ_c^t) is complete. Indeed, for any $p,q \in Y$

$$\begin{aligned} \sigma_c^t &= 2\sigma_c(p,r) - \sigma_c(p,p) - \sigma_c(r,r) \\ &= 2\max\{p,q\}(1+i) - (p+ip) - (q+iq) \\ &= |p-q| + i|p-q|. \end{aligned}$$

Thus, (Y, σ_c) is the Euclidean complex metric space which is complete. Consider the mapping $\psi \colon Y \times Y \to Y$ defined by $\psi(p,q) = \frac{|p-q|}{2}$. For any $p,q,g,h \in Y$ we have

$$\begin{aligned} \sigma_c(\psi(p,q), \psi(g,h)) &= \frac{1}{2}\max\{|p-q|, |g-h|\}(1+i) \\ &= \frac{1}{2}\max\{p-q, q-p, g-h, h-g\}(1+i) \\ &\preceq \frac{1}{2}\max\{p,q,g,h\}(1+i) \\ &= \frac{1}{2}\max\{\sigma_c(p,g), \sigma_c(q,h)\} \\ &\preceq \frac{1}{2}\max\{\sigma_c(p,g), \sigma_c(q,h), \sigma_c(\psi(p,q),p), \sigma_c(\psi(g,h),g)\}. \end{aligned}$$

Thus, ψ has a unique coupled fixed point. Here, $(0,0)$ is the unique fixed point of ψ.

4. Conclusion

In 2019, Gunaseelan and Mishra [9] proved coupled fixed point theorem on complex partial metric space. In this paper we proved coupled fixed point results on complex partial metric space using contractive condition.

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DOI: 10.1186/s13663-017-0610-3